Solutions of Matrix Games Involving Linguistic Interval-Valued Intuitionistic Fuzzy Sets

Deeba Naqvi

Indira Gandhi Delhi Technical University for Women

Rajkumar Verma

University of Chile: Universidad de Chile

Abha Aggarwal

Guru Gobind Singh Indraprastha University

Geeta Sachdev (✉ geeta.mehndiratta@rediffmail.com)

Indira Gandhi Delhi Technical University for Women <https://orcid.org/0000-0002-5387-0005>

Research Article

Keywords: matrix games, linguistic variable, intuitionistic fuzzy set, linguistic interval-valued intuitionistic fuzzy set, average aggregation operator

Posted Date: November 12th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-742814/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Solutions of Matrix Games Involving Linguistic Interval-Valued Intuitionistic Fuzzy Sets

Deeba Naqvi · Rajkumar Verma · Abha Aggarwal · Geeta Sachdev[✉]

the date of receipt and acceptance should be inserted later

Abstract In real-life decision-making challenges, experts quite frequently have a preference for expressing their perspective in natural linguistic terms rather than definite numerical format. These linguistic representation has been utilized to resolve plenty of decision-making problems. This paper displays the thorough study of matrix games where in the payoffs are characterized through linguistic interval-valued intuitionistic fuzzy sets (LIVIFSs). Solution of these matrix games are attained by resolving a duo of linear or nonlinear programming problems, originated through non-linear bi-objective programming problems. Finally, a numerical example is used to demonstrate the applicability of the suggested approach.

Keywords matrix games · linguistic variable · intuitionistic fuzzy set · linguistic interval-valued intuitionistic fuzzy set · average aggregation operator

1 Introduction

Zadeh (1965) defined fuzzy sets by employing membership degrees which only expresses the degree of affinity to the fuzzy set, under consideration. The concept of intuitionistic fuzzy sets (I-fuzzy set) was first suggested by Atanassov (1986), by introducing non-membership degrees to fuzzy sets in addition to membership degrees and reports the ambiguous viewpoints not only by degree of membership but also non-membership functions. I-fuzzy sets together with the pair of degree of membership and non-membership of a considered set describes degree of hesitancy/ unreliability. These sets are extensively utilized in many real-life problems such as, Szmidt

Deeba Naqvi
Indira Gandhi Delhi Technical University for Women, Delhi-110006, India and
Maharaja Surajmal Institute of Technology, C-4 Janakpuri, New Delhi-110058, India
E-mail: dzmh19@gmail.com

Rajkumar Verma
Department of Management Control and Information Systems, University of Chile,
Av. Diagonal Paraguay 257, 8330015, Santiago, Chile
E-mail: rkver83@gmail.com

Abha Aggarwal
University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Delhi-110078, India
E-mail: abhaaggarwal27@gmail.com

Geeta Sachdev[✉]
Indira Gandhi Delhi Technical University for Women, Delhi-110006, India
E-mail: geeta.mehndiratta@rediffmail.com

and Kacprzyk (1996); Atanassov (1999); De et al. (2001); Li and Chuntian (2002); Vlachos and Sergiadis (2007); Li (2010b); Verma and Sharma (2014, 2015); Zhang et al. (2017), to name a few.

Because real-life decision-making scenarios are more unpredictable and ambiguous, it's more practical to use linguistic variables (LVs) rather than numerical numbers. Zadeh (1975a,b) introduced the notion of linguistic variables to effectively express imprecise or ambiguous facts. Many researchers have focused their efforts on using linguistic variables to solve decision-making challenges, Herrera et al. (2000, 2009); Xu (2007); Jiang et al. (2008); Wei (2009); Zhang (2014); Chen et al. (2015); Tao et al. (2017); Xian et al. (2017), as the linguistic information process does not result in information degrading or misperception. To explore the multi-criteria decision-making problems with LVs, Herrera et al. (2000); Nan et al. (2014) remodeled the LVs into triangular fuzzy numbers/trapezoidal fuzzy numbers. But this process of transformation of LVs into fuzzy numbers may lead to information distortion or loss. Clearly, as I-fuzzy sets are advantageous, Wang and Li (2010) put forward a new term called intuitionistic linguistic variables (ILVs) by appending non-membership degrees on the basis of LVs. The I-fuzzy numbers expresses only quantitative assessment information given by decision makers whereas qualitative information are not well defined using these numbers. Zhang (2014) introduced the paradigm of a linguistic intuitionistic fuzzy set (LIFS), which combines the LVs and I-fuzzy set into a unique expression, thus, describing both qualitative and quantitative information. Subsequently Zhang (2014), adapted it to resolve the multi-attribute group decision-making problem. Furthermore, Verma (2020) generalised the postulates of linguistic fuzzy sets by instigating linguistic trapezoidal fuzzy intuitionistic fuzzy sets and considered its implementation in multiple attribute group decision making problems.

Game theory is a powerful analytical tool dealing with day-to-day problems. In some aspects, this is the study of strategy, or the optimal decision-making of independent and opposing actors/players in a strategic situation. In 1940, Neumann and Morgenstern (1944) were two of the most important architects of game theory. Many researchers have explored game theory extensively, taking into account a few Leng and Parlar (2005); Collins and Hu (2008); Kalpiński and Tamošaitiene (2010); Madani (2010); Sanchez-Soriano (2013). However, in real-life situations, we encounter uncertainty or ambiguity in the information about a problem, thus, motivating the study of fuzzy game theory. Butnariu (1978) and Aubin (1981) were the first to investigate the fuzzy game theory. Campos (1989) mainly explained the solution procedures for matrix games with fuzzy payoffs, whereas Sakawa and Nishizaki (1994) investigated zero-sum fuzzy matrix games with multi-objectives. In matrix games characterized by fuzzy payoffs, Maeda (2003) developed three types of min-max equilibrium approaches. In contrast, Bector et al. (2004a) resolved the matrix games involving fuzzy goals utilizing the fuzzy linear programming technique. Furthermore, Bector et al. (2004b) established the duality results for linear programming problems with fuzzy variables and employed these to define the solution procedure of zero-sum matrix games involving payoffs expressed by fuzzy terms. Vijay et al. (2007) used the extensive duality notion to develop the generalised representation to examine matrix games with fuzzy goals and payoffs using the fuzzy relation technique. Cevikel and Ahlatolu (2010) devised two models for analysing two person zero-sum matrix games with ambiguous payoffs and goals. Li (2012) also investigated matrix games involving triangular fuzzy payoffs. Jana and Roy (2018) recently explained the matrix game solution approach incorporating generalised trapezoidal fuzzy payoffs.

Atanassov (1995) pioneered the comprehensive analysis of matrix games involving payoffs described by I-

fuzzy sets. Li and Nan (2009) suggested the nonlinear programming technique, for deriving the solution of matrix games having I-fuzzy payoffs. Aggarwal et al. (2012a,b) extended the fuzzy duality results of Bector et al. (2004b); Vijay et al. (2005), to intuitionistic fuzzy linear programming duality and employed it on zero-sum matrix games. Thus, inspiring Khan et al. (2017) to introduce a technique to resolve indeterminacy from matrix games having intuitionistic fuzzy goals. Xia (2018) put forward the generalised procedure for matrix games involving interval-valued intuitionistic fuzzy (IVIF) payoffs utilizing generalized aggregation operators. Lately, Naqvi et al. (2019) outlined a novel Tanaka and Asai's technique to explain the solution procedure for intuitionistic fuzzy matrix games. Xing and Qiu (2019) also contributed to I-fuzzy matrix games by adopting linear accuracy function to resolve the solution of matrix games where the payoffs involved are triangular I-fuzzy numbers. Xia (2019) explored a specific sort of matrix game with cross-evaluated payoffs in depth. Several initiatives to discover solutions for intuitionistic fuzzy matrix games have been conducted so far, Seikh et al. (2013, 2015a,b); Khan et al. (2016); Bhaumik et al. (2017); Nan et al. (2010, 2017), to mention only some.

1.1 Motivation for proposed work

Fuzzy and intuitionistic fuzzy matrix games have been used to tackle a variety of real-world decision-making situations in a variety of application domains. In many realistic scenarios, however, the payoffs of a matrix game may not be described in numerical terms, but may be simply analyzed using linguistic concepts. Many eminent researchers have significantly studied the linguistic variables and their applications. Arfi (2006) described the concepts of linguistic fuzzy domination and Nash equilibrium in game theory based on linguistic fuzzy logic. In recent years, Garg and Kumar (2019) framed a distinct concept named linguistic interval-valued Atanassov's intuitionistic fuzzy sets, which is being utilized as an efficient tool to analyse the group decision-making problems. Furthermore, Verma and Aggarwal (2021) described the study of matrix games with payoffs represented by linguistic intuitionistic fuzzy numbers. Taking motivation from their work, here we make an effort to apply this tool to resolve the matrix games having payoffs described by linguistic interval-valued intuitionistic fuzzy sets (LIVIFSs), as it has not yet been explored till date.

The significance of this paper is as follows :

1. It focuses on reporting the study of solution plan for the fuzzy matrix game possessing payoffs of linguistic interval-valued intuitionistic fuzzy number, thus describing the expectation of the game as a linguistic interval-valued intuitionistic fuzzy number.
2. By using LIVIFSs, one can define its belongingness as well as its non-belongingness explicitly. Furthermore, there is a little risk of information loss or distortion during the process of solving matrix games with LIVIFSs, thus, making it convenient for the players to adopt the optimal alternative in the end.
3. To compare and aggregate the LIVIFNs payoffs, well defined comparison criteria and aggregation functions are used, which can assist and possibly prevent certain undesirable changes in the processing of information.

4. The recommended methodology suggests that by resolving a duo of primal–dual linear-programming problems, it is feasible to acquire the solutions of matrix games with LIVIFNs payoffs. The findings suggest that the introduced technique can accurately demonstrate the significance degree to which one player is preferred over another in terms of winning the game.

Extending the work of Li (2010a) for LIVIFSs, here we reduce the problems of the two players into a duo of linear bi-objective programming problems utilizing the weighted mean aggregation operator for LIVIFSs. Thus, attaining the optimal values and the mixed strategies for the two players, player I and player II, respectively.

The remaining article is assembled in the following way : In Section 2 , we analyse certain standard definitions and present concepts of LIVIFSs. In Section 3, detailed structure of matrix games with LIVIFS information is defined along with its solution concept for the two players. Section 4, presents the numerical illustration to validate the efficiency of the above said approach. Section 5, provides some conclusions and finally, in Section 6 some of the future aspects of the proposed approach are enlisted.

2 Preliminaries

This segment reviews some of the conventional definitions of intuitionistic fuzzy sets, linguistic sets and some important associated concepts from Atanassov and Gargov (1989); Herrera and Martinez (2000); Liu and Chen (2018); Martinez and Herrera (2012), respectively.

2.1 Intuitionistic Fuzzy Set (I-fuzzy set)

Definition 1 Atanassov (1986) Let X be the finite universe of discourse. An I-fuzzy set $\hat{M}$ in X is defined as

$$\hat{M} = \{ \langle x, \mu_{\hat{M}}(x), \nu_{\hat{M}}(x) \rangle \mid x \in X \},$$

where $\mu_{\hat{M}} : X \rightarrow [0, 1]$ and $\nu_{\hat{M}} : X \rightarrow [0, 1]$ defines, respectively the membership and non-membership functions, such that $\mu_{\hat{M}}(x) + \nu_{\hat{M}}(x) \leq 1$.

Definition 2 Atanassov (1986) Let $\hat{M}$ be an I-fuzzy set in X . Then the value $\pi_{\hat{M}}(x)$ given by

$$\pi_{\hat{M}}(x) = 1 - \mu_{\hat{M}}(x) - \nu_{\hat{M}}(x), \quad x \in X,$$

is called the measure of indeterminacy or undecidedness of $x \in X$.

Note that the range of undecidedness of $x \in \hat{M}$ is an interval $[\mu_{\hat{M}}(x), 1 - \nu_{\hat{M}}(x)]$, and the measure of its indeterminacy is length of this interval.

2.2 Interval Valued Intuitionistic Fuzzy Set (IVIFS)

Definition 3 Atanassov and Gargov (1989) Let X be the finite universe of discourse and $D[0, 1]$ be the set of all the closed subintervals of the interval $[0, 1]$. An interval valued intuitionistic fuzzy set (IVIFS) $\hat{V}$ in X is defined as

$$\hat{V} = \left\{ \left\langle x, \left[\mu_{\hat{V}}^-(x), \mu_{\hat{V}}^+(x) \right], \left[\nu_{\hat{V}}^-(x), \nu_{\hat{V}}^+(x) \right] \right\rangle \mid x \in X \right\}$$

where $[\mu_{\hat{V}}^-(x), \mu_{\hat{V}}^+(x)] \in D[0, 1]$ and $[\nu_{\hat{V}}^-(x), \nu_{\hat{V}}^+(x)] \in D[0, 1]$, with the condition $\mu_{\hat{V}}^+(x) + \nu_{\hat{V}}^+(x) \leq 1, \forall x \in X$. Here, the intervals $[\mu_{\hat{V}}^-(x), \mu_{\hat{V}}^+(x)]$ and $[\nu_{\hat{V}}^-(x), \nu_{\hat{V}}^+(x)]$ represents, respectively the degree of membership and the degree of non-membership of the element $x \in X$ to $\hat{V}$.

We call the interval

$$[\pi_{\hat{V}}^-(x), \pi_{\hat{V}}^+(x)] = [1 - \mu_{\hat{V}}^+(x) - \nu_{\hat{V}}^+(x), 1 - \mu_{\hat{V}}^-(x) - \nu_{\hat{V}}^-(x)],$$

as the measure of indeterminacy or hesitancy degree of interval-valued intuitionistic fuzzy set $\hat{V}$ of $x \in X$.

2.3 Linguistic Term Set (LTS)

Many decision-making challenges in real-world circumstances have qualitative characteristics that are difficult to determine using numerical figures. In such instances, linguistic variables are a more practical and appropriate instrument for evaluating qualitative data. Herrera and Martinez Herrera and Martinez (2000) expressed the Linguistic Variables (LVs) as follows.

Definition 4 Herrera and Martinez (2000) Let $\mathbb{S} = \{s_w | w = 0, 1, \dots, t\}$ be a finite linguistic term set (LTS) with an odd cardinality, where s_w represents a possible linguistic value for a linguistic variable and t be the positive integer. The LTS $\mathbb{S}$ should satisfy the following properties:

1. Well ordered: $s_p \leq s_q \Leftrightarrow p \leq q$;
2. Negation operator: $neg(s_\epsilon) = s_{t-\epsilon}$;
3. Maximum operator: $\max(s_p, s_q) = s_p \Leftrightarrow p \geq q$;
4. Minimum operator: $\min(s_p, s_q) = s_p \Leftrightarrow p \leq q$.

For example, a set of seven linguistic terms can be defined as:

$$\mathbb{S} = \{s_0 = \text{Extremely Slow}, s_1 = \text{Very Slow}, s_2 = \text{Moderately Slow}, s_3 = \text{Neither Slow Nor Fast}, \\ s_4 = \text{Moderately Fast}, s_5 = \text{Very Fast}, s_6 = \text{Extremely Fast}\}$$

To conserve all the specified information, Xu et al. (2017) broadened the discrete term set $\mathbb{S} = \{s_0, s_1, \dots, s_t\}$ to a continuous term set $\mathbb{S}_{[0,t]} = \{s_f | s_0 \leq s_f \leq s_t, f \in [0, t]\}$. The components of the set $\mathbb{S}_{[0,t]}$ exhibits all the features of the set $\mathbb{S}$. If $s_f \in \mathbb{S}$, then s_f is known as the original linguistic term, else, s_f is known as virtual linguistic term. In general, the original linguistic terms are used to provide the preference values towards the attributes/alternatives, while the virtual linguistic terms are used for calculation purpose by the decision maker. To better deal with imprecise and uncertain information in decision making, Zhang et al. (2012) introduced the definition of linguistic intuitionistic fuzzy set (LIFS), characterized by a pair of linguistic terms for a linguistic variable defining the linguistic membership degree and a linguistic non-membership degree, respectively.

Definition 5 Zhang et al. (2012) Let X be the finite universe of discourse and $\mathbb{S}_{[0,t]} = \{s_f | s_0 \leq s_f \leq s_t\}$ be a continuous LTS. A LIFS set $\hat{\mathbb{L}}$ in X is given by

$$\hat{\mathbb{L}} = \{\langle x, s_\alpha(x), s_\beta(x) \rangle | x \in X\},$$

where $s_\alpha(x), s_\beta(x) \in \mathbb{S}_{[0,t]}$ stand for the linguistic membership degree and linguistic non-membership degree of the element x to $\hat{\mathbb{L}}$, respectively with $\alpha + \beta \leq t$. For a given element $x \in X$, the pair $\langle s_\alpha, s_\beta \rangle$ is called a linguistic intuitionistic fuzzy number (LIFN), which can be simply denoted by $\hat{\gamma} = \langle s_\alpha, s_\beta \rangle$. If $s_\alpha, s_\beta \in \mathbb{S}$, then $\hat{\gamma}$ is called original LIFN, otherwise as virtual one.

Definition 6 Zhang et al. (2012) Let $\hat{\gamma} = \langle s_\alpha, s_\beta \rangle$, $\hat{\gamma}_1 = \langle s_{\alpha_1}, s_{\beta_1} \rangle$ and $\hat{\gamma}_2 = \langle s_{\alpha_2}, s_{\beta_2} \rangle$ be the three LIFN and $\lambda > 0$ be a real number. Then we have the following operational laws on LIFNs:

1. $\hat{\gamma}_1 \oplus \hat{\gamma}_2 = \left\langle s_{\alpha_1 + \alpha_2 - \frac{\alpha_1 \alpha_2}{t}}, s_{\beta_1 + \beta_2 - \frac{\beta_1 \beta_2}{t}} \right\rangle$,
2. $\hat{\gamma}_1 \otimes \hat{\gamma}_2 = \left\langle s_{\frac{\alpha_1 \alpha_2}{t}}, s_{\beta_1 + \beta_2 - \frac{\beta_1 \beta_2}{t}} \right\rangle$,
3. $\lambda \hat{\gamma} = \left\langle s_{t - t(1 - \frac{\alpha}{t})^\lambda}, s_{t(\frac{\beta}{t})^\lambda} \right\rangle$,
4. $\hat{\gamma}^\lambda = \left\langle s_{t(\frac{\alpha}{t})^\lambda}, s_{t - t(1 - \frac{\beta}{t})^\lambda} \right\rangle$.

Definition 7 Garg and Kumar (2019) Let X be the finite universe of discourse and $\mathbb{S}_{[0,t]}$ be a continuous LTS. A Linguistic Interval-Valued Intuitionistic Fuzzy set (LIVIFS) $\tilde{\mathcal{P}}$ in X is defined as

$$\tilde{\mathcal{P}} = \left\{ \langle x, s_{\xi_{\tilde{\mathcal{P}}}}(x), s_{\vartheta_{\tilde{\mathcal{P}}}}(x) \rangle \mid x \in X \right\},$$

where $s_{\xi_{\tilde{\mathcal{P}}}}(x) = [s_{\xi_{\tilde{\mathcal{P}}}^-}(x), s_{\xi_{\tilde{\mathcal{P}}}^+}(x)]$ and $s_{\vartheta_{\tilde{\mathcal{P}}}}(x) = [s_{\vartheta_{\tilde{\mathcal{P}}}^-}(x), s_{\vartheta_{\tilde{\mathcal{P}}}^+}(x)]$ are the subsets of $[s_0, s_t]$ and represents the membership degree and non-membership degree of the element $x \in X$ to $\tilde{\mathcal{P}}$, respectively and satisfies the property $s_{\xi_{\tilde{\mathcal{P}}}^+}(x) + s_{\vartheta_{\tilde{\mathcal{P}}}^+}(x) \leq s_t$, specifically $\xi_{\tilde{\mathcal{P}}}^+ + \vartheta_{\tilde{\mathcal{P}}}^+ \leq t$ exists. Usually for a given $x \in X$, the pair $\langle [s_{\xi_{\tilde{\mathcal{P}}}^-}(x), s_{\xi_{\tilde{\mathcal{P}}}^+}(x)], [s_{\vartheta_{\tilde{\mathcal{P}}}^-}(x), s_{\vartheta_{\tilde{\mathcal{P}}}^+}(x)] \rangle$ is called Linguistic Interval-Valued Intuitionistic Fuzzy number (LIVIFN), which can be simply denoted by $\tilde{\delta} = \langle [s_{\xi_{\tilde{\mathcal{P}}}^-}, s_{\xi_{\tilde{\mathcal{P}}}^+}], [s_{\vartheta_{\tilde{\mathcal{P}}}^-}, s_{\vartheta_{\tilde{\mathcal{P}}}^+}] \rangle$. The linguistic hesitation index of x to $\tilde{\mathcal{P}}$ is defined as $s_{\pi_{\tilde{\mathcal{P}}}} = [s_{\pi_{\tilde{\mathcal{P}}}^-}, s_{\pi_{\tilde{\mathcal{P}}}^+}] = [s_{t - \xi_{\tilde{\mathcal{P}}}^+ - \vartheta_{\tilde{\mathcal{P}}}^+}, s_{t - \xi_{\tilde{\mathcal{P}}}^- - \vartheta_{\tilde{\mathcal{P}}}^-}]$. Let $\mathbf{B}$ be the set of all LIVIFNs except $\langle [s_t, s_t], [s_0, s_0] \rangle$ and $\langle [s_0, s_0], [s_t, s_t] \rangle$.

Definition 8 Garg and Kumar (2019) Let $\tilde{\delta}_1 = \langle [s_{\xi_{\tilde{\mathcal{P}}_1}^-}, s_{\xi_{\tilde{\mathcal{P}}_1}^+}], [s_{\vartheta_{\tilde{\mathcal{P}}_1}^-}, s_{\vartheta_{\tilde{\mathcal{P}}_1}^+}] \rangle$ and $\tilde{\delta}_2 = \langle [s_{\xi_{\tilde{\mathcal{P}}_2}^-}, s_{\xi_{\tilde{\mathcal{P}}_2}^+}], [s_{\vartheta_{\tilde{\mathcal{P}}_2}^-}, s_{\vartheta_{\tilde{\mathcal{P}}_2}^+}] \rangle$ be two LIVIFNs, then it follows that:

1. $\tilde{\delta}_1 = \tilde{\delta}_2$ iff $\xi_{\tilde{\mathcal{P}}_1}^- = \xi_{\tilde{\mathcal{P}}_2}^-$, $\xi_{\tilde{\mathcal{P}}_1}^+ = \xi_{\tilde{\mathcal{P}}_2}^+$, $\vartheta_{\tilde{\mathcal{P}}_1}^- = \vartheta_{\tilde{\mathcal{P}}_2}^-$ and $\vartheta_{\tilde{\mathcal{P}}_1}^+ = \vartheta_{\tilde{\mathcal{P}}_2}^+$;
2. $\tilde{\delta}_1 \leq \tilde{\delta}_2$ iff $\xi_{\tilde{\mathcal{P}}_1}^- \leq \xi_{\tilde{\mathcal{P}}_2}^-$, $\xi_{\tilde{\mathcal{P}}_1}^+ \leq \xi_{\tilde{\mathcal{P}}_2}^+$, $\vartheta_{\tilde{\mathcal{P}}_1}^- \geq \vartheta_{\tilde{\mathcal{P}}_2}^-$ and $\vartheta_{\tilde{\mathcal{P}}_1}^+ \geq \vartheta_{\tilde{\mathcal{P}}_2}^+$;
3. $\tilde{\delta}_1^c = \langle [s_{\vartheta_{\tilde{\mathcal{P}}_1}^-}, s_{\vartheta_{\tilde{\mathcal{P}}_1}^+}], [s_{\xi_{\tilde{\mathcal{P}}_1}^-}, s_{\xi_{\tilde{\mathcal{P}}_1}^+}] \rangle$;
4. $\tilde{\delta}_1 \cup \tilde{\delta}_2 = \langle \left[\max \{ s_{\xi_{\tilde{\mathcal{P}}_1}^-}, s_{\xi_{\tilde{\mathcal{P}}_2}^-} \}, \max \{ s_{\xi_{\tilde{\mathcal{P}}_1}^+}, s_{\xi_{\tilde{\mathcal{P}}_2}^+} \} \right], \left[\min \{ s_{\vartheta_{\tilde{\mathcal{P}}_1}^-}, s_{\vartheta_{\tilde{\mathcal{P}}_2}^-} \}, \min \{ s_{\vartheta_{\tilde{\mathcal{P}}_1}^+}, s_{\vartheta_{\tilde{\mathcal{P}}_2}^+} \} \right] \rangle$;
5. $\tilde{\delta}_1 \cap \tilde{\delta}_2 = \langle \left[\min \{ s_{\xi_{\tilde{\mathcal{P}}_1}^-}, s_{\xi_{\tilde{\mathcal{P}}_2}^-} \}, \min \{ s_{\xi_{\tilde{\mathcal{P}}_1}^+}, s_{\xi_{\tilde{\mathcal{P}}_2}^+} \} \right], \left[\max \{ s_{\vartheta_{\tilde{\mathcal{P}}_1}^-}, s_{\vartheta_{\tilde{\mathcal{P}}_2}^-} \}, \max \{ s_{\vartheta_{\tilde{\mathcal{P}}_1}^+}, s_{\vartheta_{\tilde{\mathcal{P}}_2}^+} \} \right] \rangle$;

Definition 9 Garg and Kumar (2019) Let $\tilde{\delta} = \langle [s_{\xi_{\tilde{\mathcal{P}}}^-}, s_{\xi_{\tilde{\mathcal{P}}}^+}], [s_{\vartheta_{\tilde{\mathcal{P}}}^-}, s_{\vartheta_{\tilde{\mathcal{P}}}^+}] \rangle$ be LIVIFN. The score and accuracy values of the LIVIFN $\tilde{\delta}$ are defined as:

$$C(\tilde{\delta}) = S \left(\frac{2t + \xi_{\tilde{\mathcal{P}}}^- - \vartheta_{\tilde{\mathcal{P}}}^- + \xi_{\tilde{\mathcal{P}}}^+ - \vartheta_{\tilde{\mathcal{P}}}^+}{4} \right),$$

and

$$\mathcal{H}(\tilde{\delta}) = S \left(\frac{\xi_{\tilde{\mathcal{P}}}^- + \vartheta_{\tilde{\mathcal{P}}}^- + \xi_{\tilde{\mathcal{P}}}^+ + \vartheta_{\tilde{\mathcal{P}}}^+}{4} \right),$$

respectively.

Based on the score and accuracy function, Liu (2019) and Garg and Kumar (2019) presented the ranking approach to rank two LIVIFNs.

Let $\tilde{\delta}_1 = \left\langle \left[s_{\tilde{\delta}_1}^-, s_{\tilde{\delta}_1}^+ \right], \left[s_{\tilde{\delta}_1}^-, s_{\tilde{\delta}_1}^+ \right] \right\rangle$ and $\tilde{\delta}_2 = \left\langle \left[s_{\tilde{\delta}_2}^-, s_{\tilde{\delta}_2}^+ \right], \left[s_{\tilde{\delta}_2}^-, s_{\tilde{\delta}_2}^+ \right] \right\rangle$ be any two LIVIFNs, with $C(\tilde{\delta}_1)$ and $C(\tilde{\delta}_2)$ ($\mathcal{H}(\tilde{\delta}_1)$ and $\mathcal{H}(\tilde{\delta}_2)$) be the score values (accuracy values) of $\tilde{\delta}_1$ and $\tilde{\delta}_2$ respectively. If

1. $C(\tilde{\delta}_1) > C(\tilde{\delta}_2)$, then $\tilde{\delta}_1 > \tilde{\delta}_2$.
2. $C(\tilde{\delta}_1) = C(\tilde{\delta}_2)$ and $\mathcal{H}(\tilde{\delta}_1) > \mathcal{H}(\tilde{\delta}_2)$, then $\tilde{\delta}_1 > \tilde{\delta}_2$.
3. $C(\tilde{\delta}_1) = C(\tilde{\delta}_2)$ and $\mathcal{H}(\tilde{\delta}_1) = \mathcal{H}(\tilde{\delta}_2)$, then $\tilde{\delta}_1 = \tilde{\delta}_2$.

Definition 10 Garg and Kumar (2019) Let $\tilde{\delta} = \left\langle \left[s_{\tilde{\delta}}^-, s_{\tilde{\delta}}^+ \right], \left[s_{\tilde{\delta}}^-, s_{\tilde{\delta}}^+ \right] \right\rangle$, $\tilde{\delta}_1 = \left\langle \left[s_{\tilde{\delta}_1}^-, s_{\tilde{\delta}_1}^+ \right], \left[s_{\tilde{\delta}_1}^-, s_{\tilde{\delta}_1}^+ \right] \right\rangle$ and $\tilde{\delta}_2 = \left\langle \left[s_{\tilde{\delta}_2}^-, s_{\tilde{\delta}_2}^+ \right], \left[s_{\tilde{\delta}_2}^-, s_{\tilde{\delta}_2}^+ \right] \right\rangle$ be three LIVIFNs and $\lambda > 0$ be a real number. Then the operational laws on LIVIFNs are as follows:

1. $\tilde{\delta}_1 \oplus \tilde{\delta}_2 = \left\langle \left[s_{\tilde{\delta}_1 \oplus \tilde{\delta}_2}^-, s_{\tilde{\delta}_1 \oplus \tilde{\delta}_2}^+ \right], \left[s_{\tilde{\delta}_1 \oplus \tilde{\delta}_2}^-, s_{\tilde{\delta}_1 \oplus \tilde{\delta}_2}^+ \right] \right\rangle$;
2. $\tilde{\delta}_1 \otimes \tilde{\delta}_2 = \left\langle \left[s_{\tilde{\delta}_1 \otimes \tilde{\delta}_2}^-, s_{\tilde{\delta}_1 \otimes \tilde{\delta}_2}^+ \right], \left[s_{\tilde{\delta}_1 \otimes \tilde{\delta}_2}^-, s_{\tilde{\delta}_1 \otimes \tilde{\delta}_2}^+ \right] \right\rangle$;
3. $\lambda \tilde{\delta} = \left\langle \left[s_{\lambda \tilde{\delta}}^-, s_{\lambda \tilde{\delta}}^+ \right], \left[s_{\lambda \tilde{\delta}}^-, s_{\lambda \tilde{\delta}}^+ \right] \right\rangle$;
4. $\tilde{\delta}^\lambda = \left\langle \left[s_{\tilde{\delta}^\lambda}^-, s_{\tilde{\delta}^\lambda}^+ \right], \left[s_{\tilde{\delta}^\lambda}^-, s_{\tilde{\delta}^\lambda}^+ \right] \right\rangle$.

Remark 1 Garg and Kumar (2019) Let $\tilde{\delta} = \left\langle \left[s_{\tilde{\delta}}^-, s_{\tilde{\delta}}^+ \right], \left[s_{\tilde{\delta}}^-, s_{\tilde{\delta}}^+ \right] \right\rangle$, $\tilde{\delta}_1 = \left\langle \left[s_{\tilde{\delta}_1}^-, s_{\tilde{\delta}_1}^+ \right], \left[s_{\tilde{\delta}_1}^-, s_{\tilde{\delta}_1}^+ \right] \right\rangle$ and $\tilde{\delta}_2 = \left\langle \left[s_{\tilde{\delta}_2}^-, s_{\tilde{\delta}_2}^+ \right], \left[s_{\tilde{\delta}_2}^-, s_{\tilde{\delta}_2}^+ \right] \right\rangle$ be three LIVIFNs. For real values $\lambda, \lambda_1, \lambda_2 > 0$, the following properties holds:

1. $\tilde{\delta}_1 \oplus \tilde{\delta}_2 = \tilde{\delta}_2 \oplus \tilde{\delta}_1$;
2. $\lambda(\tilde{\delta}_1 \oplus \tilde{\delta}_2) = \lambda \tilde{\delta}_1 \oplus \lambda \tilde{\delta}_2$;
3. $\lambda_1 \tilde{\delta} \oplus \lambda_2 \tilde{\delta} = (\lambda_1 + \lambda_2) \tilde{\delta}$;
4. $\tilde{\delta}_1 \otimes \tilde{\delta}_2 = \tilde{\delta}_2 \otimes \tilde{\delta}_1$;
5. $\tilde{\delta}^{\lambda_1} \otimes \tilde{\delta}^{\lambda_2} = \tilde{\delta}^{\lambda_1 + \lambda_2}$;
6. $\tilde{\delta}_1^\lambda \otimes \tilde{\delta}_2^\lambda = (\tilde{\delta}_1 \otimes \tilde{\delta}_2)^\lambda$;

Definition 11 Garg and Kumar (2019) Let $\tilde{\delta}_r = \left\langle \left[s_{\tilde{\delta}_r}^-, s_{\tilde{\delta}_r}^+ \right], \left[s_{\tilde{\delta}_r}^-, s_{\tilde{\delta}_r}^+ \right] \right\rangle$, (where $r = 1, 2, \dots, g$) be a collection of LIVIFNs and $\omega = (\omega_1, \dots, \omega_g)^T$ be the weight vectors, such that $0 \leq \omega_r \leq 1, r = 1, 2, \dots, g, \sum_{r=1}^g \omega_r = 1$. Then the linguistic interval-valued intuitionistic fuzzy weighted average (LIVIFWA) operator is a mapping $LIVIFWA : \mathbf{B}^n \rightarrow \mathbf{B}$, and is given by

$$\begin{aligned} LIVIFWA(\tilde{\delta}_1, \tilde{\delta}_2, \dots, \tilde{\delta}_g) &= \bigoplus_{r=1}^g \omega_r \tilde{\delta}_r = \bigoplus_{r=1}^g \omega_r \left\langle \left[s_{\tilde{\delta}_r}^-, s_{\tilde{\delta}_r}^+ \right], \left[s_{\tilde{\delta}_r}^-, s_{\tilde{\delta}_r}^+ \right] \right\rangle \\ &= \left\langle \left[s_{\left(1 - \prod_{r=1}^g \left(1 - \frac{s_{\tilde{\delta}_r}^-}{t}\right)^{\omega_r}\right)}^-, s_{\left(1 - \prod_{r=1}^g \left(1 - \frac{s_{\tilde{\delta}_r}^+}{t}\right)^{\omega_r}\right)}^+ \right], \left[s_{\left(\prod_{r=1}^g \left(\frac{s_{\tilde{\delta}_r}^-}{t}\right)^{\omega_r}\right)}^-, s_{\left(\prod_{r=1}^g \left(\frac{s_{\tilde{\delta}_r}^+}{t}\right)^{\omega_r}\right)}^+ \right] \right\rangle \end{aligned}$$

Then, LIVIFWA is known as the weighted average operator for linguistic interval-valued intuitionistic fuzzy information.

3 Matrix Games with Linguistic Interval-Valued Intuitionistic Fuzzy Sets: Proposed Model

The matrix games with pay-off matrix involving LIVIFNs can be explained as follows:

Let $\chi = \{\chi_1, \chi_2, \dots, \chi_m\}$ and $\tau = \{\tau_1, \tau_2, \dots, \tau_n\}$, be two finite sets of pure strategies for player I and player II, respectively. Let $\mathcal{I} = \{1, 2, \dots, m\}$ and $\mathcal{J} = \{1, 2, \dots, n\}$. Let $\tilde{A}$ denotes a pay-off matrix with entries expressed as designated LIVIFNs. Let player I chooses pure strategy $\chi_i \in \chi$ ($i \in \mathcal{I}$) and the player II chooses pure strategy $\tau_j \in \tau$ ($j \in \mathcal{J}$). Player I gains a pay-off $\tilde{a}_{ij}$ represented by the LIVIFN $\tilde{a}_{ij} = \left\langle \left[s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \right], \left[s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \right] \right\rangle$, where, $s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+, s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \in \mathbb{S}_{[0,t]}$ and the intervals $\left[s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \right]$ and $\left[s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \right]$ are the subset of $[s_0, s_t]$ with the property $s_{\tilde{a}_{ij}}^+ + s_{\tilde{a}_{ij}}^+ \leq t$ or specifically $s_{\tilde{a}_{ij}}^+ + s_{\tilde{a}_{ij}}^+ \leq s_t$. Hence, the pay-off matrix $\tilde{A}$ of player I can be expressed as

$$\tilde{A} = \begin{bmatrix} \left\langle \left[s_{\tilde{a}_{11}}^-, s_{\tilde{a}_{11}}^+ \right], \left[s_{\tilde{a}_{11}}^-, s_{\tilde{a}_{11}}^+ \right] \right\rangle & \left\langle \left[s_{\tilde{a}_{12}}^-, s_{\tilde{a}_{12}}^+ \right], \left[s_{\tilde{a}_{12}}^-, s_{\tilde{a}_{12}}^+ \right] \right\rangle & \dots & \left\langle \left[s_{\tilde{a}_{1n}}^-, s_{\tilde{a}_{1n}}^+ \right], \left[s_{\tilde{a}_{1n}}^-, s_{\tilde{a}_{1n}}^+ \right] \right\rangle \\ \left\langle \left[s_{\tilde{a}_{21}}^-, s_{\tilde{a}_{21}}^+ \right], \left[s_{\tilde{a}_{21}}^-, s_{\tilde{a}_{21}}^+ \right] \right\rangle & \left\langle \left[s_{\tilde{a}_{22}}^-, s_{\tilde{a}_{22}}^+ \right], \left[s_{\tilde{a}_{22}}^-, s_{\tilde{a}_{22}}^+ \right] \right\rangle & \dots & \left\langle \left[s_{\tilde{a}_{2n}}^-, s_{\tilde{a}_{2n}}^+ \right], \left[s_{\tilde{a}_{2n}}^-, s_{\tilde{a}_{2n}}^+ \right] \right\rangle \\ \vdots & \vdots & & \vdots \\ \left\langle \left[s_{\tilde{a}_{m1}}^-, s_{\tilde{a}_{m1}}^+ \right], \left[s_{\tilde{a}_{m1}}^-, s_{\tilde{a}_{m1}}^+ \right] \right\rangle & \left\langle \left[s_{\tilde{a}_{m2}}^-, s_{\tilde{a}_{m2}}^+ \right], \left[s_{\tilde{a}_{m2}}^-, s_{\tilde{a}_{m2}}^+ \right] \right\rangle & \dots & \left\langle \left[s_{\tilde{a}_{mn}}^-, s_{\tilde{a}_{mn}}^+ \right], \left[s_{\tilde{a}_{mn}}^-, s_{\tilde{a}_{mn}}^+ \right] \right\rangle \end{bmatrix}$$

The player II gains a payoff $\tilde{a}_{ij}^c = \left\langle \left[s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \right], \left[s_{\tilde{a}_{ij}}^-, s_{\tilde{a}_{ij}}^+ \right] \right\rangle$. Let $\mathbf{R}^n$ denote the n -dimensional Euclidean space and $\mathbf{R}_+^n$ be its non-negative orthant. Let x_i ($i \in \mathcal{I}$) be the probability that the player I chooses the pure strategy $\chi_i \in \chi$ and y_j ($j \in \mathcal{J}$) be the probability that the player II chooses the pure strategy $\tau_j \in \tau$, then the probability vectors $x = (x_1, x_2, \dots, x_m)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ are called the mixed strategies for the player I and player II respectively, and the symbol "T" denote the transpose of a vector. Let $S^m = \{x \in \mathbf{R}_+^m \mid \sum_{i=1}^m x_i = 1\}$ and

$S^n = \{y \in \mathbf{R}_+^n \mid \sum_{j=1}^n y_j = 1\}$ be the mixed strategy spaces for the player I and player II, respectively. A Linguistic Interval-Valued Intuitionistic Fuzzy Matrix Game, is defined as

$$LIVIFMG = (\chi, S^m, \tau, S^n, \tilde{A}, \mathbb{S}_{[0,t]}).$$

If the player I and player II chooses the mixed strategies $x \in S^m$ and $y \in S^n$, respectively, the expected payoff of player I can be calculated as follows:

$$E_{\tilde{A}}(x, y) = x^T \tilde{A} y,$$

$$\begin{aligned} &= \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \otimes \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \otimes \begin{bmatrix} \left\langle \left[s_{\tilde{a}_{11}}^-, s_{\tilde{a}_{11}}^+ \right], \left[s_{\tilde{a}_{11}}^-, s_{\tilde{a}_{11}}^+ \right] \right\rangle & \left\langle \left[s_{\tilde{a}_{12}}^-, s_{\tilde{a}_{12}}^+ \right], \left[s_{\tilde{a}_{12}}^-, s_{\tilde{a}_{12}}^+ \right] \right\rangle & \dots & \left\langle \left[s_{\tilde{a}_{1n}}^-, s_{\tilde{a}_{1n}}^+ \right], \left[s_{\tilde{a}_{1n}}^-, s_{\tilde{a}_{1n}}^+ \right] \right\rangle \\ \left\langle \left[s_{\tilde{a}_{21}}^-, s_{\tilde{a}_{21}}^+ \right], \left[s_{\tilde{a}_{21}}^-, s_{\tilde{a}_{21}}^+ \right] \right\rangle & \left\langle \left[s_{\tilde{a}_{22}}^-, s_{\tilde{a}_{22}}^+ \right], \left[s_{\tilde{a}_{22}}^-, s_{\tilde{a}_{22}}^+ \right] \right\rangle & \dots & \left\langle \left[s_{\tilde{a}_{2n}}^-, s_{\tilde{a}_{2n}}^+ \right], \left[s_{\tilde{a}_{2n}}^-, s_{\tilde{a}_{2n}}^+ \right] \right\rangle \\ \vdots & \vdots & & \vdots \\ \left\langle \left[s_{\tilde{a}_{m1}}^-, s_{\tilde{a}_{m1}}^+ \right], \left[s_{\tilde{a}_{m1}}^-, s_{\tilde{a}_{m1}}^+ \right] \right\rangle & \left\langle \left[s_{\tilde{a}_{m2}}^-, s_{\tilde{a}_{m2}}^+ \right], \left[s_{\tilde{a}_{m2}}^-, s_{\tilde{a}_{m2}}^+ \right] \right\rangle & \dots & \left\langle \left[s_{\tilde{a}_{mn}}^-, s_{\tilde{a}_{mn}}^+ \right], \left[s_{\tilde{a}_{mn}}^-, s_{\tilde{a}_{mn}}^+ \right] \right\rangle \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \end{aligned}$$

From the properties of operational laws on LIVIFNs, $E_{\tilde{A}}(x, y)$ will also be a LIVIFN and is given by

$$E_{\tilde{A}}(x, y) = \left\langle \left[S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{x_i y_j} \right) \right), S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right) \right], \left[S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{x_i y_j} \right), S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right] \right\rangle \quad (3.1)$$

$$= \left\langle [S_{\xi_{a_{xy}}}^-, S_{\xi_{a_{xy}}}^+], [S_{\vartheta_{a_{xy}}}^-, S_{\vartheta_{a_{xy}}}^+] \right\rangle$$

where

$$S_{\xi_{a_{xy}}}^- = S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{x_i y_j} \right) \right) \in \mathbb{S}_{[0,t]}, \quad S_{\xi_{a_{xy}}}^+ = S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right) \in \mathbb{S}_{[0,t]},$$

$$S_{\vartheta_{a_{xy}}}^- = S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{x_i y_j} \right) \in \mathbb{S}_{[0,t]} \quad \text{and} \quad S_{\vartheta_{a_{xy}}}^+ = S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \in \mathbb{S}_{[0,t]}.$$

Further the intervals $[S_{\xi_{a_{xy}}}^-, S_{\xi_{a_{xy}}}^+]$ and $[S_{\vartheta_{a_{xy}}}^-, S_{\vartheta_{a_{xy}}}^+]$ are the subsets of $[s_0, s_t]$ with the property $\xi_{a_{xy}}^+ + \vartheta_{a_{xy}}^+ \leq t$ or $\xi_{a_{xy}}^- + \vartheta_{a_{xy}}^- \leq t$. Without loss of generality, let us assume that the player I is a maximizing player, and player II is a minimizing player. Thus, player I and player II will select the mixed strategies $x^* \in X$ and $y^* \in Y$ according to maximin and minimax principles, respectively, such that

$$E_{\tilde{A}}(x^*, y^*) = x^{*T} \tilde{A} y^* = \max_{x \in S^m} \min_{y \in S^n} \{x^T \tilde{A} y\} = \min_{y \in S^n} \max_{x \in S^m} \{x^T \tilde{A} y\}.$$

Here, it is to be noted that the expected pay-off $E_{\tilde{A}}(x^*, y^*)$ is a LIVIFN with interval valued membership and non-membership values, which are usually conflicting. Thus, the solution of the game will not be defined in terms of the optimal strategies x^* and y^* for player I and player II, respectively, as we have two person zero sum crisp matrix game.

Solving an optimization problem $\max_{x \in S^m} \min_{y \in S^n} \{x^T \tilde{A} y\}$ for player I, is equivalent to solving an optimization problem as

$$\max_{x \in S^m} \min_{y \in S^n} \left\{ \left[S_{\xi_{a_{xy}}}^-, S_{\xi_{a_{xy}}}^+ \right], [S_{\vartheta_{a_{xy}}}^-, S_{\vartheta_{a_{xy}}}^+] \right\}.$$

Similarly, solving an optimization problem $\max_{y \in S^n} \min_{x \in S^m} \{x^T \tilde{A} y\}$ for player II, is equivalent to solving an optimization problem as

$$\min_{y \in S^n} \max_{x \in S^m} \left\{ \left[S_{\xi_{a_{xy}}}^-, S_{\xi_{a_{xy}}}^+ \right], [S_{\vartheta_{a_{xy}}}^-, S_{\vartheta_{a_{xy}}}^+] \right\}.$$

Now we have the following definitions on the lines of Bector and Chandra (2005).

Definition 12 Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two LIVIFNs. If there exists $x^* \in X$ and $y^* \in Y$ such that $(x^*)^T \tilde{A} y \geq \tilde{\gamma}_1$ and $x^T \tilde{A} y^* \leq \tilde{\gamma}_2$ for any $x \in X, y \in Y$, then x^* (y^*) is called the reasonable solution of player I (player II), and $\tilde{\gamma}_1$ ($\tilde{\gamma}_2$) is called the reasonable value for player I (player II).

Let F_1 and F_2 denotes the sets of all reasonable values for player I and player II, respectively.

Definition 13 Assume that there exists $\tilde{\gamma}_1^* \in F_1$ and $\tilde{\gamma}_2^* \in F_2$. If there do not exist any $\tilde{\gamma}_1 \in F_1$ ($\tilde{\gamma}_1 \neq \tilde{\gamma}_1^*$) and $\tilde{\gamma}_2 \in F_2$ ($\tilde{\gamma}_2 \neq \tilde{\gamma}_2^*$) such that $\tilde{\gamma}_1 > \tilde{\gamma}_1^*$ and $\tilde{\gamma}_2 < \tilde{\gamma}_2^*$. Then, x^* (y^*) is called the optimal solution and the corresponding optimal value is $\tilde{\gamma}_1^*$ ($\tilde{\gamma}_2^*$) for player I (player II). Also, x^* is called a maximin strategy for player I and y^* is called a minimax strategy for player II, respectively. If

$$\tilde{V}^* = \tilde{\gamma}_1^* \cap \tilde{\gamma}_2^*.$$

then $\tilde{V}^*$ is called the value of matrix game LIVIFMG. As per the Definition 8, $\tilde{V}^*$ is a LIVIFN. Therefore, the concept of solutions of the matrix game LIVIFMG will be given in terms of the Pareto optimal solution.

3.1 Model construction and solution approach for LIVIFMG

Since player I is a maximin player, let Θ be the its minimum expected gain

$$\begin{aligned}\Theta &= \min_{y \in S^n} E_A(x, y) = \min_{y \in S^n} \langle [s_{\xi_{axy}}^-, s_{\xi_{axy}}^+], [s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+] \rangle \\ &= \left\langle \min_{y \in S^n} \{[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+]\}, \max_{y \in S^n} \{[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+]\} \right\rangle = \langle [s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+], [s_{\vartheta_{\Theta}}^-, s_{\vartheta_{\Theta}}^+] \rangle\end{aligned}$$

It may be noted that Θ is a function of x only. Player I prefers to choose a mixed strategy $x^* \in S^m$ in such a way that it will maximize its minimum expected gain. Therefore,

$$\Theta^* = \max_{x \in S^m} \Theta = \max_{x \in S^m} \langle [s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+], [s_{\vartheta_{\Theta}}^-, s_{\vartheta_{\Theta}}^+] \rangle = \left\langle \max_{x \in S^m} \min_{y \in S^n} \{[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+]\}, \min_{x \in S^m} \max_{y \in S^n} \{[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+]\} \right\rangle \quad (3.2)$$

The mixed strategy $x^* \in S^m$ is called the maximin strategy with the value of the game Θ^* for player I.

Similarly, player II is a minimax player, let Ψ be the its maximum expected loss

$$\begin{aligned}\Psi &= \max_{x \in S^m} E_A(x, y) = \max_{x \in S^m} \langle [s_{\xi_{axy}}^-, s_{\xi_{axy}}^+], [s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+] \rangle \\ &= \left\langle \max_{x \in S^m} [s_{\xi_{axy}}^-, s_{\xi_{axy}}^+], \min_{x \in S^m} [s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+] \right\rangle = \langle [s_{\xi_{\Psi}}^-, s_{\xi_{\Psi}}^+], [s_{\vartheta_{\Psi}}^-, s_{\vartheta_{\Psi}}^+] \rangle\end{aligned}$$

It may be noted that Ψ is a function of y only. Player II prefers to choose a mixed strategy $y^* \in S^n$ in such a way that it will minimize its maximum expected loss. Therefore,

$$\Psi^* = \min_{y \in S^n} \Psi = \min_{y \in S^n} \langle [s_{\xi_{\Psi}}^-, s_{\xi_{\Psi}}^+], [s_{\vartheta_{\Psi}}^-, s_{\vartheta_{\Psi}}^+] \rangle = \left\langle \min_{y \in S^n} \max_{x \in S^m} [s_{\xi_{axy}}^-, s_{\xi_{axy}}^+], \max_{y \in S^n} \min_{x \in S^m} [s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+] \right\rangle \quad (3.3)$$

The mixed strategy $y^* \in S^n$ is called the minimax strategy with the value of the game Ψ^* for player II.

3.2 Mathematical model for player I

We shall consider two cases as follows:

Case I : When $s_0 \leq s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+, s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+ < s_t$ and $s_0 < s_{\vartheta_{\Theta}}^-, s_{\vartheta_{\Theta}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \leq s_t, i \in \mathcal{I}, j \in \mathcal{J}$.

The maximin strategy x^* and the maximum gain Θ^* for player I are derived by resolving through nonlinear bi-objective interval-programming problem which is framed as follows:

$$\begin{aligned}(\text{PI-1}) \quad \Theta^* &= \left\langle \max_{x \in S^m} [s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+], \min_{x \in S^m} [s_{\vartheta_{\Theta}}^-, s_{\vartheta_{\Theta}}^+] \right\rangle \\ &\text{subject to} \\ &\left[s \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{x_i y_j} \right), s \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right] \geq [s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+] \quad \text{for any } y \in S^n \\ &\left[s \left(\prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{x_i y_j} \right), s \left(\prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right] \leq [s_{\vartheta_{\Theta}}^-, s_{\vartheta_{\Theta}}^+] \quad \text{for any } y \in S^n \\ &s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+, s_{\vartheta_{\Theta}}^-, s_{\vartheta_{\Theta}}^+, s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\ &0 \leq \xi_{\Theta}^+ + \vartheta_{\Theta}^+ \leq t, 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\ &\sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}\end{aligned}$$

Now it may be noted that the objective function is equivalent to

$$\max \{[s_{\xi_{\Theta}}^-, s_{\xi_{\Theta}}^+]\} \Leftrightarrow \min \{[s_{t-\xi_{\Theta}^+}, s_{t-\xi_{\Theta}^-}]\} \Leftrightarrow \min \left\{ \left[s_{\ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right)}, s_{\ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right)} \right] \right\},$$

$$\min \{s_{\vartheta_{\ominus}^{-}}, s_{\vartheta_{\ominus}^{+}}\} \Leftrightarrow \min \left\{ \left[s_{\ln\left(\frac{\vartheta_{\ominus}^{-}}{t}\right)}, s_{\ln\left(\frac{\vartheta_{\ominus}^{+}}{t}\right)} \right] \right\}$$

and the constraints are equivalent to

$$\begin{aligned} & \left[s_{t\left(1-\prod_{j=1}^n \prod_{i=1}^m \left(1-\frac{\xi_{a_{ij}}^{-}}{t}\right)^{x_i y_j}\right)}, s_{t\left(1-\prod_{j=1}^n \prod_{i=1}^m \left(1-\frac{\xi_{a_{ij}}^{+}}{t}\right)^{x_i y_j}\right)} \right] \geq [s_{\xi_{\ominus}^{-}}, s_{\xi_{\ominus}^{+}}] \\ & \Leftrightarrow \left[s_{\left(\prod_{j=1}^n \prod_{i=1}^m \left(1-\frac{\xi_{a_{ij}}^{+}}{t}\right)^{x_i y_j}\right)}, s_{\left(\prod_{j=1}^n \prod_{i=1}^m \left(1-\frac{\xi_{a_{ij}}^{-}}{t}\right)^{x_i y_j}\right)} \right] \leq \left[s_{1-\frac{\xi_{\ominus}^{+}}{t}}, s_{1-\frac{\xi_{\ominus}^{-}}{t}} \right] \\ & \Leftrightarrow \left[s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(1-\frac{\xi_{a_{ij}}^{+}}{t}\right)\right)}, s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(1-\frac{\xi_{a_{ij}}^{-}}{t}\right)\right)} \right] \leq \left[s_{\ln\left(1-\frac{\xi_{\ominus}^{+}}{t}\right)}, s_{\ln\left(1-\frac{\xi_{\ominus}^{-}}{t}\right)} \right], \\ & \left[s_{t\left(\prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^{-}}{t}\right)^{x_i y_j}\right)}, s_{t\left(\prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^{+}}{t}\right)^{x_i y_j}\right)} \right] \leq [s_{\vartheta_{\ominus}^{-}}, s_{\vartheta_{\ominus}^{+}}] \Leftrightarrow \left[s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(\frac{\vartheta_{a_{ij}}^{-}}{t}\right)\right)}, s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(\frac{\vartheta_{a_{ij}}^{+}}{t}\right)\right)} \right] \leq \left[s_{\ln\left(\frac{\vartheta_{\ominus}^{-}}{t}\right)}, s_{\ln\left(\frac{\vartheta_{\ominus}^{+}}{t}\right)} \right]. \end{aligned}$$

Therefore, problem **(PI-1)** becomes equivalent to

$$\begin{aligned} \textbf{(PI-2)} \quad & \min \left\{ \left[s_{\ln\left(1-\frac{\xi_{\ominus}^{+}}{t}\right)}, s_{\ln\left(1-\frac{\xi_{\ominus}^{-}}{t}\right)} \right] \right\}, \quad \min \left\{ \left[s_{\ln\left(\frac{\vartheta_{\ominus}^{-}}{t}\right)}, s_{\ln\left(\frac{\vartheta_{\ominus}^{+}}{t}\right)} \right] \right\} \\ & \text{subject to} \\ & \left[s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(1-\frac{\xi_{a_{ij}}^{+}}{t}\right)\right)}, s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(1-\frac{\xi_{a_{ij}}^{-}}{t}\right)\right)} \right] \leq \left[s_{\ln\left(1-\frac{\xi_{\ominus}^{+}}{t}\right)}, s_{\ln\left(1-\frac{\xi_{\ominus}^{-}}{t}\right)} \right], \quad \text{for any } y \in S^n \\ & \left[s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(\frac{\vartheta_{a_{ij}}^{-}}{t}\right)\right)}, s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(\frac{\vartheta_{a_{ij}}^{+}}{t}\right)\right)} \right] \leq \left[s_{\ln\left(\frac{\vartheta_{\ominus}^{-}}{t}\right)}, s_{\ln\left(\frac{\vartheta_{\ominus}^{+}}{t}\right)} \right], \quad \text{for any } y \in S^n \\ & s_{\xi_{\ominus}^{-}}, s_{\xi_{\ominus}^{+}}, s_{\vartheta_{\ominus}^{-}}, s_{\vartheta_{\ominus}^{+}}, s_{\xi_{a_{ij}}^{-}}, s_{\xi_{a_{ij}}^{+}}, s_{\vartheta_{a_{ij}}^{-}}, s_{\vartheta_{a_{ij}}^{+}} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, j \in \mathcal{J} \\ & 0 \leq \xi_{\ominus}^{+} + \vartheta_{\ominus}^{+} \leq t, \quad 0 \leq \xi_{a_{ij}}^{+} + \vartheta_{a_{ij}}^{+} \leq t, \quad i \in \mathcal{I}, j \in \mathcal{J} \\ & \sum_{i=1}^m x_i = 1, \quad x_i \geq 0, \quad i \in \mathcal{I}. \end{aligned}$$

Now, aggregating the objective function in **(PI-2)**, we get

$$\min \left\{ \left[s_{\kappa \ln\left(1-\frac{\xi_{\ominus}^{+}}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\ominus}^{-}}{t}\right)}, s_{\kappa \ln\left(1-\frac{\xi_{\ominus}^{-}}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\ominus}^{+}}{t}\right)} \right] \right\}$$

where $\kappa \in [0, 1]$ is the weight, which is suggested by the players of the game.

Similarly, the constraints in **(PI-2)** can also be aggregated as

$$\begin{aligned} & s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln\left(1-\frac{\xi_{a_{ij}}^{+}}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{a_{ij}}^{-}}{t}\right) \right)\right)} \leq s_{\left(\kappa \ln\left(1-\frac{\xi_{\ominus}^{+}}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\ominus}^{-}}{t}\right)\right)} \\ & s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln\left(1-\frac{\xi_{a_{ij}}^{-}}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{a_{ij}}^{+}}{t}\right) \right)\right)} \leq s_{\left(\kappa \ln\left(1-\frac{\xi_{\ominus}^{-}}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\ominus}^{+}}{t}\right)\right)} \end{aligned}$$

Thus, **(PI-2)** is transformed to

$$\begin{aligned}
 \text{(PI-3)} \quad & \min \left\{ \left[s_{\kappa \left(\ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) \right) + (1-\kappa) \left(\ln \left(\frac{\vartheta_{\Theta}^-}{t} \right) \right)}, s_{\kappa \left(\ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right) \right) + (1-\kappa) \left(\ln \left(\frac{\vartheta_{\Theta}^+}{t} \right) \right)} \right] \right\} \\
 & \text{subject to} \\
 & S_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) \right)} \leq S_{\left(\kappa \ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{\Theta}^-}{t} \right) \right)}, \quad \text{for any } y \in S^n \\
 & S_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \right)} \leq S_{\left(\kappa \ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{\Theta}^+}{t} \right) \right)}, \quad \text{for any } y \in S^n \\
 & s_{\xi_{\Theta}^-}, s_{\xi_{\Theta}^+}, s_{\vartheta_{\Theta}^-}, s_{\vartheta_{\Theta}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
 & 0 \leq \xi_{\Theta}^+ + \vartheta_{\Theta}^- \leq t, \quad 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^- \leq t, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
 & \sum_{i=1}^m x_i = 1, \quad x_i \geq 0, \quad i \in \mathcal{I}.
 \end{aligned}$$

Now, the objective function in **(PI-3)** is an interval-valued function.

For simplification purpose, let

$$\Gamma^- = \kappa \left(\ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) \right) + (1-\kappa) \left(\ln \left(\frac{\vartheta_{\Theta}^-}{t} \right) \right) \text{ and } \Gamma^+ = \kappa \left(\ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right) \right) + (1-\kappa) \left(\ln \left(\frac{\vartheta_{\Theta}^+}{t} \right) \right).$$

Here, $\Gamma^- \leq 0$ and $\Gamma^+ \leq 0$ because, $\kappa \in [0, 1]$ and $0 \leq \xi_{\Theta}^-, \xi_{\Theta}^+, \vartheta_{\Theta}^-, \vartheta_{\Theta}^+ \leq t$, i.e., $0 \leq 1 - \frac{\xi_{\Theta}^+}{t} \leq 1, 0 \leq \frac{\vartheta_{\Theta}^-}{t} \leq 1, 0 \leq 1 - \frac{\xi_{\Theta}^-}{t} \leq 1$ and $0 \leq \frac{\vartheta_{\Theta}^+}{t} \leq 1$.

Thus, the objective function in **(PI-3)** becomes $\min \{[s_{\Gamma^-}, s_{\Gamma^+}]\}$ which can be further expressed as interval-objective function as $\min \{[\Gamma^-, \Gamma^+]\}$, using the properties of linguistic terms. The constraints in equation **(PI-3)** can also be written as

$$\begin{aligned}
 \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) & \leq \Gamma^-, \\
 \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) & \leq \Gamma^+,
 \end{aligned}$$

Consequently, the **(PI-3)** can be modeled as follows :

$$\begin{aligned}
 \text{(PI-4)} \quad & \min \{[\Gamma^-, \Gamma^+]\} \\
 & \text{subject to} \\
 & \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right] \leq \Gamma^-, \quad \text{for any } y \in S^n \\
 & \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1-\kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \Gamma^+, \quad \text{for any } y \in S^n \\
 & s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
 & 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^- \leq t, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
 & \Gamma^- \leq 0, \Gamma^+ \leq 0, \\
 & \sum_{i=1}^m x_i = 1, \quad x_i \geq 0, \quad i \in \mathcal{I}.
 \end{aligned}$$

Since, S^n is a finite compact convex set, it is sufficient to take into account only the extreme points of the set S^n in the constraints **(PI-4)**. Therefore, **(PI-4)** is transformed into a linear programming problem involving the

interval in the objective function as follows :

$$\begin{aligned}
 \text{(PI-5)} \quad & \min \{[\Gamma^-, \Gamma^+]\} \\
 & \text{subject to} \\
 & \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right] \leq \Gamma^-, j \in \mathcal{J} \\
 & \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \Gamma^+, j \in \mathcal{J} \\
 & s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
 & 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
 & \Gamma^- \leq 0, \Gamma^+ \leq 0, \\
 & \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
 \end{aligned}$$

The centre of the interval $[\Gamma^-, \Gamma^+]$, is $\Gamma_C = \frac{\Gamma^- + \Gamma^+}{2}$. Thus using the existing approaches of Ishibuchi and Tanaka (1990); Sakawa and Nishizaki (1994); Sengupta and Pal (2000), (PI-5) is reduced to a bi-objective linear programming problem in the following way :

$$\begin{aligned}
 \text{(PI-6)} \quad & \min \{\Gamma^-\}, \min \left\{ \frac{\Gamma^- + \Gamma^+}{2} \right\} \\
 & \text{subject to} \\
 & \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right] \leq \Gamma^-, j \in \mathcal{J} \\
 & \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \Gamma^+, j \in \mathcal{J} \\
 & s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
 & 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
 & \Gamma^- \leq 0, \Gamma^+ \leq 0, \\
 & \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
(\text{PI-7}) \quad & \min \{\Gamma^-\}, \min \left\{ \frac{\Gamma^-}{2} \right\} + \min \left\{ \frac{\Gamma^+}{2} \right\} \\
& \text{subject to} \\
& \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right] \leq \Gamma^-, j \in \mathcal{J} \\
& \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \Gamma^+, j \in \mathcal{J} \\
& s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \Gamma^- \leq 0, \Gamma^+ \leq 0, \\
& \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
\end{aligned}$$

The problem (PI-7) is transformed into

$$\begin{aligned}
(\text{PI-8}) \quad & \min \{\Gamma^-\}, \min \{\Gamma^+\} \\
& \text{subject to} \\
& \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right] \leq \Gamma^-, j \in \mathcal{J} \\
& \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \Gamma^+, j \in \mathcal{J} \\
& s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \Gamma^- \leq 0, \Gamma^+ \leq 0, \\
& \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
\end{aligned}$$

which is further be written as :

$$\begin{aligned}
(\text{PI-9}) \quad & \min \{\Gamma^- + \Gamma^+\} \\
& \text{subject to} \\
& \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \Gamma^- + \Gamma^+, j \in \mathcal{J} \\
& s_{\xi_{a_{ij}}}^-, s_{\xi_{a_{ij}}}^+, s_{\vartheta_{a_{ij}}}^-, s_{\vartheta_{a_{ij}}}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \Gamma^- \leq 0, \Gamma^+ \leq 0, \\
& \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
\end{aligned}$$

Let $\mathfrak{U} = \Gamma^- + \Gamma^+$.

Since, $\Gamma^- \leq 0$ and $\Gamma^+ \leq 0$. Therefore, $\mathfrak{U} \leq 0$.

Thus, (PI-9) is transformed into a linear programming problem

$$\begin{aligned}
(\text{PI-10}) \quad & \min \{\mathfrak{U}\} \\
& \text{subject to} \\
& \sum_{i=1}^m x_i \left[\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right] \leq \mathfrak{U}, j \in \mathcal{J} \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \mathfrak{U} \leq 0, \\
& \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
\end{aligned}$$

For $s_0 \leq s_{\xi_{a_{ij}}^+} < s_t$, $s_0 \leq s_{\xi_{a_{ij}}^-} < s_t$ and $s_0 < s_{\vartheta_{a_{ij}}^-} \leq s_t$, and $s_0 < s_{\vartheta_{a_{ij}}^+} \leq s_t$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, the optimal solution obtained for the linear programming problem (PI-10) is $(x^*, \mathfrak{U}^*)$.

Case II : When $s_{\xi_{a_{ij}}^+} = s_t$, $s_{\xi_{a_{ij}}^-} = s_t$, $s_{\vartheta_{a_{ij}}^-} = s_0$, and $s_{\vartheta_{a_{ij}}^+} = s_0$, $i \in \mathcal{I}$, $j \in \mathcal{J}$.

It is significant to take into account that whenever $\xi_{a_{ij}}^+ = t$, $\vartheta_{a_{ij}}^+ = 0$ and $\xi_{a_{ij}}^- = t$ and $\vartheta_{a_{ij}}^- = 0$, then $\ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) \rightarrow -\infty$, $\ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) \rightarrow -\infty$, $\ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \rightarrow -\infty$ and $\ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \rightarrow -\infty$. Hence, at such occurrences, problem (PI-10) becomes invalid. Thus, it is reformulated as the following nonlinear programming problem

$$\begin{aligned}
(\text{PI-11}) \quad & \min \left\{ \left(1 - \frac{\xi_{\Theta}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Theta}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{\Theta}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Theta}^+}{t} \right)^{(1-\kappa)} \right\} \\
& \text{subject to} \\
& \prod_{i=1}^m \left[\left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{(1-\kappa)} \right]^{x_i} \leq \left(1 - \frac{\xi_{\Theta}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Theta}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{\Theta}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Theta}^+}{t} \right)^{(1-\kappa)}, \\
& \text{for any } y \in S^n \\
& s_{\xi_{\Theta}^-}, s_{\xi_{\Theta}^+}, s_{\vartheta_{\Theta}^-}, s_{\vartheta_{\Theta}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{\Theta}^+ + \vartheta_{\Theta}^+ \leq t, 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
\end{aligned}$$

$$\text{Let } \mathcal{V} = \left(1 - \frac{\xi_{\Theta}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Theta}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{\Theta}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Theta}^+}{t} \right)^{(1-\kappa)}.$$

Since $\kappa \in [0, 1]$, $0 \leq 1 - \frac{\xi_{\Theta}^+}{t} \leq 1$, $0 \leq \frac{\xi_{\Theta}^+}{t} \leq 1$, $0 \leq 1 - \frac{\xi_{\Theta}^-}{t} \leq 1$ and $0 \leq 1 - \frac{\vartheta_{\Theta}^-}{t} \leq 1$. Therefore, $0 \leq \mathcal{V} \leq 1$.

Thus, (PI-11) is rewritten as :

$$\begin{aligned}
(\text{PI-12}) \quad & \min \{\mathcal{V}\} \\
& \text{subject to} \\
& \prod_{i=1}^m \left[\left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{(1-\kappa)} \right]^{x_i} \leq \mathcal{V}, \text{ for any } y \in S^n \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \mathcal{V} \leq 1, \\
& \sum_{i=1}^m x_i = 1, x_i \geq 0, i \in \mathcal{I}.
\end{aligned}$$

The optimal solution of **(PI-12)** is $(x^*, \mathcal{V}^*)$. It is evident from **(PI-10)** and **(PI-12)**, that $\mathcal{V}^* = e^{\mathcal{U}^*}$, when $(x^*, \mathcal{U}^*)$ exists as the optimal solutions of **(PI-10)**.

3.3 Mathematical model for player II

Case I : If $s_0 \leq s_{\xi_{\psi}^-}, s_{\xi_{\psi}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+} < s_t$ and $s_0 < s_{\vartheta_{\psi}^-}, s_{\vartheta_{\psi}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \leq s_t, i \in \mathcal{I}, j \in \mathcal{J}$.

For player II, the minmax strategy y^* and the minimum loss Ψ^* in (3.3) can be obtained by using (3.1) and Definitions 12 and 13. The nonlinear bi-objective interval-programming problem for player II is as follows :

$$\begin{aligned}
 \text{(PII-1)} \quad & \Psi^* = \left\langle \min \left\{ [s_{\xi_{\psi}^-}, s_{\xi_{\psi}^+}] \right\}, \max \left\{ [s_{\vartheta_{\psi}^-}, s_{\vartheta_{\psi}^+}] \right\} \right\rangle \\
 & \text{subject to} \\
 & \left[S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{x_i y_j} \right) \right), S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right) \right] \leq [s_{\xi_{\psi}^-}, s_{\xi_{\psi}^+}], \quad \text{for any } x \in S^m \\
 & \left[S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{x_i y_j} \right), S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right] \geq [s_{\vartheta_{\psi}^-}, s_{\vartheta_{\psi}^+}], \quad \text{for any } x \in S^m \\
 & s_{\xi_{\psi}^-}, s_{\xi_{\psi}^+}, s_{\vartheta_{\psi}^-}, s_{\vartheta_{\psi}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
 & 0 \leq \xi_{\psi}^+ + \vartheta_{\psi}^+ \leq t, 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
 & \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}.
 \end{aligned}$$

It may be noted that

$$\begin{aligned}
 \min \left\{ [s_{\xi_{\psi}^-}, s_{\xi_{\psi}^+}] \right\} & \Leftrightarrow \max \left\{ [s_{t-\xi_{\psi}^+}, s_{t-\xi_{\psi}^-}] \right\} \Leftrightarrow \max \left\{ \left[s_{\ln \left(1 - \frac{\xi_{\psi}^+}{t} \right)}, s_{\ln \left(1 - \frac{\xi_{\psi}^-}{t} \right)} \right] \right\}, \\
 \max \left\{ [s_{\vartheta_{\psi}^-}, s_{\vartheta_{\psi}^+}] \right\} & \Leftrightarrow \max \left\{ \left[s_{\frac{\vartheta_{\psi}^-}{t}}, s_{\frac{\vartheta_{\psi}^+}{t}} \right] \right\} \Leftrightarrow \max \left\{ \left[s_{\ln \left(\frac{\vartheta_{\psi}^-}{t} \right)}, s_{\ln \left(\frac{\vartheta_{\psi}^+}{t} \right)} \right] \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left[S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{x_i y_j} \right) \right), S \left(t \left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right) \right] \leq [s_{\xi_{\psi}^-}, s_{\xi_{\psi}^+}] \\
 & \Leftrightarrow \left[S \left(\prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{x_i y_j} \right), S \left(\prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{x_i y_j} \right) \right] \geq \left[s_{\left(1 - \frac{\xi_{\psi}^+}{t} \right)}, s_{\left(1 - \frac{\xi_{\psi}^-}{t} \right)} \right] \\
 & \Leftrightarrow \left[S \left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) \right), S \left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) \right) \right] \geq \left[s_{\ln \left(1 - \frac{\xi_{\psi}^+}{t} \right)}, s_{\ln \left(1 - \frac{\xi_{\psi}^-}{t} \right)} \right], \\
 & \left[S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{x_i y_j} \right), S \left(t \prod_{j=1}^n \prod_{i=1}^m \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{x_i y_j} \right) \right] \geq [s_{\vartheta_{\psi}^-}, s_{\vartheta_{\psi}^+}] \\
 & \Leftrightarrow \left[S \left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right), S \left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \right] \geq \left[s_{\ln \left(\frac{\vartheta_{\psi}^-}{t} \right)}, s_{\ln \left(\frac{\vartheta_{\psi}^+}{t} \right)} \right]
 \end{aligned}$$

Therefore, problem **(PII-1)** on operating logarithmic function gets reduced to :

$$\begin{aligned}
\text{(PII-2)} \quad & \max \left\{ \left[s_{\ln\left(1-\frac{\xi_{\Psi}^+}{t}\right)}, s_{\ln\left(1-\frac{\xi_{\Psi}^-}{t}\right)} \right], \max \left\{ \left[s_{\ln\left(\frac{\vartheta_{\Psi}^-}{t}\right)}, s_{\ln\left(\frac{\vartheta_{\Psi}^+}{t}\right)} \right] \right\} \right\} \\
& \text{subject to} \\
& \left[s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(1-\frac{\xi_{a_{ij}}^+}{t}\right)\right)}, s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(1-\frac{\xi_{a_{ij}}^-}{t}\right)\right)} \right] \geq \left[s_{\ln\left(1-\frac{\xi_{\Psi}^-}{t}\right)}, s_{\ln\left(1-\frac{\xi_{\Psi}^+}{t}\right)} \right], \quad \text{for any } x \in S^m \\
& \left[s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(\frac{\vartheta_{a_{ij}}^-}{t}\right)\right)}, s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \ln\left(\frac{\vartheta_{a_{ij}}^+}{t}\right)\right)} \right] \geq \left[s_{\ln\left(\frac{\vartheta_{\Psi}^-}{t}\right)}, s_{\ln\left(\frac{\vartheta_{\Psi}^+}{t}\right)} \right], \quad \text{for any } x \in S^m \\
& s_{\xi_{\Psi}^-}, s_{\xi_{\Psi}^+}, s_{\vartheta_{\Psi}^-}, s_{\vartheta_{\Psi}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{\Psi}^+ + \vartheta_{\Psi}^+ \leq t, \quad 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
& \sum_{j=1}^n y_j = 1, \quad y_j \geq 0, \quad j \in \mathcal{J}.
\end{aligned}$$

Employing aggregation technique, the objective function of **(PII-2)** is transformed as:

$$\max \left\{ \left[s_{\kappa \ln\left(1-\frac{\xi_{\Psi}^+}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\Psi}^-}{t}\right)}, s_{\kappa \ln\left(1-\frac{\xi_{\Psi}^-}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\Psi}^+}{t}\right)} \right] \right\} \quad (3.4)$$

where $\kappa \in [0, 1]$ is the weight, which is suggested by the players of the game.

Similarly, aggregating the constraints in **(PII-2)** which gets reduced to:

$$\begin{aligned}
& s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln\left(1-\frac{\xi_{a_{ij}}^+}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{a_{ij}}^-}{t}\right) \right)\right)} \geq s_{\left(\kappa \ln\left(1-\frac{\xi_{\Psi}^+}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{\Psi}^-}{t}\right)\right)} \\
& s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln\left(1-\frac{\xi_{a_{ij}}^-}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{a_{ij}}^+}{t}\right) \right)\right)} \geq s_{\left(\kappa \ln\left(1-\frac{\xi_{\Psi}^-}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{\Psi}^+}{t}\right)\right)}
\end{aligned} \quad (3.5)$$

Consequently, using equations (3.4) and (3.5), **(PII-2)** is remodeled as:

$$\begin{aligned}
\text{(PII-3)} \quad & \max \left\{ \left[s_{\kappa \ln\left(1-\frac{\xi_{\Psi}^+}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\Psi}^-}{t}\right)}, s_{\kappa \ln\left(1-\frac{\xi_{\Psi}^-}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\Psi}^+}{t}\right)} \right] \right\} \\
& \text{subject to} \\
& s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln\left(1-\frac{\xi_{a_{ij}}^+}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{a_{ij}}^-}{t}\right) \right)\right)} \geq s_{\left(\kappa \ln\left(1-\frac{\xi_{\Psi}^+}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{\Psi}^-}{t}\right)\right)}, \quad \text{for any } x \in S^m \\
& s_{\left(\sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln\left(1-\frac{\xi_{a_{ij}}^-}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{a_{ij}}^+}{t}\right) \right)\right)} \geq s_{\left(\kappa \ln\left(1-\frac{\xi_{\Psi}^-}{t}\right) + (1-\kappa) \ln\left(1-\frac{\vartheta_{\Psi}^+}{t}\right)\right)}, \quad \text{for any } x \in S^m \\
& s_{\xi_{\Psi}^-}, s_{\xi_{\Psi}^+}, s_{\vartheta_{\Psi}^-}, s_{\vartheta_{\Psi}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{\Psi}^+ + \vartheta_{\Psi}^+ \leq t, \quad 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
& \sum_{j=1}^n y_j = 1, \quad y_j \geq 0, \quad j \in \mathcal{J}.
\end{aligned}$$

Thus, the objective function in **(PII-3)** is an interval-valued function.

For sake of convenience, let $\Omega^- = \kappa \ln\left(1-\frac{\xi_{\Psi}^+}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\Psi}^-}{t}\right)$ and $\Omega^+ = \kappa \ln\left(1-\frac{\xi_{\Psi}^-}{t}\right) + (1-\kappa) \ln\left(\frac{\vartheta_{\Psi}^+}{t}\right)$.

Then, $\Omega^- \leq 0$ and $\Omega^+ \leq 0$. This is because, $\kappa \in [0, 1]$ and $0 \leq \xi_{\Psi}^-, \xi_{\Psi}^+, \vartheta_{\Psi}^-, \vartheta_{\Psi}^+ \leq t$, which imply $0 \leq 1 - \frac{\xi_{\Psi}^-}{t} \leq 1$, $0 \leq 1 - \frac{\xi_{\Psi}^+}{t} \leq 1$, $0 \leq 1 - \frac{\vartheta_{\Psi}^-}{t} \leq 1$ and $0 \leq 1 - \frac{\vartheta_{\Psi}^+}{t} \leq 1$.

Therefore, the objective function in **(PII-3)**, becomes $\max \{[s_{\Omega^-}, s_{\Omega^+}]\}$, which can be equivalently expressed as $\max \{[\Omega^-, \Omega^+]\}$.

Then the constraints in **(PII-3)** are transformed as :

$$\begin{aligned} \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) &\geq \Omega^-, \\ \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) &\geq \Omega^+, \end{aligned}$$

Consequently, **(PII-3)** can be re-framed as :

$$\begin{aligned} \textbf{(PII-4)} \quad & \max \{[\Omega^-, \Omega^+]\} \\ & \text{subject to} \\ & \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) \geq \Omega^-, \text{ for any } x \in S^m \\ & \sum_{j=1}^n \sum_{i=1}^m x_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \geq \Omega^+, \text{ for any } x \in S^m \\ & s_{\xi_{a_{ij}}^-}^-, s_{\xi_{a_{ij}}^+}^+, s_{\vartheta_{a_{ij}}^-}^-, s_{\vartheta_{a_{ij}}^+}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\ & 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\ & \Omega^- \leq 0, \Omega^+ \leq 0, \\ & \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}. \end{aligned}$$

It is sufficient to analyze only the extreme points of the set S^m in the constraints of **(PII-4)**, as S^m is a finite compact convex set.

Therefore, **(PII-4)** is transformed into the following linear programming problem :

$$\begin{aligned} \textbf{(PII-5)} \quad & \max \{[\Omega^-, \Omega^+]\} \\ & \text{subject to} \\ & \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) \geq \Omega^-, i \in \mathcal{I} \\ & \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \leq \Omega^+, i \in \mathcal{I} \\ & s_{\xi_{a_{ij}}^-}^-, s_{\xi_{a_{ij}}^+}^+, s_{\vartheta_{a_{ij}}^-}^-, s_{\vartheta_{a_{ij}}^+}^+ \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\ & 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\ & \Omega^- \leq 0, \Omega^+ \leq 0, \\ & \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}. \end{aligned}$$

The centre of the interval $[\Omega^-, \Omega^+]$ is $\Omega_C = \frac{\Omega^- + \Omega^+}{2}$. Using existing approaches of Ishibuchi and Tanaka (1990); Sakawa and Nishizaki (1994); Sengupta and Pal (2000), **(PII-5)** is modified into a bi-objective linear programming problem as follows :

$$\begin{aligned}
\text{(PII-6)} \quad & \max \{\Omega^-\}, \max \left\{ \frac{\Omega^- + \Omega^+}{2} \right\} \\
& \text{subject to} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) \geq \Omega^-, i \in \mathcal{I} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \geq \Omega^+, i \in \mathcal{I} \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \Omega^- \leq 0, \Omega^+ \leq 0, \\
& \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}.
\end{aligned}$$

Further, the problem (PII-6) is analogous to the bi-objective linear programming problem:

$$\begin{aligned}
\text{(PII-7)} \quad & \max \{\Omega^-\}, \max \left\{ \frac{\Omega^-}{2} \right\} + \max \left\{ \frac{\Omega^+}{2} \right\} \\
& \text{subject to} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) \geq \Omega^-, i \in \mathcal{I} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \geq \Omega^+, i \in \mathcal{I} \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \Omega^- \leq 0, \Omega^+ \leq 0, \\
& \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}.
\end{aligned}$$

The above problem is equivalent to :

$$\begin{aligned}
\text{(PII-8)} \quad & \max \{\Omega^-\}, \max \{\Omega^+\} \\
& \text{subject to} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) \geq \Omega^-, i \in \mathcal{I} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \geq \Omega^+, i \in \mathcal{I} \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& \Omega^- \leq 0, \Omega^+ \leq 0, \\
& \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}.
\end{aligned}$$

On aggregating the constraints in (PII-8), we get the following linear programming problem :

$$\begin{aligned}
(\text{PII-9}) \quad & \max \{\Omega^- + \Omega^+\} \\
& \text{subject to} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \geq \Omega^- + \Omega^+, \quad i \in \mathcal{I} \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, \quad j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, \quad i \in \mathcal{I}, \quad j \in \mathcal{J} \\
& \Omega^- \leq 0, \quad \Omega^+ \leq 0, \\
& \sum_{j=1}^n y_j = 1, \quad y_j \geq 0, \quad j \in \mathcal{J}.
\end{aligned}$$

Let $\mathfrak{F} = \Omega^- + \Omega^+$.

Since, $\Omega^- \leq 0$ and $\Omega^+ \leq 0$, therefore $\mathfrak{F} \leq 0$.

Thus, (PII-9) is remodeled into a linear programming problem given below :

$$\begin{aligned}
(\text{PII-10}) \quad & \max \{\mathfrak{F}\} \\
& \text{subject to} \\
& \sum_{j=1}^n y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) \geq \mathfrak{F}, \quad i \in \mathcal{I} \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, \quad i \in \mathcal{I}, \quad j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, \quad i \in \mathcal{I}, \quad j \in \mathcal{J} \\
& \mathfrak{F} \leq 0, \\
& \sum_{j=1}^n y_j = 1, \quad y_j \geq 0, \quad j \in \mathcal{J}.
\end{aligned}$$

For any specified weight $\kappa \in [0, 1]$, as stated in Theorem 4 (proved later), Definitions 12 and 13, a minimax strategy y^* for player II, could be obtained using (PII-10). Therefore, $(y^*, \mathfrak{F}^*)$ is the optimal solution of the linear programming problem (PII-10).

Case II : If $s_{\xi_{a_{ij}}^+} = s_t$, $s_{\xi_{a_{ij}}^-} = s_t$, $s_{\vartheta_{a_{ij}}^-} = s_0$, and $s_{\vartheta_{a_{ij}}^+} = s_0$, $i \in \mathcal{I}$, $j \in \mathcal{J}$.

In this particular case, it is significant to take into account that whenever $\xi_{a_{ij}}^- = t$, $\xi_{a_{ij}}^+ = t$, $\vartheta_{a_{ij}}^- = 0$, and $\vartheta_{a_{ij}}^+ = 0$, then $\ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) \rightarrow -\infty$, $\ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) \rightarrow -\infty$ and $\ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \rightarrow -\infty$, $\ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \rightarrow -\infty$. Therefore, at such occurrences, the (PII-10) becomes invalid. Thus, it is remodeled as the following nonlinear programming problems :

$$\begin{aligned}
\text{(PII-11)} \quad & \max \left\{ \left(1 - \frac{\xi_{\Psi}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Psi}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{\Psi}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Psi}^+}{t} \right)^{(1-\kappa)} \right\} \\
& \text{subject to} \\
& \prod_{j=1}^n \left[\left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{(1-\kappa)} \right]^{y_j} \geq \left(1 - \frac{\xi_{\Psi}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Psi}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{\Psi}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Psi}^+}{t} \right)^{(1-\kappa)}, \\
& \text{for any } x \in S^m
\end{aligned}$$

$$s_{\xi_{\Psi}^-}, s_{\xi_{\Psi}^+}, s_{\vartheta_{\Psi}^-}, s_{\vartheta_{\Psi}^+}, s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J}$$

$$0 \leq \xi_{\Psi}^+ + \vartheta_{\Psi}^+ \leq t, 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J}$$

$$\sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}.$$

$$\text{Let } \mathcal{W} = \left(1 - \frac{\xi_{\Psi}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Psi}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{\Psi}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{\Psi}^+}{t} \right)^{(1-\kappa)}.$$

Since $\kappa \in [0, 1]$, $0 \leq 1 - \frac{\xi_{\Psi}^-}{t} \leq 1$, $0 \leq \frac{\vartheta_{\Psi}^+}{t} \leq 1$, $0 \leq 1 - \frac{\xi_{\Psi}^+}{t} \leq 1$ and $0 \leq \frac{\vartheta_{\Psi}^-}{t} \leq 1$. Therefore, $0 \leq \mathcal{W} \leq 1$.

Consequently, (PII-11) is transformed as :

$$\begin{aligned}
\text{(PII-12)} \quad & \max \{\mathcal{W}\} \\
& \text{subject to} \\
& \prod_{j=1}^n \left[\left(1 - \frac{\xi_{a_{ij}}^+}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^-}{t} \right)^{(1-\kappa)} \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right)^{\kappa} \left(\frac{\vartheta_{a_{ij}}^+}{t} \right)^{(1-\kappa)} \right]^{y_j} \geq \mathcal{W}, \text{ for any } x \in S^m \\
& s_{\xi_{a_{ij}}^-}, s_{\xi_{a_{ij}}^+}, s_{\vartheta_{a_{ij}}^-}, s_{\vartheta_{a_{ij}}^+} \in \mathbb{S}_{[0,t]}, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \xi_{a_{ij}}^+ + \vartheta_{a_{ij}}^+ \leq t, i \in \mathcal{I}, j \in \mathcal{J} \\
& 0 \leq \mathcal{W} \leq 1, \\
& \sum_{j=1}^n y_j = 1, y_j \geq 0, j \in \mathcal{J}.
\end{aligned}$$

The optimal solution for (PII-12) is $(y^*, \mathcal{W}^*)$. Since, $\mathcal{W}^* = e^{\tilde{\delta}^*}$. Therefore, $(y^*, \tilde{\gamma}^*)$ is the optimal solution of the linear programming problem (PII-10).

Since, (PI-10) and (PII-10) are dual of each other, therefore $\mathcal{U}^* = \tilde{\gamma}^*$. Hence, $\mathcal{V}^* = \mathcal{W}^*$ because $\mathcal{V}^* = e^{\mathcal{U}^*}$ and $\mathcal{W}^* = e^{\tilde{\delta}^*}$.

Theorem 1 Let Θ^* be the maximin for player I and Ψ^* be the minmax for player II, respectively. Then Θ^* and Ψ^* are LIVIFNs, such that $\Theta^* \leq \Psi^*$.

Proof From (3.1), $E_{\tilde{A}}(x, y) = \left\langle [s_{\xi_{a_{xy}}^-}, s_{\xi_{a_{xy}}^+}], [s_{\vartheta_{a_{xy}}^-}, s_{\vartheta_{a_{xy}}^+}] \right\rangle$ is an LIVIFN.

where, $s_{\xi_{a_{xy}}^-}, s_{\xi_{a_{xy}}^+}, s_{\vartheta_{a_{xy}}^-}, s_{\vartheta_{a_{xy}}^+} \in \mathbb{S}_{[0,t]}$, such that $s_{\xi_{a_{xy}}^+} + s_{\vartheta_{a_{xy}}^+} \leq t$.

Therefore,

$$0 \leq \xi_{a_{xy}}^+ + \vartheta_{a_{xy}}^+ \leq t. \quad (3.6)$$

$$\text{and } \max_{x \in S^m} \min_{y \in S^n} \{s_{\xi_{a_{xy}}^+}\}, \min_{x \in S^m} \max_{y \in S^n} \{s_{\vartheta_{a_{xy}}^+}\} \in \mathbb{S}_{[0,t]}. \quad (3.7)$$

From (3.6), we obtain

$$0 \leq \xi_{a_{xy}}^+ + \min_{y \in S^n} \{\vartheta_{a_{xy}}^+\} \leq \xi_{a_{xy}}^+ + \max_{y \in S^n} \{\vartheta_{a_{xy}}^+\} \leq t$$

and

$$0 \leq \min_{y \in S^n} \xi_{axy}^+ + \min_{y \in S^n} \{\vartheta_{axy}^+\} \leq \xi_{axy}^+ + \max_{y \in S^n} \{\vartheta_{axy}^+\} \leq t$$

It gives,

$$0 \leq \min_{y \in S^n} \{\xi_{axy}^+\} + \min_{x \in S^m} \min_{y \in S^n} \{\vartheta_{axy}^+\} \leq \min_{y \in S^n} \{\xi_{axy}^+\} + \min_{x \in S^m} \max_{y \in S^n} \{\vartheta_{axy}^+\} \leq \xi_{axy}^+ + \min_{y \in S^n} \max_{y \in S^n} \{\vartheta_{axy}^+\} \leq t$$

Therefore,

$$0 \leq \max_{x \in S^m} \min_{y \in S^n} \{\xi_{axy}^+\} + \min_{x \in S^m} \min_{y \in S^n} \{\vartheta_{axy}^+\} \leq \max_{x \in S^m} \min_{y \in S^n} \{\xi_{axy}^+\} + \min_{x \in S^m} \max_{y \in S^n} \{\vartheta_{axy}^+\} \leq \max_{x \in S^m} \{\xi_{axy}^+\} + \min_{y \in S^n} \max_{y \in S^n} \{\vartheta_{axy}^+\} \leq t \quad (3.8)$$

i.e.,

$$0 \leq \max_{x \in S^m} \min_{y \in S^n} \{\xi_{axy}^+\} + \min_{x \in S^m} \max_{y \in S^n} \{\vartheta_{axy}^+\} \leq t \quad (3.9)$$

Thus, from (3.2), (3.6) and (3.9), we get $0 \leq \xi_{\Theta^*}^+ + \vartheta_{\Theta^*}^+ \leq t$, where $s_{\xi_{\Theta^*}^+}, s_{\vartheta_{\Theta^*}^+} \in \mathbb{S}_{[0,t]}$. Thus, Θ^* is an LIVIFN.

Likewise, ϑ^* can be established as LIVIFN.

For any $x \in S^m$ and $y \in S^n$:

$$\min_{y \in S^n} \{s_{\xi_{axy}}^-\} \leq s_{\xi_{axy}}^- \leq \max_{x \in S^m} \{s_{\xi_{axy}}^-\}$$

Hence,

$$\min_{y \in S^n} \{s_{\xi_{axy}}^-\} \leq s_{\xi_{axy}}^- \leq \min_{y \in S^n} \max_{x \in S^m} \{s_{\xi_{axy}}^-\}$$

Subsequently,

$$\max_{x \in S^m} \min_{y \in S^n} \{s_{\xi_{axy}}^-\} \leq \min_{y \in S^n} \max_{x \in S^m} \{s_{\xi_{axy}}^-\} \quad (3.10)$$

Similarly, the following can also be obtained:

$$\max_{x \in S^m} \min_{y \in S^n} \{s_{\xi_{axy}}^+\} \leq \min_{y \in S^n} \max_{x \in S^m} \{s_{\xi_{axy}}^+\} \quad (3.11)$$

Combining (3.10) and (3.11), we obtain that

$$\max_{x \in S^m} \min_{y \in S^n} \left\{ \left[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+ \right] \right\} \leq \min_{y \in S^n} \max_{x \in S^m} \left\{ \left[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+ \right] \right\}. \quad (3.12)$$

Similarly, for any $x \in S^m$ and $y \in S^n$

$$\min_{x \in S^m} \{s_{\vartheta_{axy}}^-\} \leq s_{\vartheta_{axy}}^- \leq \max_{y \in S^n} \{s_{\vartheta_{axy}}^-\}$$

Hence,

$$\max_{y \in S^n} \min_{x \in S^m} \{s_{\vartheta_{axy}}^-\} \leq \max_{y \in S^n} \{s_{\vartheta_{axy}}^-\}$$

and therefore,

$$\max_{y \in S^n} \min_{x \in S^m} \{s_{\vartheta_{axy}}^-\} \leq \min_{x \in S^m} \max_{y \in S^n} \{s_{\vartheta_{axy}}^-\} \quad (3.13)$$

Also, we can established that

$$\max_{y \in S^n} \min_{x \in S^m} \{s_{\vartheta_{axy}}^+\} \leq \min_{x \in S^m} \max_{y \in S^n} \{s_{\vartheta_{axy}}^+\} \quad (3.14)$$

Therefore, combining (3.13) and (3.14), we get

$$\max_{y \in S^n} \min_{x \in S^m} \left\{ \left[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+ \right] \right\} \leq \min_{x \in S^m} \max_{y \in S^n} \left\{ \left[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+ \right] \right\} \quad (3.15)$$

Now, using (3.12), (3.15) and Definition 8, we get

$$\max_{x \in S^m} \min_{y \in S^n} \left\{ \left\langle [S_{a_{xy}}^-, S_{a_{xy}}^+], [S_{a_{xy}}^-, S_{a_{xy}}^+] \right\rangle \right\} \leq \min_{y \in S^n} \max_{x \in S^m} \left\{ \left\langle [S_{a_{xy}}^-, S_{a_{xy}}^+], [S_{a_{xy}}^-, S_{a_{xy}}^+] \right\rangle \right\}.$$

Therefore, $\Theta^* \leq \Psi^*$. $\square$

Theorem 2 For specific $\kappa \in [0, 1]$, the matrix game LIVIFMG involving payoffs characterized by LIVIFNs always possess a solution $(x^*, y^*, x^{*T} \tilde{A} y^*)$.

Proof For any $\kappa \in [0, 1]$ be a given weight. Then, it is evident that (PI-10) and (PII-10) forms a pair of primal-dual linear programming problems equivalent to the matrix game having payoffs matrix as:

$$\left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right)_{m \times n},$$

Following the minimax theorem for matrix games Owen (1982), (PI-10) and (PII-10) will always have optimal solutions. Here, $(x^*, \mathfrak{U}^*)$ and $(y^*, \mathfrak{F}^*)$ are the optimal solution of (PI-10) and (PII-10), respectively, where $\mathfrak{U}^* = \mathfrak{F}^*$. Hence, the matrix game LIVIFMG having payoff matrix $\tilde{A}$ characterized by LIVIFNs always possess a solution $(x^*, y^*, x^{*T} \tilde{A} y^*)$, for specified weight $\kappa \in [0, 1]$. $\square$

For each $\kappa \in [0, 1]$, $\mathfrak{U}$ and $\mathfrak{F}$ are defined in terms of the weight κ , and retains the following property:

Theorem 3 $\mathfrak{U}$ and $\mathfrak{F}$ are monotonically non-decreasing functions of $\kappa \in [0, 1]$.

Proof From problem (PI-10)

$$\mathfrak{U} = \Gamma^- + \Gamma^+ = \kappa \ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\Theta}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\Theta}^+}{t} \right)$$

where $0 \leq 1 - \frac{\xi_{\Theta}^+}{t} \leq 1$, $0 \leq \frac{\vartheta_{\Theta}^-}{t} \leq 1$, $0 \leq 1 - \frac{\xi_{\Theta}^-}{t} \leq 1$, $0 \leq \frac{\vartheta_{\Theta}^+}{t} \leq 1$ and $t > 0$.

Since, $0 \leq \frac{\xi_{\Theta}^-}{t} \leq \frac{\xi_{\Theta}^+}{t} \leq 1$, therefore $0 \leq 1 - \frac{\xi_{\Theta}^+}{t} \leq 1 - \frac{\xi_{\Theta}^-}{t} \leq 1$,

which further implies that $\ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) \leq \ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right) \leq 0$.

Also, $0 \leq \frac{\vartheta_{\Theta}^-}{t} \leq \frac{\vartheta_{\Theta}^+}{t} \leq 1$, implies $\ln \left(\frac{\vartheta_{\Theta}^-}{t} \right) \leq \ln \left(\frac{\vartheta_{\Theta}^+}{t} \right) \leq 0$.

Now,

$$\begin{aligned} \frac{\partial \mathfrak{U}}{\partial \kappa} &= \ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) - \ln \left(\frac{\vartheta_{\Theta}^-}{t} \right) + \ln \left(1 - \frac{\xi_{\Theta}^-}{t} \right) - \ln \left(\frac{\vartheta_{\Theta}^+}{t} \right) \\ &\geq 2 \ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) - 2 \ln \left(\frac{\vartheta_{\Theta}^+}{t} \right) \end{aligned}$$

Here, $0 \leq \xi_{\Theta}^+ + \vartheta_{\Theta}^+ \leq t$, which implies that $0 \leq \frac{\xi_{\Theta}^+}{t} + \frac{\vartheta_{\Theta}^+}{t} \leq 1$ and hence $1 - \frac{\xi_{\Theta}^+}{t} \geq \frac{\vartheta_{\Theta}^+}{t}$. Thus, $\ln \left(1 - \frac{\xi_{\Theta}^+}{t} \right) \geq \ln \left(\frac{\vartheta_{\Theta}^+}{t} \right)$.

Thus, $\frac{\partial \mathfrak{U}}{\partial \kappa} \geq 0$, except for $\frac{\vartheta_{\Theta}^+}{t} = 0$.

Hence, $\mathfrak{U}$ is monotonically non-decreasing function for each $\kappa \in [0, 1]$.

Analogously, $\mathfrak{F}$ can also be proved a monotonically non-decreasing function for $\kappa \in [0, 1]$. $\square$

Finally, a well defined relationship between optimal solutions of (PI-10) and (PII-10) and the solution of the matrix game LIVIFMG having payoffs characterized by LIVIFNs $\tilde{A}$, are explained as:

Theorem 4 Suppose $(x^*, \mathfrak{U}^*)$ and $(y^*, \mathfrak{Y}^*)$ being optimal solutions of (PI-10) and (PII-10) with $0 < \kappa < 1$, respectively. Then (x^*, Θ^*) and (y^*, Ψ^*) are the Pareto-optimal solutions of (PI-1) and (PII-1), where

$$\Theta^* = \left\langle \max_{x \in S^m} \min_{y \in S^n} \{[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+]\}, \min_{x \in S^m} \max_{y \in S^n} \{[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+]\} \right\rangle$$

and

$$\Psi^* = \left\langle \min_{y \in S^n} \max_{x \in S^m} \{[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+]\}, \max_{y \in S^n} \min_{x \in S^m} \{[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+]\} \right\rangle$$

Proof Assuming that (x^*, Θ^*) is not a Pareto optimal solution for (PI-1). Then there exists a feasible solution $(\hat{x}, \hat{\Theta})$, for $\hat{x} \in S^m$ and $\hat{\Theta} = \langle [s_{\xi_{\hat{\Theta}}}^-, s_{\xi_{\hat{\Theta}}}^+], [s_{\vartheta_{\hat{\Theta}}}^-, s_{\vartheta_{\hat{\Theta}}}^+] \rangle$ such that

$$s_{\xi_{\hat{\Theta}}}^- \leq s_{\xi_{\hat{\Theta}}}^-, \text{ for any } y \in S^n$$

$$s_{\xi_{\hat{\Theta}}}^+ \leq s_{\xi_{\hat{\Theta}}}^+, \text{ for any } y \in S^n$$

$$s_{\vartheta_{\hat{\Theta}}}^- \leq s_{\vartheta_{\hat{\Theta}}}^-, \text{ for any } y \in S^n$$

$$s_{\vartheta_{\hat{\Theta}}}^+ \leq s_{\vartheta_{\hat{\Theta}}}^+, \text{ for any } y \in S^n$$

$$s_{\xi_{\hat{\Theta}}}^-, s_{\xi_{\hat{\Theta}}}^+, s_{\vartheta_{\hat{\Theta}}}^-, s_{\vartheta_{\hat{\Theta}}}^+ \in \mathbb{S}_{[0,t]},$$

$$0 \leq \xi_{\hat{\Theta}}^+ + \vartheta_{\hat{\Theta}}^+ \leq t$$

$$\sum_{i=1}^m \hat{x}_i = 1, \hat{x}_i \geq 0, i \in \mathcal{I}.$$

where

$$\hat{\Theta} = \left\langle \min_{y \in S^n} \{[s_{\xi_{axy}}^-, s_{\xi_{axy}}^+]\}, \max_{y \in S^n} \{[s_{\vartheta_{axy}}^-, s_{\vartheta_{axy}}^+]\} \right\rangle = \langle [s_{\xi_{\hat{\Theta}}}^-, s_{\xi_{\hat{\Theta}}}^+], [s_{\vartheta_{\hat{\Theta}}}^-, s_{\vartheta_{\hat{\Theta}}}^+] \rangle$$

Also, $[s_{\xi_{\hat{\Theta}}}^-, s_{\xi_{\hat{\Theta}}}^+] \geq [s_{\xi_{\Theta^*}}^-, s_{\xi_{\Theta^*}}^+]$ and $[s_{\vartheta_{\hat{\Theta}}}^-, s_{\vartheta_{\hat{\Theta}}}^+] \leq [s_{\vartheta_{\Theta^*}}^-, s_{\vartheta_{\Theta^*}}^+]$, where either of the two relations can be a strict inequality.

Using ordering property of LTS and further aggregation of the above set of inequalities for $0 < \kappa < 1$ yields :

$$\begin{aligned} \sum_{j=1}^n \sum_{i=1}^m \hat{x}_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) &\leq \kappa \ln \left(1 - \frac{\xi_{\hat{\Theta}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\hat{\Theta}}^-}{t} \right), \text{ for any } y \in S^n \\ \sum_{j=1}^n \sum_{i=1}^m \hat{x}_i y_j \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) &\leq \kappa \ln \left(1 - \frac{\xi_{\hat{\Theta}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\hat{\Theta}}^+}{t} \right), \text{ for any } y \in S^n \end{aligned} \quad (3.16)$$

$$s_{\xi_{\hat{\Theta}}}^-, s_{\xi_{\hat{\Theta}}}^+, s_{\vartheta_{\hat{\Theta}}}^-, s_{\vartheta_{\hat{\Theta}}}^+ \in \mathbb{S}_{[0,t]},$$

$$0 \leq \xi_{\hat{\Theta}}^+ + \vartheta_{\hat{\Theta}}^+ \leq t$$

$$\sum_{i=1}^m \hat{x}_i = 1, \hat{x}_i \geq 0, i \in \mathcal{I}.$$

Furthermore, R.H.S. of (3.16) implies the following relations

$$\begin{aligned} \kappa \ln \left(1 - \frac{\xi_{\hat{\Theta}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\hat{\Theta}}^-}{t} \right) &\leq \kappa \ln \left(1 - \frac{\xi_{\Theta^*}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\Theta^*}^-}{t} \right) \\ \kappa \ln \left(1 - \frac{\xi_{\hat{\Theta}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\hat{\Theta}}^+}{t} \right) &\leq \kappa \ln \left(1 - \frac{\xi_{\Theta^*}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\Theta^*}^+}{t} \right) \end{aligned} \quad (3.17)$$

Let $\hat{\Gamma}^- = \kappa \ln \left(1 - \frac{\xi_{\ominus}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\ominus}^-}{t} \right)$ and $\hat{\Gamma}^+ = \kappa \ln \left(1 - \frac{\xi_{\ominus}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{\ominus}^+}{t} \right)$.

From (3.17), we get

$$\hat{\Gamma}^- \leq \Gamma^{-*} \text{ and } \hat{\Gamma}^+ \leq \Gamma^{+*}. \quad (3.18)$$

where one of these relations can be a strict inequality.

Therefore,

$$\hat{\Gamma}^- + \hat{\Gamma}^+ \leq \Gamma^{-*} + \Gamma^{+*} \text{ or } \hat{\mathfrak{U}} \leq \mathfrak{U}^*. \quad (3.19)$$

Also, S^n being a finite compact convex set, it is sufficient to take into account only the extreme points of the set S^n . Thus, the constraints (3.16) can be remodeled as:

$$\begin{aligned} \sum_{i=1}^m \hat{x}_i \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) \right) &\leq \hat{\Gamma}^-, \text{ for any } j \in \mathcal{J} \\ \sum_{i=1}^m \hat{x}_i \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) &\leq \hat{\Gamma}^+, \text{ for any } j \in \mathcal{J} \\ \hat{\Gamma}^- &\leq 0, \hat{\Gamma}^+ \leq 0, \\ \sum_{i=1}^m \hat{x}_i &= 1, \hat{x}_i \geq 0, , i \in \mathcal{I}. \end{aligned}$$

Further the above constraints are reformulated as :

$$\left. \begin{aligned} \sum_{i=1}^m \hat{x}_i \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) &\leq \hat{\Gamma}^- + \hat{\Gamma}^+, \text{ for any } j \in \mathcal{J} \\ \hat{\Gamma}^- &\leq 0, \hat{\Gamma}^+ \leq 0, \\ \sum_{i=1}^m \hat{x}_i &= 1, \hat{x}_i \geq 0, , i \in \mathcal{I}. \end{aligned} \right\} \quad (3.20)$$

Defining $\hat{\mathfrak{U}} = \hat{\Gamma}^- + \hat{\Gamma}^+$. Then, (3.20) is transformed as follows:

$$\left. \begin{aligned} \sum_{i=1}^m \hat{x}_i \left(\kappa \ln \left(1 - \frac{\xi_{a_{ij}}^+}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^-}{t} \right) + \kappa \ln \left(1 - \frac{\xi_{a_{ij}}^-}{t} \right) + (1 - \kappa) \ln \left(\frac{\vartheta_{a_{ij}}^+}{t} \right) \right) &\leq \hat{\mathfrak{U}}, \text{ for any } j \in \mathcal{J} \\ \hat{\mathfrak{U}} &\leq 0, \\ \sum_{i=1}^m \hat{x}_i &= 1, \hat{x}_i \geq 0, , i \in \mathcal{I}. \end{aligned} \right\}$$

Therefore, $(\hat{x}, \hat{\mathfrak{U}})$ is a feasible solution of **(PI-10)**.

Also, $\hat{\mathfrak{U}} \leq \mathfrak{U}^*$ from (3.19), which is contrary to the statement $(x^*, \mathfrak{U}^*)$ exists as the optimal solution for **(PI-10)**.

Thus, (x^*, Θ^*) is the Pareto optimal solution of **(PI-1)**.

On the similar lines, it can be proved that (y^*, Ψ^*) is the Pareto optimal solution of **(PII-1)**. $\square$

4 Numerical Illustration

Assume that the two companies, $\mathcal{K}_1$ and $\mathcal{K}_2$, manufacture air conditioners (ACs) and compete for the supply of window ACs to a newly established university. It is assumed that the demand for air conditioners is largely

consistent. When one company's profit increases, at the same time, the other's profit decreases. The two companies need to choose their marketing strategies wisely in pursuit of competitive advantage. The company $\mathcal{K}_1$ having a good deal of constraints, still prioritize three basics alternatives:

- (i) giving special discount offer (σ_1),
- (ii) giving additional warranty (σ_2),
- (iii) producing energy-efficient ACs (σ_3).

On the other hand, company $\mathcal{K}_2$ too have three alternatives on priority basis:

- (i) to provide special price offer (ζ_1),
- (ii) generating energy-efficient air conditioners (ζ_2),
- (iii) to provide one time free servicing to users (ζ_3).

The two companies have limited funds, thus leaving them with only one possibility to select. Hence, the aforementioned challenging choice issue is referred to as a matrix game. The companies $\mathcal{K}_1$ and $\mathcal{K}_2$ are believed to be two players, player I and player II, respectively. The payoffs of game are the market gains that a company expects to receive based on the strategy that the individual company chooses. Moreover, the market facts are scarcely known/unknown and are frequently uncertain. Thus, a unique numeric value for the payoff cannot model it realistically. A committee of experts assesses the state of the financial markets and expresses their conclusions in linguistic terms, i.e. LIVINs. The evaluation of the above-mentioned alternatives for each player (companies) according to LTSs is defined as follows:

$S = \{s_0 = \text{no gain}, s_1 = \text{very low gain}, s_2 = \text{low gain}, s_3 = \text{slightly low gain}, s_4 = \text{moderate gain}, s_5 = \text{slightly good gain}, s_6 = \text{good gain}, s_7 = \text{very good gain}, s_8 = \text{excellent gain}\}$.

Under this paradigm, based on expert's evaluation, the payoff matrix $\tilde{A}$ for player I (i.e. company $\mathcal{K}_1$), with payoffs as LIVINs is as follows :

$$\tilde{A} = \begin{bmatrix} \langle [s_5, s_6]; [s_1, s_2] \rangle & \langle [s_4, s_5]; [s_1, s_2] \rangle & \langle [s_1, s_3]; [s_4, s_5] \rangle \\ \langle [s_0, s_2]; [s_3, s_6] \rangle & \langle [s_7, s_7]; [s_1, s_1] \rangle & \langle [s_3, s_6]; [s_1, s_2] \rangle \\ \langle [s_0, s_1]; [s_2, s_2] \rangle & \langle [s_1, s_2]; [s_4, s_6] \rangle & \langle [s_2, s_7]; [s_1, s_1] \rangle \end{bmatrix}$$

The problem for player I is:

(PI-1.1) $\min \{\mathfrak{U}\}$
 subject to

$$\begin{aligned} & x_1 \left[\kappa \ln \left(1 - \frac{6}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{5}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] + \\ & x_2 \left[\kappa \ln \left(1 - \frac{2}{8} \right) + (1 - \kappa) \ln \left(\frac{3}{8} \right) + \kappa \ln \left(1 - \frac{0}{8} \right) + (1 - \kappa) \ln \left(\frac{6}{8} \right) \right] + \\ & x_3 \left[\kappa \ln \left(1 - \frac{1}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) + \kappa \ln \left(1 - \frac{0}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] \leq \mathfrak{U}, \\ & x_1 \left[\kappa \ln \left(1 - \frac{5}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{4}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] + \\ & x_2 \left[\kappa \ln \left(1 - \frac{7}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{7}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) \right] + \\ & x_3 \left[\kappa \ln \left(1 - \frac{2}{8} \right) + (1 - \kappa) \ln \left(\frac{4}{8} \right) + \kappa \ln \left(1 - \frac{1}{8} \right) + (1 - \kappa) \ln \left(\frac{6}{8} \right) \right] \leq \mathfrak{U}, \\ & x_1 \left[\kappa \ln \left(1 - \frac{3}{8} \right) + (1 - \kappa) \ln \left(\frac{4}{8} \right) + \kappa \ln \left(1 - \frac{1}{8} \right) + (1 - \kappa) \ln \left(\frac{5}{8} \right) \right] + \\ & x_2 \left[\kappa \ln \left(1 - \frac{6}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{3}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] + \\ & x_3 \left[\kappa \ln \left(1 - \frac{7}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{2}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) \right] \leq \mathfrak{U}, \\ & x_1 + x_2 + x_3 = 1, \\ & \mathfrak{U} \leq 0, \quad x_1, x_2, x_3 \geq 0, \end{aligned}$$

and the problem for player II is :

(PII-1.2) $\max \{\mathfrak{F}\}$
 such that

$$\begin{aligned} & y_1 \left[\kappa \ln \left(1 - \frac{6}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{5}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] + \\ & y_2 \left[\kappa \ln \left(1 - \frac{5}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{4}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] + \\ & y_3 \left[\kappa \ln \left(1 - \frac{3}{8} \right) + (1 - \kappa) \ln \left(\frac{4}{8} \right) + \kappa \ln \left(1 - \frac{1}{8} \right) + (1 - \kappa) \ln \left(\frac{5}{8} \right) \right] \geq \mathfrak{F}, \\ & y_1 \left[\kappa \ln \left(1 - \frac{2}{8} \right) + (1 - \kappa) \ln \left(\frac{3}{8} \right) + \kappa \ln \left(1 - \frac{0}{8} \right) + (1 - \kappa) \ln \left(\frac{6}{8} \right) \right] + \\ & y_2 \left[\kappa \ln \left(1 - \frac{7}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{7}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) \right] + \\ & y_3 \left[\kappa \ln \left(1 - \frac{6}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{3}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] \geq \mathfrak{F}, \\ & y_1 \left[\kappa \ln \left(1 - \frac{1}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) + \kappa \ln \left(1 - \frac{0}{8} \right) + (1 - \kappa) \ln \left(\frac{2}{8} \right) \right] + \\ & y_2 \left[\kappa \ln \left(1 - \frac{2}{8} \right) + (1 - \kappa) \ln \left(\frac{4}{8} \right) + \kappa \ln \left(1 - \frac{1}{8} \right) + (1 - \kappa) \ln \left(\frac{6}{8} \right) \right] + \\ & y_3 \left[\kappa \ln \left(1 - \frac{7}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) + \kappa \ln \left(1 - \frac{2}{8} \right) + (1 - \kappa) \ln \left(\frac{1}{8} \right) \right] \geq \mathfrak{F}, \\ & y_1 + y_2 + y_3 = 1, \\ & \mathfrak{F} \leq 0, \quad y_1, y_2, y_3 \geq 0. \end{aligned}$$

We can solve the problems **(PI-1.1)** and **(PII-1.2)** for distinct values of $\kappa \in [0, 1]$, using existing methods for linear programming problems. The optimal solutions and the corresponding expected values of the problems **(PI-1.1)** and **(PII-1.2)** are illustrated in Table-1.

Analogously, for **(PI-1.1)** and **(PII-1.2)**, the corresponding nonlinear programming problems so obtained are as follows :

(PI-1.3) $\min \{\mathcal{V}\}$
 subject to

$$\left[\left(1 - \frac{6}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{5}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{x_1} * \left[\left(1 - \frac{2}{8}\right)^\kappa \left(\frac{3}{8}\right)^{(1-\kappa)} \left(1 - \frac{0}{8}\right)^\kappa \left(\frac{6}{8}\right)^{(1-\kappa)} \right]^{x_2} * \left[\left(1 - \frac{1}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \left(1 - \frac{0}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{x_3} \leq \mathcal{V},$$

$$\left[\left(1 - \frac{5}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{4}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{x_1} * \left[\left(1 - \frac{7}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{7}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \right]^{x_2} * \left[\left(1 - \frac{2}{8}\right)^\kappa \left(\frac{4}{8}\right)^{(1-\kappa)} \left(1 - \frac{1}{8}\right)^\kappa \left(\frac{6}{8}\right)^{(1-\kappa)} \right]^{x_3} \leq \mathcal{V},$$

$$\left[\left(1 - \frac{3}{8}\right)^\kappa \left(\frac{4}{8}\right)^{(1-\kappa)} \left(1 - \frac{1}{8}\right)^\kappa \left(\frac{5}{8}\right)^{(1-\kappa)} \right]^{x_1} * \left[\left(1 - \frac{6}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{3}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{x_2} * \left[\left(1 - \frac{7}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{2}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \right]^{x_3} \leq \mathcal{V},$$

$$x_1 + x_2 + x_3 = 1,$$

$$0 \leq \mathcal{V} \leq 1, x_1, x_2, x_3 \geq 0.$$

and

(PII-1.4) $\max \{\mathcal{W}\}$
 subject to

$$\left[\left(1 - \frac{6}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{5}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{y_1} * \left[\left(1 - \frac{5}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{4}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{y_2} * \left[\left(1 - \frac{3}{8}\right)^\kappa \left(\frac{4}{8}\right)^{(1-\kappa)} \left(1 - \frac{1}{8}\right)^\kappa \left(\frac{5}{8}\right)^{(1-\kappa)} \right]^{y_3} \geq \mathcal{W},$$

$$\left[\left(1 - \frac{2}{8}\right)^\kappa \left(\frac{3}{8}\right)^{(1-\kappa)} \left(1 - \frac{0}{8}\right)^\kappa \left(\frac{6}{8}\right)^{(1-\kappa)} \right]^{y_1} * \left[\left(1 - \frac{7}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{7}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \right]^{y_2} * \left[\left(1 - \frac{6}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{3}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{y_3} \geq \mathcal{W},$$

$$\left[\left(1 - \frac{1}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \left(1 - \frac{0}{8}\right)^\kappa \left(\frac{2}{8}\right)^{(1-\kappa)} \right]^{y_1} * \left[\left(1 - \frac{2}{8}\right)^\kappa \left(\frac{4}{8}\right)^{(1-\kappa)} \left(1 - \frac{1}{8}\right)^\kappa \left(\frac{6}{8}\right)^{(1-\kappa)} \right]^{y_2} * \left[\left(1 - \frac{7}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \left(1 - \frac{2}{8}\right)^\kappa \left(\frac{1}{8}\right)^{(1-\kappa)} \right]^{y_3} \geq \mathcal{W},$$

$$y_1 + y_2 + y_3 = 1,$$

$$0 \leq \mathcal{W} \leq 1, y_1, y_2, y_3 \geq 0.$$

Problems **(PII-1.3)** and **(PII-1.4)** are solved using existing methods for solving non-linear programming methods. Their results have been tabulated for the optimal values and their respective expected values in Table-2. The results summarized in Tables 1 and 2 clearly indicates that as the value of κ changes, different mixed strategies are obtained corresponding to the Players I and II, respectively. It is worth mentioning that the values $\mathcal{U}^*$, $\mathcal{F}^*$, $\mathcal{V}^*$ and $\mathcal{W}^*$ are monotonically non-decreasing with respect to κ . Also, maxmin strategies x^* and minmax strategies y^* obtained by the problems **(PI-1.1)** and **(PII-1.2)**, are respectively, similar to those obtained by the problems given in **(PI-1.3)** and **(PII-1.4)**. Furthermore, $\mathcal{V}^* = e^{\mathcal{U}^*}$ and $\mathcal{W}^* = e^{\mathcal{F}^*}$, with $s_{\xi_{a_{ij}}}^+ \neq s_t$, $s_{\xi_{a_{ij}}}^- \neq s_t$, $s_{\vartheta_{a_{ij}}}^+ \neq s_0$ and $s_{\vartheta_{a_{ij}}}^- \neq s_0$ ($i, j = 1, 2, 3$).

Player I			Player II		$E_{\tilde{A}}(x^*, y^*)$
κ	x^*	$\mathfrak{U}^*$	y^*	$\tilde{\delta}^*$	
0.1	(0.4318,0.2034,0.3649)	-2.6021	(0.4273, 0.2452, 0.3276)	-2.6021	$\langle [s_{2.9249}, s_{4.8839}], [s_{1.6867}, s_{2.4555}] \rangle$
0.2	(0.4404,0.1862,0.3734)	-2.4673	(0.4360,0.2234,0.3406)	-2.4673	$\langle [s_{2.8526}, s_{4.8795}], [s_{1.6907}, s_{2.4448}] \rangle$
0.3	(0.4503,0.1690,0.3807)	-2.3308	(0.4423,0.2011,0.3566)	-2.3308	$\langle [s_{2.7872}, s_{4.8852}], [s_{1.6939}, s_{2.4315}] \rangle$
0.4	(0.4614,0.1520,0.3866)	-2.1931	(0.4459,0.1787,0.3755)	-2.1931	$\langle [s_{2.7266}, s_{4.8995}], [s_{1.6978}, s_{2.4161}] \rangle$
0.5	(0.4736,0.1355,0.3909)	-2.0545	(0.4464,0.1563,0.3973)	-2.0545	$\langle [s_{2.6711}, s_{4.9195}], [s_{1.70331}, s_{2.4022}] \rangle$
0.6	(0.4868,0.1197,0.3955)	-1.9154	(0.4436,0.1344,0.4220)	-1.9154	$\langle [s_{2.6247}, s_{4.9641}], [s_{1.7311}, s_{2.3823}] \rangle$
0.7	(0.5010,0.1050,0.3941)	-1.7765	(0.4374,0.1133,0.4494)	-1.7765	$\langle [s_{2.5736}, s_{4.9773}], [s_{1.7215}, s_{2.3802}] \rangle$
0.8	(0.5159,0.0915,0.3926)	-1.6384	(0.4276,0.0933,0.4792)	-1.6384	$\langle [s_{2.5309}, s_{5.0078}], [s_{1.7372}, s_{2.3777}] \rangle$
0.9	(0.5314,0.0796,0.3890)	-1.5015	(0.4142,0.0748,0.5110)	-1.5015	$\langle [s_{2.4874}, s_{5.0381}], [s_{1.7580}, s_{2.3813}] \rangle$

Table 1 Optimal solutions for (PI-1.1) and (PII-1.2)

Player I			Player II		$E_{\tilde{A}}(x^*, y^*)$
κ	x^*	$\mathcal{V}^*$	y^*	$\mathcal{W}^*$	
0.1	(0.4317,0.2034,0.3649)	0.0741	(0.4273, 0.2452, 0.3276)	0.0741	$\langle [s_{2.9249}, s_{4.8839}], [s_{1.6867}, s_{2.4555}] \rangle$
0.2	(0.4404,0.1862,0.3734)	0.0848	(0.4360,0.2234,0.3406)	0.0848	$\langle [s_{2.8526}, s_{4.8795}], [s_{1.6907}, s_{2.4448}] \rangle$
0.3	(0.4503,0.1690,0.3807)	0.0972	(0.4423,0.2011,0.3566)	0.0972	$\langle [s_{2.7872}, s_{4.8852}], [s_{1.6939}, s_{2.4315}] \rangle$
0.4	(0.4613,0.1520,0.3866)	0.1116	(0.4459,0.1787,0.3755)	0.1116	$\langle [s_{2.7266}, s_{4.8995}], [s_{1.6978}, s_{2.4161}] \rangle$
0.5	(0.4735,0.1355,0.3910)	0.1281	(0.4464,0.1563,0.3973)	0.1281	$\langle [s_{2.6711}, s_{4.9195}], [s_{1.70331}, s_{2.4022}] \rangle$
0.6	(0.4868,0.1197,0.3955)	0.1473	(0.4437,0.1344,0.4220)	0.1473	$\langle [s_{2.6247}, s_{4.9641}], [s_{1.7311}, s_{2.3823}] \rangle$
0.7	(0.5010,0.1050,0.3941)	0.1692	(0.4374,0.1133,0.4494)	0.1692	$\langle [s_{2.5736}, s_{4.9773}], [s_{1.7215}, s_{2.3802}] \rangle$
0.8	(0.5159,0.0915,0.3926)	0.1943	(0.4276,0.0933,0.4792)	0.1943	$\langle [s_{2.5309}, s_{5.0078}], [s_{1.7372}, s_{2.3777}] \rangle$
0.9	(0.5314,0.0796,0.3890)	0.2228	(0.4142,0.0747,0.5110)	0.2228	$\langle [s_{2.4874}, s_{5.0381}], [s_{1.7580}, s_{2.3813}] \rangle$

Table 2 Optimal solutions of the nonlinear programming problems (PII-1.3) and (PII-1.4)

5 Conclusion

In this study, we examine the matrix games under qualitative information environment rather than quantitative. Inspired by Garg and Kumar (2019), the matrix games involving payoffs described by linguistic interval valued intuitionistic fuzzy terms have been formulated. All the properties of matrix games involving LIVIFSs have been investigated. The linguistic interval valued intuitionistic fuzzy weighted average operator (LIV-IFWA) has been utilized to calculate the game's expected value. Furthermore, using the order relation between LIVIFNs, the nonlinear bi-objective programming problems for each player are reduced to a corresponding pair of linear/nonlinear programming problems, to determine the maxmin and minmax strategies for the two players. Finally, the solution technique was validated by illustrating a real-life decision-making problem. In future, the proposed approach can even be employed to investigate multiobjective matrix games involving LIVIFSs.

Compliance with ethical standards

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Funding Details No funding received.

Conflict of Interest The authors declare that they have no conflict of interest.

Informed Consent The authors declare that they have seen and been given the opportunity to read both the Material and the Article to be published by the journal. All of the coauthors of this paper have discussed this consent.

Authorship Contributions The first and second authors conceived the presented idea. The first author performed the analytic calculations, numerical simulations and prepared the manuscript with support from third and fourth authors. The second, third and fourth authors developed the theory and supervised the findings of this work. All the coauthors verified the analytical methods, encouraged the first author to investigate and supervise the findings of this work. All authors discussed the results and contributed to the final manuscript.

References

- Aggarwal A, Dubey D, Chandra S, Mehra A (2012a) Application of Atanassov's I-fuzzy set theory to matrix games with fuzzy goals and fuzzy payoffs. *Fuzzy Information and Engineering* 4:401–414
- Aggarwal A, Mehra A, Chandra S (2012b) Application of linear programming with I-fuzzy sets to matrix games with I-fuzzy goals. *Fuzzy Optimization and Decision Making* 11:465–480
- Arfi B (2006) Linguistic fuzzy-logic game theory. *Journal of Conflict Resolution* 50(1):28–57
- Atanassov KT (1986) Intuitionistic fuzzy sets. *Fuzzy Sets and Systems* 20:87–96
- Atanassov KT (1995) Ideas for intuitionistic fuzzy equations, inequalities and optimization. *Notes on Intuitionistic Fuzzy Sets* 1:17–24
- Atanassov KT (1999) *Intuitionistic Fuzzy Sets: Theory and Applications*. Physica-Verlag HD
- Atanassov KT, Gargov G (1989) Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems* 31:343–349
- Aubin J (1981) Cooperative fuzzy games. *Mathematics of Operations Research* 6(1):1–13
- Bector C, Chandra S (2005) *Fuzzy mathematical programming and fuzzy matrix games*. Springer-Verlag
- Bector C, Chandra S, Vidyottama V (2004a) Matrix games with fuzzy goals and fuzzy linear programming duality. *Fuzzy Optimization and Decision Making* 3:255–269
- Bector CR, Chandra S, Vijay V (2004b) Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs. *Fuzzy Sets and Systems* 146:253–269
- Bhaumik A, Roy SK, Li D (2017) Analysis of triangular intuitionistic fuzzy matrix games using robust ranking. *Journal of Intelligent and Fuzzy Systems* 33(1):327–336
- Butnariu D (1978) Fuzzy games: A description of the concept. *Fuzzy Sets and Systems* 1(3):181–192
- Campos L (1989) Fuzzy linear programming models to solve fuzzy matrix games. *Fuzzy Sets and Systems* 32:275–289
- Cevikel AC, Ahlatolu (2010) Solutions for fuzzy matrix games. *Computers and Mathematics with Applications* 60(3)
- Chen Z, Liu P, Pei Z (2015) An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *International Journal of Computational Intelligence System* 8:747–760

- Collins WD, Hu C (2008) Studying interval valued matrix games with fuzzy logic. *Soft Computing* 12:147–155
- De SK, Biswas R, Roy AR (2001) An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy Sets and Systems* 117:209–213
- Garg H, Kumar K (2019) Linguistic interval-valued Atanassov intuitionistic fuzzy sets and their applications to group decision-making problems. *IEEE Transactions on Fuzzy Sets* 27:2302–2311
- Herrera F, Martinez L (2000) A 2-tuple fuzzy linguistic represent model for computing with words. *IEEE Transactions for Fuzzy Systems* 8:746–752
- Herrera F, Herrera-Viedma, Martinez E (2000) A fusion approach for managing multi-granularity linguistic term sets in decision making. *Fuzzy Sets and Systems* 114:43–58
- Herrera F, Alonso S, Chiclana F, Herrera-Viedma E (2009) Computing with words in decision making: foundations, trends and prospects. *Fuzzy Optimization Decision Making* 8:337–364
- Ishibuchi H, Tanaka H (1990) Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research* 48:219–225
- Jana J, Roy SK (2018) Solution of matrix games with generalized trapezoidal fuzzy payoffs. *Fuzzy Information and Engineering* 10(2):213–224
- Jiang Y, Fan Z, Ma J (2008) A method for group decision making with multi-granularity linguistic assessment information. *Information Sciences* 178:1098–1109
- Kalpiński G, Tamošaitienė (2010) Game theory applications in construction engineering and management. *Ukio Technologinis ir Ekonominis Vystymas* 16(2):348–363
- Khan I, Aggarwal A, Mehra A (2016) Solving I-fuzzy bi-matrix games with I-fuzzy goals by resolving indeterminacy. *Journal of Uncertain Systems* 10:204–222
- Khan I, Aggarwal A, Mehra A, Chandra S (2017) Solving matrix games with Atanassov's I-fuzzy goals via indeterminacy resolution approach. *Journal of Information and Optimization Sciences* 38:259–287
- Leng M, Parlar M (2005) Game theoretic applications in supply chain management: A review. *INFOR* 43(3):187–220
- Li DF (2010a) Mathematical-programming approach to matrix games with payoffs represented by Atanassov's interval-valued intuitionistic fuzzy sets. *IEEE Transactions on Fuzzy Systems* 18:1112–1128
- Li DF (2010b) TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets. *IEEE Transactions on Fuzzy Systems* 18:299–311
- Li DF (2012) A fast approach to compute fuzzy values of matrix games with payoffs of triangular fuzzy numbers. *European Journal of Operational Research* 223:421–429
- Li DF, Chuntian C (2002) New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognition Letters* 23:221–225
- Li DF, Nan JX (2009) A nonlinear programming approach to matrix games with payoffs of Atanassov's intuitionistic fuzzy sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 17:585–607
- Liu P, Chen SM (2018) Multiattribute group decision making based on intuitionistic 2-tuple linguistic information. *Information Sciences* 430:599–619

- Liu QX P (2019) A new decision-making method based on interval-valued linguistic intuitionistic fuzzy information. *Cognitive Computation* 11:125–144
- Madani K (2010) Game theory and water resources. *Journal of Hydrology* 381(3-4):225–238
- Maeda T (2003) On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs. *Fuzzy Sets and Systems* 139:283–296
- Martinez L, Herrera F (2012) An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences* 207:1–18
- Nan JX, Li DF, Zhang MJ (2010) A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers. *International Journal of Computational Intelligence Systems* 3:280–289
- Nan JX, Zhang MJ, Li DF (2014) Intuitionistic fuzzy programming models for matrix games with payoffs of trapezoidal intuitionistic fuzzy numbers. *International Journal of Fuzzy Systems* 16:444–456
- Nan JX, Li DF, An JJ (2017) Solving bi-matrix games with intuitionistic fuzzy goals and intuitionistic fuzzy payoffs. *Journal of Intelligent Fuzzy Systems* 33:3723–3732
- Naqvi D, Aggarwal A, Sachdev G, Khan I (2019) Solving I-fuzzy two person zero-sum matrix games: Tanaka and Asai approach. *Granular Computing* pp 80–86, DOI <https://doi.org/10.1007/s41066-019-00200-7>
- Neumann V, Morgenstern O (1944) *Theory of games and economic behavior*. Princeton University Press
- Owen G (1982) *Game Theory*. New York: Academic Press
- Sakawa M, Nishizaki I (1994) Max-min solutions for multiobjective matrix games. *Fuzzy Sets Systems* 61:265–275
- Sanchez-Soriano J (2013) An overview on game theory applications to engineering. *International Game Theory Review* 15(3):134009
- Seikh MR, Nayak PK, Pal M (2013) Matrix games in intuitionistic fuzzy environment. *International Journal of Mathematics in Operational Research* 5:693–708
- Seikh MR, Nayak PK, Pal M (2015a) Matrix games with intuitionistic fuzzy pay-offs. *Journal of Information and Optimization Sciences* 36:159–181
- Seikh MR, Nayak PK, Pal M (2015b) Solving bi-matrix games with pay-offs of triangular intuitionistic fuzzy numbers. *European Journal of Pure and Applied Mathematics* 8:153–171
- Sengupta A, Pal T (2000) On comparing interval numbers. *European Journal of Operational Research* 127:28–43
- Szmidt E, Kacprzyk J (1996) Remarks on some applications of intuitionistic fuzzy sets in decision making. *Notes on Intuitionistic Fuzzy Sets* 2:22–31
- Tao Z, Liu J, Chen H, Zhou L (2017) Using new version of extended t-norms and s-norms for aggregating interval linguistic labels. *IEEE Transactions on Systems, Man and Cybernetics: Systems* 47(12):3184–3298
- Verma R (2020) On aggregation operators for linguistic trapezoidal fuzzy intuitionistic fuzzy sets and their application to multiple attribute group decision making. *Journal of Intelligent and Fuzzy Systems* 38:2907–2950
- Verma R, Aggarwal A (2021) Matrix games with linguistic intuitionistic fuzzy payoffs: Basic results and solution methods. *Artificial Intelligence Review* Doi:10.1007/s10462-021-10014-2
- Verma R, Sharma BD (2014) Trapezoid fuzzy linguistic prioritized weighted average operators and their application to multiple attribute group decision making. *Journal of Uncertainty Analysis and Applications*

2:1–19

- Verma R, Sharma BD (2015) Intuitionistic fuzzy Einstein prioritized weighted average operators and their application to multiple attribute group decision making. *Applied Mathematics and Information Sciences* 9(6):3095–3107
- Vijay V, Chandra S, Bector CR (2005) Matrix games with fuzzy goals and fuzzy payoffs. *Omega* 33(5):425–429
- Vijay V, Mehra A, Chandra S, Bector CR (2007) Fuzzy matrix games via a fuzzy relation approach. *Fuzzy Optimization and Decision Making* 6:299–314
- Vlachos IK, Sergiadis GD (2007) Intuitionistic fuzzy information–applications to pattern recognition. *Pattern Recognition Letters* 28:197–206
- Wang JQ, Li HB (2010) Multi-criteria decision making method based on aggregation operators for intuitionistic linguistic fuzzy numbers. *Control Decisions* 25(10):1571–1574
- Wei G (2009) Uncertain linguistic hybrid geometric mean operator and its application to group decision making under uncertain linguistic environment. *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems* 17(2)
- Xia M (2018) Interval-valued intuitionistic fuzzy matrix games based on archimedian t-conorm and t-norm. *International Journal of General Systems* 47:278–293, DOI <https://doi.org/10.1080/03081079.2017.141310>
- Xia M (2019) Methods for solving matrix games with cross-evaluated payoffs. *Soft Computing* 23:1123–1140
- Xian S, Dong Y, Liu Y, Jing N (2017) A novel approach for linguistic group decision making based on generalised interval-valued intuitionistic fuzzy linguistic induced hybrid operator and TOPSIS. *International Journal of Intelligent Systems* pp 1–27
- Xing Y, Qiu D (2019) Solving triangular intuitionistic fuzzy matrix game by applying the accuracy function method. *Symmetry* 11(10):1258, DOI: 10.3390/sym11101258
- Xu C, Meng F, Zhang Q (2017) PN equilibrium strategy for matrix games with fuzzy payoffs. *Journal of Intelligent and Fuzzy Systems* 32:2195–2206
- Xu Z (2007) A method for multiple attribute decision making with incomplete weight information in linguistic setting. *Knowledge-Based Systems* 20(8):719–725
- Zadeh LA (1965) Fuzzy sets. *Information and Control* 8:338–353
- Zadeh LA (1975a) The concept of a linguistic variable and its application approximate reasoning-I. *Information Sciences* 8(3):199–240
- Zadeh LA (1975b) The concept of a linguistic variable and its application approximate reasoning-II. *Information Sciences* 8(4):301–357
- Zhang H (2014) Linguistic intuitionistic fuzzy sets and application in MAGDM. *Journal of Applied Mathematics* 56:1–11
- Zhang W, Ju Y, Liu X (2017) Interval-valued intuitionistic fuzzy programming technique for multicriteria group decision making based on shapley values and incomplete preference information. *Soft Computing* 21:5787–5804
- Zhang Y, Ma H, Liu B, Liu J (2012) Group decision making with 2-tuple intuitionistic fuzzy linguistic preference relations. *Soft Computing* 16:1439–1446