N-Bipolar Soft Sets and Their Application in Decision Making

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N -bipolar soft sets and their application in decision making.

Muhammad Shabir* and Javaria Fatima†

Abstract

The concept of soft set was extended to N -soft set by Fatimah et al. and used as grading system. Bipolar soft sets gave the concept of a binary model of grading. Kamacı and Petchimuchu defined bipolar N -soft set but our approach is different from their approach. We defined N -bipolar soft set which extends the concept of bipolar soft set. We explained the notions through some important examples. We discussed some vital definitions and were motivated towards their use and need. We also described some basic algebraic definitions and with their help, we developed the algebraic structure of our proposed model. We give decision making algorithms and applied them to real life examples to motivate towards its application. Conflict analysis has been a vast topic for research. It was first given by Pawlak. The first extension to this model was given by Pawlak itself. Then many researchers extended his idea. We also discussed here the application of N -bipolar soft set to conflict analysis. The combination of N -bipolar soft set and conflict analysis can give user the best way to decide suitable and feasible action.

Keywords N -bipolar soft set, algebraic structure, decision making and application to conflict analysis.

1 Introduction

The concept of the soft set was introduced by Molodstov [29], to deal with the problem of inadequate parameterization. Pawlak [31] introduced the concept of soft set in a different and new way. His approach towards soft sets can be thought of as complementary to fuzzy sets. Maji et al. [27], [28] defined some operations on soft sets. Then Ali et al. [5], [7] corrected these operations and defined some more operations on soft sets. The intuitive definitions and interpretations relating extended union, extended intersection, restricted union and restricted intersections discussed in [5] are given in [2], [6], [8] and [35] among them.

The bipolar soft set was proposed by Shabir and Naz [36]. This concept was built to distinguish between preferred and adverse sides of the data. Dubois and Prade [13] introduced the role of polarity to give the reason for the positive and negative sides of alternatives. Extension of this, bipolar fuzzy soft set is given in [1]. The use of neural networks and fuzzy logic systems in different theories have been very important for researchers. With the use of neural networks and barrier Luyponav Function BLF, Time-varying IBLFs-based adaptive control of uncertain nonlinear systems with full constraints, neural networks based adaptive event trigger control for a class of electromagnetic suspension systems and integral barrier Lyapunov functions-based adaptive control for switched nonlinear systems has been developed in [24], [25] and [26].

N -binary valued information system was introduced by Herawan and Deris [22] in soft sets, but it did not define rank orders like

the cases in [9]. Astericks were used by Hakim et al. [20], [21] as the ranking system to evaluate the objects parameters of soft sets.

In view of the importance of grading or ranking system and to develop a non binary model Fatimah et al. [14] introduced the concept of N -soft set. Fatimah et al. [14] discussed some algebraic operations on N -soft sets and application to decision making. Fatimah et al. [14] also related N -soft set with real life problems in a very effective way. Alcantud et al. [3] further worked on N -soft sets by introducing rough structures and approximations. Alcantud et al. [3] derived Pawlak's rough set, tolerance rough sets and multigranular rough set from N -soft set and conversaly. Bipolar soft set [36] gives two parametrized families of subsets of the universe $\mathcal{U}$ and the condition $\tilde{F}(e) \cap \tilde{G}(\neg e) = \phi \forall e \in \tilde{A}$ and $\neg e \in \neg \tilde{A}$ as a consistency constraint. It is of vital importance that an object lacking the property e , may not have the opposite property $\neg e$, so one may have $\tilde{G}(\neg e) \neq \mathcal{U} - \tilde{F}(e)$ for some $e \in \tilde{A}$. This is known as degree of reluctance, which occurs due to inadequate knowledge, incomplete information or hesitancy in deciding for an object to have an attribute e or $\neg e$ as discussed in [36]. We proposed the new extended model. We first proceed to motivate the need for a new extended model of soft sets with some examples in Section 3. Further after defining N -Bipolar soft set, we explained all the notions or conditions used in the definitions with examples. In its explanation, we have given the concept of $r_e + r_{\neg e} \leq N - 1$ which is worth noticing.

Kamacı and Petchimuchu in [23] gave an idea of bipolar grading of attributes of N -soft set with positive and negative assessment

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space but the parameters set taken by them was the same for positive and negative grading. They did not impose any conditions in defining the bipolar N -soft set. They did not discussed the positive and negative aspects on which the grading is made. This model is also not effective in cases when the user has to decide between the alternatives for examples as we do in deciding to choose house in green surrounding and commercial area. We, therefore, introduced a new model which is extended form of bipolar soft set that not only grades the positive and negative aspects of the attributes explaining the reason to grade the parameter but also helps to choose between the alternative parameters. We also describe the algebraic structure of our proposed model. Using the concepts in [5], [14] on N -soft sets, and Shabir and Naz [36] on the bipolar soft sets we developed these concepts on N -bipolar soft sets in Section 4.

In Section 5 we developed some algorithms for decision making. The algorithm based on T-choice value helps to make a perfect decision.

The conflict was first introduced by Pawlak [30], [32]. He extended his own concept and worked to find the conflicting attribute that is the reason or cause of the conflict. After that, Pawlak [30] gave example of the Palestine dispute as a conflicting issue and developed conflicting function to find the degree of conflict between the countries acting as agents due to some conflicting attributes. Deja [10], [11] extended the work of Pawlak and arose 3 questions, for classical Pawlak conflict model [30], that became a keen interest for the upcoming researchers. These three questions are also discussed in [12]. It is worth noticing that the work of Pawlak [30], [32] and Deja [10], [11] were not multi decisional. Sun and Ma [37] then used Dominance based rough structure introduced by Greco et al. [15], [16], [17], [18] and [19] and worked on multi-agent and multi decisional conflict problems and answered second and third questions of Deja [10], [11], [12]. Sun and Ma [37] developed an algorithm in which they found that feasible consensus or agent at which maximum decision makers agreed, not all of them. Ali et al. [4] using the concept of [33], [34] introduced the concept of dominance based optimistic and pessimistic multi-granulation rough set, then formulated the algorithm that worked on the flaws of the algorithm introduced in [37]. From this Ali et al. [4] was able to find that action at which all the decision makers were agreed and answered the second and third question of Deja [11] deliberately. Our work in Section 6 is basically to solve the conflict model in view of N -bipolar soft sets. We used the algorithm of [4] and transform it according to N -bipolar soft sets to find the optimal consensus which is agreed by all decision makers and has very low or least degree of disagreement as for grades.

A comparison is also given in Section 6 between

- Bipolar N -soft sets and N -bipolar soft sets.
- Bipolar soft sets and N -bipolar soft sets.

2 Preliminaries

Before the discussion on our work we discuss some important notions related to our work.

2.1 Soft sets

Definition 2.1. [29] Let $\mathcal{U}$ be a universe of objects and $\check{E}$ be a set of attributes, $\check{A} \subseteq \check{E}$. A soft set is a pair $(\check{F}, \check{A})$, where $\check{F}$ is a mapping

$$\check{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U})$$

where $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$.

Definition 2.2. [27] Let $\check{E} = \{e_1, e_2, \dots, e_n\}$ be a set of attributes. The NOT set of the set of parameters is defined as $\neg\check{E} = \{\neg e_1, \neg e_2, \dots, \neg e_n\}$, where $\neg e_i = \text{not } e_i$ for all i .

Definition 2.3. [27] The complement of a soft set $(\check{F}, \check{A})$ is represented by $(\check{F}, \check{A})^c$ and is defined by $(\check{F}, \check{A})^c = (\check{F}^c, \neg\check{A})$, where $\check{F}^c$ is a mappings defined by $\check{F}^c : \neg\check{A} \rightarrow \mathcal{P}(\mathcal{U})$ given by $\check{F}^c(e) = \mathcal{U} - \check{F}(\neg e)$ for all $e \in \neg\check{A}$. Note that $((\check{F}, \check{A})^c)^c = (\check{F}, \check{A})$.

2.2 N -soft set

We discuss here some important definitions related to N -soft set taken from [14].

Definition 2.4. Let $\mathcal{U}$ be a universe of objects and $\check{E}$ be a set of attributes and $\check{A} \subseteq \check{E}$. Let $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \dots\}$. An N -soft set is $(\check{F}, \check{A}, N)$ over $\mathcal{U}$, where

$$\check{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \check{R})$$

with the property that for each $e \in \check{A}$ there exists a unique $(u, r_e) \in \mathcal{U} \times \check{R}$ such that $(u, r_e) \in \check{F}(e)$, where $u \in \mathcal{U}$ and this is also denoted by $\check{F}(e)(u) = r_e$, where $\mathcal{P}(\mathcal{U} \times \check{R})$ is the power set of $\mathcal{U} \times \check{R}$.

Definition 2.5. Let $\mathcal{U}$ be a universe set of objects and $\check{E}$ be a set of attributes and $\check{A} \subseteq \check{E}$. Let $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \dots\}$. An incomplete N -soft is $(\check{F}, \check{A}, N)$ on $\mathcal{U}$, where

$$\check{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \check{R})$$

with the property that for each $e \in \check{A}$, there exists atmost one (u, r_e) such that $(u, r_e) \in \check{F}(e)$.

Definition 2.6. An N -soft set is said to be efficient if $\check{F}(e_j)(u_i) = N-1$ for some $e_j \in \check{A}$ and $u_i \in \mathcal{U}$.

Definition 2.7. The normalized N -soft set $(\check{F}^0, \check{S}, N)$ of the N -soft set $(\check{F}, \check{A}, N)$ over $\mathcal{U}$ is defined by the expression : for all $e_j \in \check{A}$, $u_i \in \mathcal{U}$, $\check{F}^0(e_j)(u_i) = \check{F}(e_j)(u_i) - m$, where $\check{S} = \{1, 2, \dots, s\}$ is the index set for attributes and $m = \min r_{ij} = \min \check{F}(e_j)(u_i)$.

Definition 2.8. A weak compliment of the N -soft set $(\check{F}, \check{A}, N)$ over $\mathcal{U}$ is any N -soft set $(\check{F}^c, \check{A}, N)$ over $\mathcal{U}$, where $\check{F}^c(e) \cap \check{F}(e) = \phi$ for each $e \in \check{A}$.

Definition 2.9. The top weak compliment of the N -soft set $(\tilde{F}, \check{A}, N)$ over the fixed universe $\mathcal{U}$ is denoted by $(\tilde{F}^t, \check{A}, N)$ such that

$$\tilde{F}^t(e_j)(u_i) = \begin{cases} N-1 & \text{if } \tilde{F}(e_j)(u_i) < N-1, \\ 0 & \text{if } \tilde{F}(e_j)(u_i) = N-1, \end{cases} \quad (1)$$

for all $u_i \in \mathcal{U}$ and $e_j \in \check{A}$.

Definition 2.10. The bottom weak compliment of the N -soft set $(\tilde{F}, \check{A}, N)$ over the fixed universe $\mathcal{U}$ is denoted by $(\tilde{F}^b, \check{A}, N)$ such that

$$\tilde{F}^b(e_j)(u_i) = \begin{cases} 0 & \text{if } \tilde{F}(e_j)(u_i) > 0, \\ N-1 & \text{if } \tilde{F}(e_j)(u_i) = 0, \end{cases} \quad (2)$$

for all $u_i \in \mathcal{U}$ and $e_j \in \check{A}$.

Definition 2.11. Let $0 < T < N$ be a threshold and $(\tilde{F}, \check{A}, N)$ be an N -soft set over the universe $\mathcal{U}$. The soft set associated with $(\tilde{F}, \check{A}, N)$, denoted by $(\tilde{F}^T, \check{A})$ and is defined by the expression, $u \in \tilde{F}^T(e)$ if and only if $r_e \geq T$ when $(u, r_e) \in \tilde{F}(e)$, that is

$$\tilde{F}^T(e)(u) = \begin{cases} 1 & \text{if } (u, r_e) \in \tilde{F}(e) \text{ and } r_e \geq T, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

for all $e \in \check{A}$ and $u \in \mathcal{U}$.

Definition 2.12. Let $\mathcal{U}$ be a fixed universe of discourse. The restricted intersection of two N -soft sets $(\tilde{F}, \check{A}, N_1)$ and $(\tilde{F}_1, \check{B}, N_2)$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{A}, N_1) \cap_{\mathbb{R}} (\tilde{F}_1, \check{B}, N_2) = (\tilde{H}, \check{A} \cap \check{B}, \min(N_1, N_2))$, where for all $e \in \check{A} \cap \check{B} \neq \phi$ and $u \in \mathcal{U}$, $(u, r_e) \in \tilde{H}(e)$ if and only if $r_e = \min(r_e^1, r_e^2)$, where $(u, r_e^1) \in \tilde{F}(e)$ and $(u, r_e^2) \in \tilde{F}_1(e)$.

Definition 2.13. Let $\mathcal{U}$ be a fixed universe of discourse. The extended intersection of two N -soft sets $(\tilde{F}, \check{A}, N_1)$ and $(\tilde{F}_1, \check{B}, N_2)$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{A}, N_1) \cap_{\mathbb{E}} (\tilde{F}_1, \check{B}, N_2) = (\tilde{J}, \check{A} \cup \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$

$$\tilde{J}(e)(u) = \begin{cases} \tilde{F}(e)(u) & \text{if } e \in \check{A} - \check{B}, \\ \tilde{F}_1(e)(u) & \text{if } e \in \check{B} - \check{A}, \\ r_e & \text{if } e \in \check{A} \cap \check{B}, \end{cases} \quad (4)$$

where $r_e = \min(r_e^1, r_e^2)$ if $\tilde{F}(e)(u) = r_e^1$ and $\tilde{F}_1(e)(u) = r_e^2$.

Definition 2.14. Let $\mathcal{U}$ be a fixed universe of discourse. The restricted union of two N -soft sets $(\tilde{F}, \check{A}, N_1)$ and $(\tilde{F}_1, \check{B}, N_2)$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{A}, N_1) \cup_{\mathbb{R}} (\tilde{F}_1, \check{B}, N_2) = (\tilde{L}, \check{A} \cap \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cap \check{B} \neq \phi$ and $u \in \mathcal{U}$, $(u, r_e) \in \tilde{L}(e)$ if and only if $r_e = \max(r_e^1, r_e^2)$ if $(u, r_e^1) \in \tilde{F}(e)$ and $(u, r_e^2) \in \tilde{F}_1(e)$.

Definition 2.15. Let $\mathcal{U}$ be a fixed universe of discourse. The extended union of two N -soft sets $(\tilde{F}, \check{A}, N_1)$ and $(\tilde{F}_1, \check{B}, N_2)$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{A}, N_1) \cup_{\mathbb{E}} (\tilde{F}_1, \check{B}, N_2) = (\tilde{M}, \check{A} \cup \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$, $(u, r_e) \in \tilde{M}(e)$ if and only if $r_e = \max(r_e^1, r_e^2)$ if $(u, r_e^1) \in \tilde{F}(e)$ and $(u, r_e^2) \in \tilde{F}_1(e)$.

$(\tilde{F}_1, \check{B}, N_2) = (\tilde{N}, \check{A} \cup \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$

$$\tilde{N}(e)(u) = \begin{cases} \tilde{F}(e)(u) & \text{if } e \in \check{A} - \check{B}, \\ \tilde{F}_1(e)(u) & \text{if } e \in \check{B} - \check{A}, \\ r_e & \text{if } e \in \check{A} \cap \check{B}, \end{cases} \quad (5)$$

where $r_e = \max(r_e^1, r_e^2)$ if $\tilde{F}(e)(u) = r_e^1$ and $\tilde{F}_1(e)(u) = r_e^2$.

2.3 Bipolar soft sets

We mention here some basic definitions and results for the Bipolar soft sets which are necessary for our work. The material given in this section is taken from [36]

Definition 2.16. Let $\mathcal{U}$ be a universe of objects and $\check{E}$ be a set of attributes and $\check{A} \subseteq \check{E}$. A Bipolar soft set is a triplet $(\tilde{F}, \check{G}, \check{A})$, where

$$\tilde{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U})$$

$$\check{G} : \neg\check{A} \rightarrow \mathcal{P}(\mathcal{U})$$

with the property that $\tilde{F}(e) \cap \check{G}(\neg e) = \phi$, for all $e \in \check{A}$ and $\mathcal{P}(\mathcal{U})$ denotes the power set of $\mathcal{U}$.

Definition 2.17. The complement of a bipolar soft set $(\tilde{F}, \check{G}, \check{A})$ is represented by $(\tilde{F}, \check{G}, \check{A})^c$ and is defined by $(\tilde{F}, \check{G}, \check{A})^c = (\tilde{F}^c, \check{G}^c, \check{A})$, where $\tilde{F}^c$ and $\check{G}^c$ are mappings defined by $\tilde{F}^c(e) = \check{G}(\neg e)$ and $\check{G}^c(\neg e) = \tilde{F}(e)$ for all $e \in \check{A}$.

Definition 2.18. Let $\mathcal{U}$ be a fixed universe of discourse. The restricted intersection of two bipolar soft sets $(\tilde{F}, \check{G}, \check{A})$ and $(\tilde{F}_1, \check{G}_1, \check{B})$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{G}, \check{A}) \cap_{\mathbb{R}} (\tilde{F}_1, \check{G}_1, \check{B}) = (\tilde{H}, \check{I}, \check{A} \cap \check{B})$, where for all $e \in \check{A} \cap \check{B} \neq \phi$, $\tilde{H}(e) = \tilde{F}(e) \cap \tilde{F}_1(e)$ and $\check{I}(\neg e) = \check{G}(\neg e) \cup \check{G}_1(\neg e)$.

Definition 2.19. Let $\mathcal{U}$ be a fixed universe of discourse. The extended intersection of two bipolar soft sets $(\tilde{F}, \check{G}, \check{A})$ and $(\tilde{F}_1, \check{G}_1, \check{B})$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{G}, \check{A}) \cap_{\mathbb{E}} (\tilde{F}_1, \check{G}_1, \check{B}) = (\tilde{J}, \check{K}, \check{A} \cup \check{B})$, where

$$\tilde{J}(e) = \begin{cases} \tilde{F}(e) & \text{if } e \in \check{A} - \check{B}, \\ \tilde{F}_1(e) & \text{if } e \in \check{B} - \check{A}, \\ \tilde{F}(e) \cap \tilde{F}_1(e) & \text{if } e \in \check{A} \cap \check{B}. \end{cases} \quad (6)$$

and

$$\check{K}(\neg e) = \begin{cases} \check{G}(\neg e) & \text{if } \neg e \in \neg\check{A} - \neg\check{B}, \\ \check{G}_1(\neg e) & \text{if } \neg e \in \neg\check{B} - \neg\check{A}, \\ \check{G}(\neg e) \cup \check{G}_1(\neg e) & \text{if } \neg e \in \neg\check{A} \cap \neg\check{B}. \end{cases} \quad (7)$$

Definition 2.20. Let $\mathcal{U}$ be a fixed universal set of objects. The restricted union of two bipolar soft sets $(\tilde{F}, \check{G}, \check{A})$ and $(\tilde{F}_1, \check{G}_1, \check{B})$ over the common universe $\mathcal{U}$ is denoted and defined by $(\tilde{F}, \check{G}, \check{A}) \cup_{\mathbb{R}} (\tilde{F}_1, \check{G}_1, \check{B}) = (\tilde{L}, \check{M}, \check{A} \cap \check{B})$, where for all $e \in \check{A} \cap \check{B} \neq \phi$, $\tilde{L}(e) = \tilde{F}(e) \cup \tilde{F}_1(e)$ and $\check{M}(\neg e) = \check{G}(\neg e) \cap \check{G}_1(\neg e)$.

Definition 2.21. Let $\mathcal{U}$ be a fixed universal set of objects. The extended union of two bipolar soft sets $(\check{F}, \check{G}, \check{A})$ and $(\check{F}_1, \check{G}_1, \check{B})$ over the common universe $\mathcal{U}$ is denoted and defined by $(\check{F}, \check{G}, \check{A}) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}) = (\check{N}, \check{O}, \check{A} \cup \check{B})$, where

$$\check{N}(e) = \begin{cases} \check{F}(e) & \text{if } e \in \check{A} - \check{B}, \\ \check{F}_1(e) & \text{if } e \in \check{B} - \check{A}, \\ \check{F}(e) \cup \check{F}_1(e) & \text{if } e \in \check{A} \cap \check{B}. \end{cases} \quad (8)$$

and

$$\check{O}(\neg e) = \begin{cases} \check{G}(\neg e) & \text{if } \neg e \in \neg \check{A} - \neg \check{B}, \\ \check{G}_1(\neg e) & \text{if } \neg e \in \neg \check{B} - \neg \check{A}, \\ \check{G}(\neg e) \cap \check{G}_1(\neg e) & \text{if } \neg e \in \neg \check{A} \cap \neg \check{B}. \end{cases} \quad (9)$$

Definition 2.22. A Bipolar soft set over a universe $\mathcal{U}$ is said to be a relative null bipolar soft set denoted by $(\check{\phi}, \check{\mathbb{U}}, \check{A})$ if $\check{\phi}(e) = \phi$ for all $e \in \check{A}$ and $\check{\mathbb{U}}(\neg e) = \mathcal{U}$ for all $\neg e \in \neg \check{A}$.

Definition 2.23. A Bipolar soft set over a universe $\mathcal{U}$ is said to be a relative absolute bipolar soft set denoted by $(\check{\mathbb{U}}, \check{\phi}, \check{A})$ if $\check{\mathbb{U}}(e) = \mathcal{U}$ for all $e \in \check{A}$ and $\check{\phi}(\neg e) = \phi$ for all $\neg e \in \neg \check{A}$.

Definition 2.24. A bipolar soft set $(\check{F}_1, \check{G}_1, \check{B})$ is called a Bipolar soft subset of $(\check{F}, \check{G}, \check{A})$ over a common universal set $\mathcal{U}$ if
(i) $\check{B} \subseteq \check{A}$
(ii) $\check{F}_1(e) \subseteq \check{F}(e)$ for all $e \in \check{B}$ and $\check{G}_1(\neg e) \subseteq \check{G}(\neg e)$ for all $\neg e \in \neg \check{B}$.

Now consider the examples given in Shabir and Naz [36]. The following example of bipolar soft set represents the houses on demand of Mr. X.

Example 2.1. Suppose Mr. X wants to buy a house. Let $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ be the universe of houses and $\check{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\} = \{\text{in green surroundings, wooden, cheap, in good repair, furnished, traditional}\}$ be the set of attributes also $\neg \check{E} = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6\} = \{\text{in commercial area, marbled, expensive, in bad repair, non-furnished, modern}\}$, take $\check{A} = \{e_1, e_2, e_3, e_6\}$. He will decide to buy one of the houses from the universe of discourse on the basis of the requirements and the presence or absence of the attributes.

$\check{F}(e_1) = \{u_2, u_3\}$, $\check{F}(e_2) = \{u_1, u_2, u_5\}$, $\check{F}(e_3) = \{u_1, u_3\}$, $\check{F}(e_6) = \{u_2, u_3, u_5\}$, $\check{G}(\neg e_1) = \{u_4, u_5\}$, $\check{G}(\neg e_2) = \{u_3, u_4\}$, $\check{G}(\neg e_3) = \{u_2, u_4\}$ and $\check{G}(\neg e_6) = \{u_4\}$.

Mr. X demands a house according to the parameters, "in green surroundings, wooden, cheap and traditional". The houses are parametrized accordingly, either having the said attributes or its opposite. It is observed that, Mr. X thinks that houses u_2 and u_3 are situated in green surroundings, while u_4 and u_5 are situated entirely in commercial area. The house u_1 is neither parametrized according to e_1 nor according to $\neg e_1$, this shows that u_1 is partly located in green surrounding and partly in commercial area or not as much green.

Shabir and Naz [36] mentioned a very vital example of Bipolar disorder. It is a severe psychological illness. Problematic careers and relationship, and suicidal tendencies, especially if not treated earlier. The bipolar soft set in Example 2.2 represents the observation of patient over a week for bipolar disorder.

Example 2.2. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ be the universe of days in which the record has been maintained and $\check{E} = \{e_1, e_2, e_3, e_4, e_5\} = \{\text{Severe Mania, Severe depression, Anxiety, Medication, Side effects}\}$ and $\neg \check{E} = \{\text{Mild Mania, Mild Depression, No Anxiety, No Medication, No Side effects}\}$. Let the bipolar soft set $(\check{F}, \check{G}, \check{E})$ describes the daily record of the behavior of patient. Then suppose that $\check{F}(e_1) = \{1, 5\}$, $\check{F}(e_2) = \{1, 2, 3, 4, 7\}$, $\check{F}(e_3) = \{2, 4, 5, 6\}$, $\check{F}(e_4) = \{1, 2, 4, 5, 6, 7\}$, $\check{F}(e_5) = \{2, 3, 5, 7\}$, $\check{G}(\neg e_1) = \{2, 6, 7\}$, $\check{G}(\neg e_2) = \{6\}$, $\check{G}(\neg e_3) = \{1, 7\}$, $\check{G}(\neg e_4) = \{3\}$ and $\check{G}(\neg e_5) = \{1, 4, 6\}$.

We can observe that on third day of observation the patient has neither anxiety nor no anxiety, we can also say that may be the patient has not as much anxiety.

3 N-bipolar soft set

Recall Example 2.1, search of houses according to the requirements and the extent to which the mentioned attributes are present, we need a grading system for both the positive and negative sides of attributes, which help and make easier for decision process. For Bipolar mood charts in Example 2.2, if we model this problem to N-bipolar soft set, it would be very easy to detect the symptoms and determination of proper treatment. These examples motivate us to define N-bipolar soft set. We define N-bipolar soft set as follows.

Definition 3.1. Let $\mathcal{U}$ be a universe of objects and $\check{E}$ be a set of attributes and $\check{A} \subseteq \check{E}$. Let $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \dots\}$. An N-bipolar soft set over $\mathcal{U}$ is a quadruple $(\check{F}, \check{G}, \check{A}, N)$, where

$$\check{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \check{R})$$

$$\check{G} : \neg \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \check{R})$$

with the property that for each $e \in \check{A}$ there exists a unique $(u, r_e) \in \mathcal{U} \times \check{R}$ and for each $\neg e \in \neg \check{A}$ there exists a unique $(u, r_{\neg e}) \in (\mathcal{U} \times \check{R})$ such that $(u, r_e) \in \check{F}(e)$ and $(u, r_{\neg e}) \in \check{G}(\neg e)$, where $u \in \mathcal{U}$, $r_e, r_{\neg e} \in \check{R}$, with a condition that $r_e + r_{\neg e} \leq N-1$. Where $\mathcal{P}(\mathcal{U} \times \check{R})$ denotes the power set of $\mathcal{U} \times \check{R}$.

Given each attribute $e \in \check{A}$, every object $u \in \mathcal{U}$ receives unique evaluations r_e and $r_{\neg e} \in \check{R}$, for which $(u, r_e), (u, r_{\neg e}) \in \mathcal{U} \times \check{R}$ such that $(u, r_e) \in \check{F}(e)$ and $(u, r_{\neg e}) \in \check{G}(\neg e)$. From now onwards, we consider that both $\mathcal{U} = \{u_i, i = 1, 2, 3, \dots, q\}$ and $\check{A} = \{e_j, j = 1, 2, \dots, s\}$ are finite sets. We can represent an N-bipolar soft set by two tables, one for the parameter set $\check{A}$ and one for parameter set $\neg \check{A}$, where $r_{i,j_{e_j}}$ and $r_{i,j_{\neg e_j}}$ means $(u_i, r_{i,j_{e_j}}) \in \check{F}(e_j)$ and $(u_i, r_{i,j_{\neg e_j}}) \in \check{G}(\neg e_j)$. Also,

$r_{i,j e_j} = \check{F}(e_j)(u_i)$ and $r_{i,j \neg e_j} = \check{G}(\neg e_j)(u_i)$.

Now we will explain two points through examples.

- The condition $\check{F}(e) \cap \check{G}(\neg e) \neq \phi$ may or may not hold.
- $r_e + r_{\neg e} \leq N - 1$.

Reconsider the Example 2.1, we model it by N -bipolar soft set as follows.

Example 3.1. Let $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of houses under consideration and $\check{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\} = \{\text{in green surroundings, wooden, cheap, in good repair, furnished, traditional}\}$ be a set of parameters. Then $\neg \check{E} = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6\} = \{\text{in commercial area, marbled, expensive, in bad repair, non-furnished, modern}\}$. Let $\check{R} = \{0, 1, 2, 3, 4, 5, 6\}$.
 0 = not having the attribute, 1 = lower, 2 = low, 3 = medium, 4 = high, 5 = higher and 6 = full

Table 1: 7-soft set

$(\check{F}, \check{E}, 7)$	e_1	e_2	e_3	e_4	e_5	e_6
u_1	3	3	4	5	4	2
u_2	4	0	3	6	6	3
u_3	1	1	2	1	3	1
u_4	6	2	1	2	1	0
u_5	1	3	4	3	1	1

Table 2: 7-soft set

$(\check{G}, \neg \check{E}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$
u_1	3	2	2	1	2	4
u_2	2	6	3	0	0	3
u_3	3	5	4	5	3	5
u_4	0	0	5	4	5	6
u_5	3	1	2	3	5	5

Now the N -Bipolar soft set is given in Tables 1 and 2. By looking at Table 1 we can see that the u_1 house is partially in green surroundings so u_1 is given grade 3 under the attribute e_1 and to the same extent it is in commercial area thus for the not set u_1 is again given grade 3 under the attribute $\neg e_1$, this gives us that,

$$\check{F}(e_1)(u_1) = 3$$

$$\check{G}(\neg e_1)(u_1) = 3$$

that is $(u_1, 3) \in \check{F}(e_1)$ and $(u_1, 3) \in \check{G}(\neg e_1)$. This example shows that $\check{F}(e) \cap \check{G}(\neg e) \neq \phi$.

It is important to consider that the above mentioned condition may or may not hold, in Example 3.3, we can see that $\check{F}(e) \cap \check{G}(\neg e) = \phi$.

To explain the condition $r_e + r_{\neg e} \leq N - 1$, see the following example.

Example 3.2. Suppose that a prestigious restaurant wants to hire a chef. They test the candidates by the dish prepared by them. Let $\mathcal{U} = \{c_1, c_2, c_3\}$ be the universe of candidates preparing a common dish. $\check{A}$ be the set of parameters. $\check{A} = \{\text{presentable, spice and salt, delicious}\} = \{e_1, e_2, e_3\}$ and $\neg \check{A} = \{\text{unpresentable, unflavoured, not delicious}\} = \{\neg e_1, \neg e_2, \neg e_3\}$. The result of above test in the form of N -bipolar soft set is given in Table 3 and 4. By looking at the Tables 3 and 4, we came to know that there is a hesitancy for the attributes. Since thinking for the dish prepared by chef c_2 having grade 3 for e_1 and 4 for $\neg e_1$ is a bizzare thought, because none of the dish can be highly presentable along with fully unpresentable, Similarly we cannot believe for a dish prepared by chef c_1 having grade 2 for e_2 and 4 for $\neg e_2$ because a dish which is unflavored can not be spicy to a great extent, that is why $r_e + r_{\neg e} \leq N - 1$.

Table 3: 5-soft set

$(\check{F}, \check{A}, 5)$	e_1	e_2	e_3
c_1	1	2	3
c_2	3	0	2
c_3	2	2	3

Table 4: 5-soft set

$(\check{G}, \neg \check{A}, 5)$	$\neg e_1$	$\neg e_2$	$\neg e_3$
c_1	3	0	1
c_2	1	3	2
c_3	2	2	1

Remark 3.1. A bipolar soft set is a 2-bipolar soft set, with $N = 2$

It is natural to identify a 2-bipolar soft set by a bipolar soft set and vice versa. A 2-bipolar soft set $(\check{F}, \check{G}, \check{A}, 2)$ can be converted into a bipolar soft set $(\acute{F}, \acute{G}, \acute{A})$ by defining

$$\acute{F} : \acute{A} \rightarrow \mathcal{P}(\mathcal{U})$$

$$\acute{G} : \neg \acute{A} \rightarrow \mathcal{P}(\mathcal{U})$$

by

$$\acute{F}(e) = \{u \in \mathcal{U}; \check{F}(e)(u) = 1\}$$

$$\acute{G}(\neg e) = \{u \in \mathcal{U}; \check{G}(\neg e)(u) = 1\}$$

If $(\acute{F}, \acute{G}, \acute{A})$ is a bipolar soft set then we can construct a 2-bipolar soft set $(\check{F}, \check{G}, \check{A}, 2)$ by defining

$$\check{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \{0, 1\})$$

$$\check{G} : \neg \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \{0, 1\})$$

$$\check{F}(e) = \{(u, r_e) : u \in \mathcal{U} \text{ and } r_e = 1 \text{ if } u \in \acute{F}(e) \text{ and } r_e = 0 \text{ if } u \notin \acute{F}(e)\}$$

$$\check{G}(\neg e) = \{(u, r_{\neg e}) : u \in \mathcal{U} \text{ and } r_{\neg e} = 1 \text{ if } u \in \acute{G}(\neg e) \text{ and } r_{\neg e} = 0 \text{ if } u \notin \acute{G}(\neg e)\}.$$

This is explained by the following examples.

Example 3.3. Let $\mathcal{U} = \{s_1, s_2, s_3\}$ be a universe of objects and $\check{A} = \{e_1, e_2, e_3\}$ be a set of parameters. $\neg \check{A} = \{\neg e_1, \neg e_2, \neg e_3\}$. Consider the 2-bipolar soft set $(\check{F}, \check{G}, \check{A}, 2)$ defined as:
 $\check{F}(e_1) = \{(s_1, 1), (s_2, 1), (s_3, 0)\}$, $\check{F}(e_2) = \{(s_1, 1), (s_2, 0), (s_3, 1)\}$,
 $\check{F}(e_3) = \{(s_1, 0), (s_2, 0), (s_3, 1)\}$, $\check{G}(\neg e_1) = \{(s_1, 0), (s_2, 0), (s_3, 1)\}$,
 $\check{G}(\neg e_2) = \{(s_1, 0), (s_2, 1), (s_3, 0)\}$ and $\check{G}(\neg e_3) = \{(s_1, 1), (s_2, 1), (s_3, 0)\}$

$\{(s_1, 1), (s_2, 1), (s_3, 0)\}$.

A 2-bipolar soft set for this is represented in Table 5 and 6. This 2-bipolar soft set $(\tilde{F}, \check{G}, \check{A}, 2)$ can be identified with the bipolar soft set $(\tilde{F}, \check{G}, \check{A})$ which is given by

$\tilde{F}(e_1) = \{s_1, s_2\}$, $\tilde{F}(e_2) = \{s_1, s_3\}$, $\tilde{F}(e_3) = \{s_3\}$, $\check{G}(\neg e_1) = \{s_3\}$, $\check{G}(\neg e_2) = \{s_2\}$, $\check{G}(\neg e_3) = \{s_1, s_2\}$.

Table 5: 2-soft set

Table 6: 2-soft set

$(\tilde{F}, \check{A}, 2)$	e_1	e_2	e_3
s_1	1	1	0
s_2	1	0	0
s_3	0	1	1

$(\check{G}, \neg \check{A}, 2)$	$\neg e_1$	$\neg e_2$	$\neg e_3$
s_1	0	0	1
s_2	0	1	1
s_3	1	0	0

Example 3.4. Let $\mathcal{U} = \{u_1, u_2, u_3\}$ be a universe of objects and $\check{A}$ be a set of parameters, $\check{A} = \{e_1, e_2, e_3, e_4\}$. Then $\neg \check{A} = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4\}$. Consider the bipolar soft set $(\tilde{F}, \check{G}, \check{A})$ defined as follows.

$\tilde{F}(e_1) = \{u_1, u_3\}$, $\tilde{F}(e_2) = \{u_1, u_3\}$, $\tilde{F}(e_3) = \{u_2, u_3\}$ and $\tilde{F}(e_4) = \{u_1, u_2\}$.

$\check{G}(\neg e_1) = \{u_2\}$, $\check{G}(\neg e_2) = \{u_2\}$, $\check{G}(\neg e_3) = \{u_1\}$ and $\check{G}(\neg e_4) = \{u_3\}$.

This bipolar soft set $(\tilde{F}, \check{G}, \check{A})$ is identified with the 2-bipolar soft set $(\tilde{F}, \check{G}, \check{A}, 2)$ which is given by

$\tilde{F}(e_1) = \{(u_1, 1), (u_2, 0), (u_3, 1)\}$, $\tilde{F}(e_2) = \{(u_1, 1), (u_2, 0), (u_3, 1)\}$, $\tilde{F}(e_3) = \{(u_1, 0), (u_2, 1), (u_3, 1)\}$ and $\tilde{F}(e_4) = \{(u_1, 1), (u_2, 1), (u_3, 0)\}$.

$\check{G}(\neg e_1) = \{(u_1, 0), (u_2, 1), (u_3, 0)\}$, $\check{G}(\neg e_2) = \{(u_1, 0), (u_2, 1), (u_3, 0)\}$, $\check{G}(\neg e_3) = \{(u_1, 1), (u_2, 0), (u_3, 0)\}$ and $\check{G}(\neg e_4) = \{(u_1, 0), (u_2, 0), (u_3, 1)\}$.

- In definition the grade $0 \in \check{R}$ does not mean that there is incomplete information or no information. It means the lowest grade or it represents that the individual does not possess the given attribute for example in Table 3 and 4. To interpret the above fact we consider the example of students and we model it by N -bipolar soft set.

Example 3.5. Let $\mathcal{U} = \{s_1, s_2, s_3\}$ be a universe of students and $\check{A}$ be a set of attributes "evaluation of students by the given parameters".

$e_1 = \text{legible handwriting}$, $\neg e_1 = \text{illegible handwriting}$, $e_2 = \text{cleanliness}$, $\neg e_2 = \text{dirty}$, $e_3 = \text{polite}$ and $\neg e_3 = \text{rude}$.

Evaluation is done by giving grades to students on their performance. The graded evaluation is identified by numbers $\check{R} = \{0, 1, 2, 3, 4, 5\}$ where $0 = \text{lowest}$, $1 = \text{lower}$, $2 = \text{low}$, $3 = \text{high}$, $4 = \text{higher}$, $5 = \text{highest}$.

The 6-bipolar soft set for this is represented in Table 7 and 8. Looking at the Tables 7, 8 and considering s_2 the "–" tells us that this student may not brought the copy, so the teacher cannot evaluate the handwriting for that student having. The most important point is that 0 here means lowest grade.

Table 7: 6-soft set

Table 8: 6-soft set

$(\tilde{F}, \check{A}, 6)$	e_1	e_2	e_3
s_1	1	4	1
s_2	–	3	0
s_3	2	4	1

$(\check{G}, \neg \check{A}, 6)$	$\neg e_1$	$\neg e_2$	$\neg e_3$
s_1	4	1	4
s_2	–	2	5
s_3	3	1	4

- For incomplete information we redefine as follows.

Definition 3.2. Let $\mathcal{U}$ be a universe set of objects and E be a set of attributes, $\check{A} \subseteq \check{E}$. Let $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \dots\}$. An incomplete N -bipolar soft is a quadruple $(\tilde{F}, \check{G}, \check{A}, N)$ on $\mathcal{U}$, where

$$\tilde{F} : \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \check{R})$$

$$\check{G} : \neg \check{A} \rightarrow \mathcal{P}(\mathcal{U} \times \check{R})$$

with the property that for each $e \in \check{A}$ and $\neg e \in \neg \check{A}$, there exist atmost one $(u, r_e), (u, r_{\neg e}) \in \mathcal{U} \times \check{R}$ with $(u, r_e) \in \tilde{F}(e)$ and $(u, r_{\neg e}) \in \check{G}(\neg e)$, where $u \in \mathcal{U}$, $r_e, r_{\neg e} \in \check{R}$, with a condition that $r_e + r_{\neg e} \leq N-1$. Where $\mathcal{P}(\mathcal{U} \times \check{R})$ denotes the power set of $\mathcal{U} \times \check{R}$.

Example 3.5 is Incomplete 6-bipolar soft set.

Remark 3.2. Any N -bipolar soft set can be considered, an arbitrary M -bipolar soft set with $M > N$.

For example, the 6-bipolar soft set given in Tables 9 and 10 can be considered as 7-bipolar soft set and so on over the same universe and with the same attributes.

Top grades sometimes appear in $(\tilde{F}, \check{A}, N)$ but not in $(\check{G}, \neg \check{A}, N)$ and vice versa. Sometimes top grades are available in both of the sets. Motivated from this, we define

Definition 3.3. An N -bipolar soft set is said to be a positive efficient over universe $\mathcal{U}$ if $\tilde{F}(e_j)(u_i) = N-1$ and thus the corresponding value is $\check{G}(\neg e_j)(u_i) = 0$ for some $e_j \in \check{A}$, $\neg e_j \in \neg \check{A}$ and $u_i \in \mathcal{U}$

Definition 3.4. An N -bipolar soft set is said to be a negative efficient over universe $\mathcal{U}$ if $\check{G}(\neg e_j)(u_i) = N-1$ and thus the corresponding value is $\tilde{F}(e_j)(u_i) = 0$, for some $e_j \in \check{A}$, $\neg e_j \in \neg \check{A}$ and $u_i \in \mathcal{U}$.

Example 3.6. Let $\mathcal{U} = \{u_1, u_2, u_3\}$ be a universe of objects. $\check{A} = \{e_1, e_2, e_3\}$ be the set of parameters and $\neg \check{A} = \{\neg e_1, \neg e_2, \neg e_3\}$. Consider the 6-Bipolar soft set $(\tilde{F}, \check{G}, \check{A}, 6)$ as given in Tables 9 and 10. This N -bipolar soft set is negative efficient.

Negative efficient 6-bipolar soft set

Table 9

Table 10

$(\tilde{F}, \check{A}, 6)$	e_1	e_2	e_3
u_1	1	4	1
u_2	3	3	0
u_3	2	4	1

$(\check{G}, \neg \check{A}, 6)$	$\neg e_1$	$\neg e_2$	$\neg e_3$
u_1	4	1	4
u_2	2	2	5
u_3	3	1	4

Definition 3.5. An N -bipolar soft set is said to be a total efficient over universe $\mathcal{U}$ if $\tilde{F}(e_j)(u_i) = N-1$ and the corresponding value is $\check{G}(\neg e_j)(u_i) = 0$, for some $e_j \in \check{A}$ and $\neg e_j \in \neg \check{A}$. Also, we have $\check{G}(\neg e_k)(u_i) = N-1$ with the corresponding value $\tilde{F}(e_k)(u_i) = 0$, for some $e_k \in \check{A}$ and $\neg e_k \in \neg \check{A}$, where $u_i, u_l \in \mathcal{U}$.

A total efficient N -bipolar soft set is that in which highest available grade is utilized in both positive as well as negative attributes. Example 3.1 is total efficient.

We now present the formalization of an idea for the bottom grade

Definition 3.6. The normalized N -bipolar soft set $(\tilde{F}^0, \tilde{G}^0, \check{\mathfrak{S}}, N)$ of an N -bipolar soft set $(\tilde{F}, \tilde{G}, \check{A}, N)$ over $\mathcal{U}$ is defined by the expression; for all $e_j \in \check{A}$, $\neg e_j \in \neg \check{A}$ and $u_i \in \mathcal{U}$, $\tilde{F}^0(e_j)(u_i) = \tilde{F}(e_j)(u_i) - m$ and $\tilde{G}^0(\neg e_j)(u_i) = \tilde{G}(\neg e_j)(u_i) - k$, where $m = \min \tilde{F}(e_j)(u_i)$, $k = \min \tilde{G}(\neg e_j)(u_i)$ and $\check{\mathfrak{S}} = \{1, 2, \dots, s\}$ is the index set for attributes.

Example 3.7. Let $\mathcal{U} = \{u_1, u_2\}$ be a universe of businesses. Suppose that Mr. Y wants to invest in a business, so in order to decide the suitable one, he wants to know about the level of presence of given attributes $\check{A} = \{e_1, e_2\} = \{\text{profit margins, expensive resources}\}$ and $\neg \check{A} = \{\neg e_1, \neg e_2\} = \{\text{loss margins, cheap resources}\}$, where $\check{R} = \{0, 1, 2, 3, 4, 5\}$. The 6-bipolar soft set for this is given in Tables 11 and 12.

Table 11: 6-soft set

$(\tilde{F}, \check{A}, 6)$	e_1	e_2
u_1	4	2
u_2	1	1

Table 12: 6-soft set

$(\tilde{G}, \neg \check{A}, 6)$	$\neg e_1$	$\neg e_2$
u_1	1	3
u_2	4	4

The normalized 6-bipolar soft set of above 6-bipolar soft set is given in Tables 13 and 14.

Table 13

$(\tilde{F}^0, \check{\mathfrak{S}}, 6)$	1	2
u_1	3	1
u_2	0	0

Table 14

$(\tilde{G}^0, \check{\mathfrak{S}}, 6)$	1	2
u_1	0	2
u_2	3	3

Definition 3.7. Two N -bipolar soft sets $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{F}_1, \tilde{G}_1, \check{B}, \check{N})$ over a common universe $\mathcal{U}$ are equal if and only if $\tilde{F} = \tilde{F}_1$, $\tilde{G} = \tilde{G}_1$, $\check{A} = \check{B}$ and $\check{N} = N$. We denote it by $(\tilde{F}, \tilde{G}, \check{A}, N) = (\tilde{F}_1, \tilde{G}_1, \check{B}, \check{N})$.

Definition 3.8. Two N -bipolar soft sets $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{P}, \tilde{S}, \check{B}, N)$ over a common universe $\mathcal{U}$ are said to be equivalent if $(\tilde{F}^0, \tilde{G}^0, \check{\mathfrak{S}}_1, N) = (\tilde{P}^0, \tilde{S}^0, \check{\mathfrak{S}}_2, N)$, that is there normalized N -bipolar soft sets are equal.

Definition 3.9. A weak compliment of an N -bipolar soft set $(\tilde{F}, \tilde{G}, \check{A}, N)$ over universe $\mathcal{U}$ is any N -bipolar soft set $(\tilde{F}^c, \tilde{G}^c, \check{A}, N)$ over $\mathcal{U}$, where $\tilde{F}^c(e) \cap \tilde{F}(e) = \phi$ and $\tilde{G}^c(\neg e) \cap \tilde{G}(\neg e) = \phi$.

Definition 3.10. The top weak compliment of an N -bipolar soft set $(\tilde{F}, \tilde{G}, \check{A}, N)$ over universe $\mathcal{U}$ is defined by $(\tilde{F}^t, \tilde{G}^t, \check{A}, N)$, where

$$\tilde{F}^t(e_j)(u_i) = \begin{cases} N-1 & \text{if } \tilde{F}(e_j)(u_i) < N-1, \\ 0 & \text{if } \tilde{F}(e_j)(u_i) = N-1. \end{cases} \quad (10)$$

and

$$\tilde{G}^t(\neg e_j)(u_i) = \begin{cases} 0 & \text{if } \tilde{G}(\neg e_j)(u_i) > 0, \\ 0 & \text{if } \tilde{G}(\neg e_j)(u_i) = 0 \text{ and } \tilde{F}(e_j)(u_i) < N-1, \\ N-1 & \text{otherwise} \end{cases} \quad (11)$$

for all $e_j \in \check{A}$, $\neg e_j \in \neg \check{A}$ and $u_i \in \mathcal{U}$.

Definition 3.11. The bottom weak compliment of an N -Bipolar soft set $(\tilde{F}, \tilde{G}, \check{A}, N)$ over universe $\mathcal{U}$ is defined by $(\tilde{F}^b, \tilde{G}^b, \check{A}, N)$, where

$$\tilde{F}^b(e_j)(u_i) = \begin{cases} 0 & \text{if } \tilde{F}(e_j)(u_i) > 0, \\ 0 & \text{if } \tilde{F}(e_j)(u_i) = 0 \text{ and } \tilde{G}(\neg e_j)(u_i) < N-1, \\ N-1 & \text{otherwise} \end{cases} \quad (12)$$

and

$$\tilde{G}^b(\neg e_j)(u_i) = \begin{cases} N-1 & \text{if } \tilde{G}(\neg e_j)(u_i) < N-1, \\ 0 & \text{if } \tilde{G}(\neg e_j)(u_i) = N-1. \end{cases} \quad (13)$$

for all $e_j \in \check{A}$, $\neg e_j \in \neg \check{A}$ and $u_i \in \mathcal{U}$.

Example 3.8. Weak, top weak and bottom weak compliments of the N -bipolar soft set $(\tilde{F}, \tilde{G}, \check{A}, 7)$ of Example 3.1 are given below.

Weak compliment of 7-bipolar soft set

Table 15

$(\tilde{F}^c, \check{A}, 7)$	e_1	e_2	e_3	e_4	e_5	e_6
u_1	4	2	1	4	5	1
u_2	1	3	2	5	4	2
u_3	0	3	1	2	2	2
u_4	2	4	2	1	2	1
u_5	3	1	5	2	2	2

Table 16

$(\tilde{G}^c, \neg \check{A}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$
u_1	2	4	5	2	1	5
u_2	5	3	4	1	2	4
u_3	6	3	5	4	4	4
u_4	4	2	4	5	4	5
u_5	2	5	1	4	4	4

Top weak compliment of 7-bipolar soft set

Table 17

$(\check{F}^t, \check{A}, 7)$	e_1	e_2	e_3	e_4	e_5	e_6
u_1	6	6	6	6	6	6
u_2	6	6	6	0	0	6
u_3	6	6	6	6	6	6
u_4	0	6	6	6	6	6
u_5	6	6	6	6	6	6

Table 18

$(\check{G}^t, \neg\check{A}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$
u_1	0	0	0	0	0	0
u_2	0	0	0	6	6	0
u_3	0	0	0	0	0	0
u_4	6	0	0	0	0	0
u_5	0	0	0	0	0	0

Bottom weak compliment of 7-bipolar soft set

Table 19

$(\check{F}^b, \check{A}, 7)$	e_1	e_2	e_3	e_4	e_5	e_6
u_1	0	0	0	0	0	0
u_2	0	6	0	0	0	0
u_3	0	0	0	0	0	0
u_4	0	0	0	0	0	6
u_5	0	0	0	0	0	0

Table 20

$(\check{G}^b, \neg\check{A}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$
u_1	6	6	6	6	6	6
u_2	6	0	6	6	6	6
u_3	6	6	6	6	6	6
u_4	6	6	6	6	6	0
u_5	6	6	6	6	6	6

- Each N -bipolar soft set can be associated with bipolar soft set for each $0 < T < N$.

Definition 3.12. Let $0 < T < N$ be a threshold and $(\check{F}, \check{G}, \check{A}, N)$ be an N -bipolar soft set over a universe $\mathcal{U}$. The bipolar soft set associated with $(\check{F}, \check{G}, \check{A}, N)$, denoted by $(\check{F}^T, \check{G}^T, \check{A}, N)$ and is defined by the expressions:

$$\check{F}^T(e)(u) = \begin{cases} 1 & \text{if } \check{F}(e)(u) \geq T, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

and

$$\check{G}^T(\neg e)(u) = \begin{cases} 1 & \text{if } \check{G}(\neg e)(u) \geq N - T, \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

for all $e \in \check{A}$, $\neg e \in \neg\check{A}$ and $u \in \mathcal{U}$.

Example 3.9. The bipolar soft set associated with 6-bipolar soft set of Example 3.7 is given in the following tables.

Bipolar soft set associated with 6-bipolar soft set when $T = 3$

Table 21

$(\check{F}^{T=3}, \check{A})$	e_1	e_2
u_1	1	0
u_2	0	0

Table 22

$(\check{G}^{T=3}, \neg\check{A})$	$\neg e_1$	$\neg e_2$
u_1	0	1
u_2	1	1

Remark 3.3. $(\check{F}^1, \check{G}^1, \check{A})$ when $T = 1$, is the bottom bipolar soft set associated with $(\check{F}, \check{G}, \check{A}, N)$, and $(\check{F}^{N-1}, \check{G}^{N-1}, \check{A}, N)$ when $T = N - 1$, is the top bipolar soft set associated with $(\check{F}, \check{G}, \check{A}, N)$.

4 Algebraic operations on N -Bipolar soft sets

If two persons are going to buy house, as given in Example 3.1 for the house of their common interest they can use restricted intersection, restricted union, extended intersection and extended union.

We here initiate these concepts on the N -bipolar soft sets.

Definition 4.1. Let $\mathcal{U}$ be a fixed universe of objects, the restricted intersection of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over $\mathcal{U}$ is denoted and defined by $(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{H}, \check{I}, \check{A} \cap \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cap \check{B} \neq \phi$ and $u \in \mathcal{U}$, $(u, r_e) \in \check{H}(e)$ if and only if $r_e = \min(r_e^1, r_e^2)$ if $(u, r_e^1) \in \check{F}(e)$ and $(u, r_e^2) \in \check{F}_1(e)$ also $(u, r_{\neg e}) \in \check{I}(\neg e)$ if and only if $r_{\neg e} = \max(r_{\neg e}^1, r_{\neg e}^2)$ if $(u, r_{\neg e}^1) \in \check{G}(\neg e)$ and $(u, r_{\neg e}^2) \in \check{G}_1(\neg e)$.

Definition 4.2. Let $\mathcal{U}$ be a fixed universe of objects. The extended intersection of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over $\mathcal{U}$ is denoted and defined by $(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{J}, \check{K}, \check{A} \cup \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$

$$\check{J}(e)(u) = \begin{cases} \check{F}(e)(u) & \text{if } e \in \check{A} - \check{B} \\ \check{F}_1(e)(u) & \text{if } e \in \check{B} - \check{A} \\ r_e & \text{if } e \in \check{A} \cap \check{B} \end{cases} \quad (16)$$

where $r_e = \min(r_e^1, r_e^2)$ such that $r_e^1 = \check{F}(e)(u)$ and $r_e^2 = \check{F}_1(e)(u)$

and

$$\check{K}(\neg e)(u) = \begin{cases} \check{G}(\neg e)(u) & \text{if } \neg e \in \neg\check{A} - \neg\check{B} \\ \check{G}_1(\neg e)(u) & \text{if } \neg e \in \neg\check{B} - \neg\check{A} \\ r_{\neg e} & \text{if } \neg e \in \neg\check{A} \cap \neg\check{B} \end{cases} \quad (17)$$

where $r_{-e} = \max(r_{-e}^1, r_{-e}^2)$ such that $r_{-e}^1 = \check{G}(-e)(u)$ and $r_{-e}^2 = \check{G}_1(-e)(u)$.

Definition 4.3. Let $\mathcal{U}$ be a fixed universe of objects. The restricted union of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over $\mathcal{U}$ is denoted and defined by $(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{L}, \check{M}, \check{A} \cap \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cap \check{B} \neq \phi$ and $u \in \mathcal{U}$,
 $(u, r_e) \in \check{L}(e)$ if and only if $r_e = \max(r_e^1, r_e^2)$ if $(u, r_e^1) \in \check{F}(e)$ and $(u, r_e^2) \in \check{F}_1(e)$ also $(u, r_{-e}) \in \check{M}(-e)$ if and only if $r_{-e} = \min(r_{-e}^1, r_{-e}^2)$ if $(u, r_{-e}^1) \in \check{G}(-e)$ and $(u, r_{-e}^2) \in \check{G}_1(-e)$.

Definition 4.4. Let $\mathcal{U}$ be the fixed universe of objects. The extended union of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over $\mathcal{U}$ is denoted and defined by $(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{N}, \check{O}, \check{A} \cup \check{B}, \max(N_1, N_2))$, where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$

$$\check{N}(e)(u) = \begin{cases} \check{F}(e)(u) & \text{if } e \in \check{A} - \check{B}, \\ \check{F}_1(e)(u) & \text{if } e \in \check{B} - \check{A}, \\ r_e & \text{if } e \in \check{A} \cap \check{B} \end{cases} \quad (18)$$

where $r_e = \max(r_e^1, r_e^2)$ such that $r_e^1 = \check{F}(e)(u)$ and $r_e^2 = \check{F}_1(e)(u)$.

and

$$\check{O}(-e)(u) = \begin{cases} \check{G}(-e)(u) & \text{if } -e \in \neg \check{A} - \neg \check{B}, \\ \check{G}_1(-e)(u) & \text{if } -e \in \neg \check{B} - \neg \check{A}, \\ r_{-e} & \text{if } -e \in \neg \check{A} \cap \neg \check{B} \end{cases} \quad (19)$$

where $r_{-e} = \min(r_{-e}^1, r_{-e}^2)$ such that $r_{-e}^1 = \check{G}(-e)(u)$ and $r_{-e}^2 = \check{G}_1(-e)(u)$.

Example 4.1. Let $\mathcal{U} = \{u_1, u_2, u_3\}$ be a universe. Let the attribute set be $\check{E} = \{e_1, e_2, e_3, e_4\}$. Let $\check{A} = \{e_1, e_2, e_3\}$ and $\check{B} = \{e_1, e_2, e_4\}$, $\neg \check{A} = \{\neg e_1, \neg e_2, \neg e_3\}$ and $\neg \check{B} = \{\neg e_1, \neg e_2, \neg e_4\}$. Consider the two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, 6)$ and $(\check{F}_1, \check{G}_1, \check{B}, 7)$ over $\mathcal{U}$ given in Tables 23, 24, 25 and 26. The extended and restricted union and intersection are given in the Tables 27, 28, 29, 30, 31, 32, 33 and 34.

6-Bipolar soft set

Table 23: 6-soft set

$(\check{F}, \check{A}, 6)$	e_1	e_2	e_3
u_1	1	4	2
u_2	2	3	0
u_3	2	1	3

Table 24: 6-soft set

$(\check{G}, \neg \check{A}, 6)$	$\neg e_1$	$\neg e_2$	$\neg e_3$
u_1	3	1	3
u_2	1	2	5
u_3	3	3	2

7-Bipolar soft set

Table 25: 7-soft set

$(\check{F}_1, \check{B}, 7)$	e_1	e_2	e_4
u_1	3	0	6
u_2	2	1	2
u_3	4	5	1

Table 26: 7-soft set

$(\check{G}_1, \neg \check{B}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_4$
u_1	2	5	0
u_2	3	2	3
u_3	1	0	3

Restricted intersection

Table 27

$(\check{H}, \check{A} \cap \check{B}, \max(6, 7))$	e_1	e_2
u_1	1	0
u_2	2	1
u_3	2	1

Table 28

$(\check{I}, \neg \check{A} \cap \neg \check{B}, 7)$	$\neg e_1$	$\neg e_2$
u_1	3	5
u_2	3	2
u_3	3	3

Restricted union

Table 29

$(\check{L}, \check{A} \cap \check{B}, \max(6, 7))$	e_1	e_2
u_1	3	4
u_2	2	3
u_3	4	5

Table 30

$(\check{M}, \neg \check{A} \cap \neg \check{B}, 7)$	$\neg e_1$	$\neg e_2$
u_1	2	1
u_2	1	2
u_3	1	0

Extended intersection

Table 31

$(\check{J}, \check{A} \cup \check{B}, 7)$	e_1	e_2	e_3	e_4
u_1	1	0	2	6
u_2	2	1	0	2
u_3	2	1	3	1

Table 32

$(\check{K}, \neg \check{A} \cup \neg \check{B}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$
u_1	3	5	3	0
u_2	3	2	5	3
u_3	3	3	2	3

Extended union

Table 33

$(\check{N}, \check{A} \cup \check{B}, 7)$	e_1	e_2	e_3	e_4
u_1	3	4	2	6
u_2	2	3	0	2
u_3	4	5	3	1

Table 34

$(\check{O}, \neg \check{A} \cup \neg \check{B}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$
u_1	2	1	3	0
u_2	1	2	5	3
u_3	1	0	2	3

Definition 4.5. The restricted T -intersection of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over a common universe $\mathcal{U}$, where $T < \min(N_1, N_2)$, is the restricted intersection of two bipolar soft sets $(\check{F}^T, \check{G}^T, \check{A})$ and $(\check{F}_1^T, \check{G}_1^T, \check{B})$. It is denoted by

$$(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}}^T (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{F}^T, \check{G}^T, \check{A}) \cap_{\mathbb{R}} (\check{F}_1^T, \check{G}_1^T, \check{B}).$$

Similarly T -extended intersection of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over a common universe $\mathcal{U}$, where $T < \min(N_1, N_2)$, is the extended intersection of two bipolar soft sets $(\check{F}^T, \check{G}^T, \check{A})$ and $(\check{F}_1^T, \check{G}_1^T, \check{B})$. It is denoted by

$$(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}}^T (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{F}^T, \check{G}^T, \check{A}) \cap_{\mathbb{E}} (\check{F}_1^T, \check{G}_1^T, \check{B}).$$

Definition 4.6. The restricted T -union of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over a common universe $\mathcal{U}$, where $T < \min(N_1, N_2)$, is the restricted union of two bipolar soft sets $(\check{F}^T, \check{G}^T, \check{A})$ and $(\check{F}_1^T, \check{G}_1^T, \check{B})$. It is denoted by

$$(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}}^T (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{F}^T, \check{G}^T, \check{A}) \cup_{\mathbb{R}} (\check{F}_1^T, \check{G}_1^T, \check{B}).$$

Similarly T -extended union of two N -bipolar soft sets $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ over a common universe $\mathcal{U}$, where $T < \min(N_1, N_2)$, is the extended union of two bipolar soft sets $(\check{F}^T, \check{G}^T, \check{A})$ and $(\check{F}_1^T, \check{G}_1^T, \check{B})$. It is denoted by

$$(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}}^T (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{F}^T, \check{G}^T, \check{A}) \cup_{\mathbb{E}} (\check{F}_1^T, \check{G}_1^T, \check{B}).$$

Definition 4.7. An N -bipolar soft set $(\check{\phi}, \check{\mathfrak{U}}, \check{A}, N)$ over $\mathcal{U}$ is said to be a relative null N -bipolar soft set if for all $e \in \check{A}$ and $\neg e \in \neg \check{A}$, $\check{\phi}(e)(u) = 0$ and $\check{\mathfrak{U}}(\neg e)(u) = N - 1$ for all $u \in \mathcal{U}$.

Definition 4.8. An N -bipolar soft set $(\check{\mathfrak{U}}, \check{\phi}, \check{A}, N)$ over $\mathcal{U}$ is said to be a relative absolute N -bipolar soft set if for all $e \in \check{A}$ and $\neg e \in \neg \check{A}$, $\check{\mathfrak{U}}(e)(u) = N - 1$ and $\check{\phi}(\neg e)(u) = 0$ for all $u \in \mathcal{U}$.

Conventionally we write if $\check{A} \cap \check{B} = \phi$, $(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{\phi}, \check{\mathfrak{U}}, \phi, \max(N_1, N_2))$. Similarly if $\check{A} \cap \check{B} = \phi$, $(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{\phi}, \check{\mathfrak{U}}, \phi, \max(N_1, N_2))$.

Definition 4.9. An N -bipolar soft set $(\alpha, \beta, \check{B}, N)$ is called the N -bipolar soft subset of $(\check{F}, \check{G}, \check{A}, N)$ over a common universe $\mathcal{U}$ if

$$(i) \check{B} \subseteq \check{A}.$$

$$(ii) \alpha(e)(u) \leq \check{F}(e)(u) \text{ for all } e \in \check{B} \text{ and } \beta(\neg e)(u) \geq \check{G}(\neg e)(u) \text{ for all } \neg e \in \neg \check{B} \text{ and } u \in \mathcal{U}.$$

It is denoted by $(\alpha, \beta, \check{B}, N) \hat{\subseteq} (\check{F}, \check{G}, \check{A}, N)$.

The distributive laws which hold in N -bipolar soft sets are described in the following theorem.

Theorem 4.1. Let $(\check{F}, \check{G}, \check{A}, N_1)$, $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ and $(\check{F}_3, \check{G}_3, \check{C}, N_3)$ are the three N -bipolar soft sets over the common universe $\mathcal{U}$. Then the following are true.

- (i) $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (ii) $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{R}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (iii) $[(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cup_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cap_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cup_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (iv) $[(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cap_{\mathbb{R}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (v) $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{R}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (vi) $[(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cap_{\mathbb{R}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (vii) $[(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cap_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (viii) $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cup_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cup_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (ix) $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3) \hat{\subseteq} [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$
- (x) $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cap_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \hat{\subseteq} [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cup_{\mathbb{E}} (\check{F}_3, \check{G}_3, \check{C}, N_3).$

Proof. (i) **L.H.S**

Let $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] = (\check{I}, \check{J}, \check{A} \cup \check{B}, \max(N_1, N_2))$ where for each $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$, we have

$$\check{I}(e)(u) \begin{cases} \check{F}(e)(u) & \text{if } e \in \check{A} - \check{B} = \check{A} \cap \check{B}^c \\ \check{F}_1(e)(u) & \text{if } e \in \check{B} - \check{A} = \check{A}^c \cap \check{B} \\ \max(\check{F}(e)(u), \check{F}_1(e)(u)) & \text{if } e \in \check{A} \cap \check{B} \end{cases} \quad (20)$$

and

$$\check{J}(\neg e)(u) \begin{cases} \check{G}(\neg e)(u) & \text{if } \neg e \in \neg \check{A} - \neg \check{B} \\ \check{G}_1(\neg e)(u) & \text{if } \neg e \in \neg \check{B} - \neg \check{A} \\ \min(\check{G}(\neg e)(u), \check{G}_1(\neg e)(u)) & \text{if } \neg e \in \neg \check{A} \cap \neg \check{B} \end{cases} \quad (21)$$

Let $[(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)$
 $= (\check{I}, \check{J}, \check{A} \cup \check{B}, \max(N_1, N_2)) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)$
 $= (\check{K}, \check{L}, (\check{A} \cup \check{B}) \cap \check{C}, \max(\max(N_1, N_2), N_3))$
 $= (\check{K}, \check{L}, (\check{A} \cup \check{B}) \cap \check{C}, \max(N_1, N_2, N_3))$
 where for each $e \in (\check{A} \cup \check{B}) \cap \check{C} = (\check{A} \cap \check{C}) \cup (\check{B} \cap \check{C})$ and $u \in \mathcal{U}$, we have

$$\check{K}(e)(u) = \min(\check{I}(e)(u), \check{F}_3(e)(u))$$

Now for each $e \in \check{A} \cap \check{B}^c \cap \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{K}(e)(u) = \min(\check{F}(e)(u), \check{F}_3(e)(u))$$

where for each $e \in \check{A}^c \cap \check{B} \cap \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{K}(e)(u) = \min(\check{F}_1(e)(u), \check{F}_3(e)(u))$$

where for each $e \in \check{A} \cap \check{B} \cap \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{K}(e)(u) = \min(\max(\check{F}(e)(u), \check{F}_1(e)(u)), \check{F}_3(e)(u))$$

and

Now for each $\neg e \in (\neg \check{A} \cup \neg \check{B}) \cap \neg \check{C} = (\neg \check{A} \cap \neg \check{C}) \cup (\neg \check{B} \cap \neg \check{C})$ and $u \in \mathcal{U}$, we have

$$\check{L}(\neg e)(u) = \max(\check{J}(\neg e)(u), \check{G}_3(\neg e)(u))$$

where for each $\neg e \in \neg \check{A} \cap \neg \check{B}^c \cap \neg \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{L}(\neg e)(u) = \max(\check{G}(\neg e)(u), \check{G}_3(\neg e)(u))$$

where for each $\neg e \in \neg \check{A}^c \cap \neg \check{B} \cap \neg \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{L}(\neg e)(u) = \max(\check{G}_1(\neg e)(u), \check{G}_3(\neg e)(u))$$

where for each $\neg e \in \neg \check{A} \cap \neg \check{B} \cap \neg \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{L}(\neg e)(u) = \max(\min(\check{G}(\neg e)(u), \check{G}_1(\neg e)(u)), \check{G}_3(\neg e)(u)).$$

R.H.S

$$\text{Let } [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] = (\check{M}, \check{R}, \check{A} \cap \check{C}, \max(N_1, N_3))$$

where for all $e \in \check{A} \cap \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{M}(e)(u) = \min(\check{F}(e)(u), \check{F}_3(e)(u))$$

$$\check{R}(\neg e)(u) = \max(\check{G}(\neg e)(u), \check{G}_3(\neg e)(u)).$$

$$\text{Let } [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] = (\check{P}, \check{Q}, \check{B} \cap \check{C}, \max(N_2, N_3))$$

where for all $e \in \check{B} \cap \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{P}(e)(u) = \min(\check{F}_1(e)(u), \check{F}_3(e)(u))$$

$$\check{Q}(\neg e)(u) = \max(\check{G}_1(\neg e)(u), \check{G}_3(\neg e)(u)).$$

$$\text{Now } [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{E}} [(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)]$$

$$= (\check{M}, \check{R}, \check{A} \cap \check{C}, \max(N_1, N_3)) \cup_{\mathbb{E}} (\check{P}, \check{Q}, \check{B} \cap \check{C}, \max(N_2, N_3))$$

$$= (\check{S}, \check{T}, (\check{A} \cap \check{C}) \cup (\check{B} \cap \check{C}), \max(\max(N_1, N_3), \max(N_2, N_3)))$$

$$= (\check{S}, \check{T}, (\check{A} \cup \check{B}) \cap \check{C}, \max(N_1, N_2, N_3))$$

where for $e \in (\check{A} \cap \check{C}) - (\check{B} \cap \check{C}) = (\check{A} \cap \check{C}) \cap (\check{B} \cap \check{C})^c$

$$= (\check{A} \cap \check{C}) \cap (\check{B}^c \cup \check{C}^c) = (\check{A} \cap \check{C}) \cap \check{B}^c \text{ and } u \in \mathcal{U}, \text{ we have}$$

$$\check{S}(e)(u) = \check{M}(e)(u) = \min(\check{F}(e)(u), \check{F}_3(e)(u)).$$

$$\text{For } e \in (\check{B} \cap \check{C}) - (\check{A} \cap \check{C}) = (\check{B} \cap \check{C}) \cap (\check{A} \cap \check{C})^c =$$

$$(\check{B} \cap \check{C}) \cap (\check{A}^c \cup \check{C}^c) = (\check{B} \cap \check{A}^c) \cap \check{C}$$

and $u \in \mathcal{U}$, we have

$$\check{S}(e)(u) = \check{P}(e)(u) = \min(\check{F}_1(e)(u), \check{F}_3(e)(u)).$$

For $e \in (\check{A} \cap \check{C}) \cap (\check{B} \cap \check{C}) = \check{A} \cap \check{B} \cap \check{C}$ and $u \in \mathcal{U}$, we have

$$\check{S}(e)(u) = \max(\check{M}(e)(u), \check{P}(e)(u))$$

$$= \max(\min(\check{F}(e)(u), \check{F}_3(e)(u)), \min(\check{F}_1(e)(u), \check{F}_3(e)(u))).$$

$$\text{Now for } \neg e \in (\neg \check{A} \cap \neg \check{C}) - (\neg \check{B} \cap \neg \check{C}) = (\neg \check{A} \cap \neg \check{C}) \cap (\neg \check{B} \cap \neg \check{C})^c =$$

$$(\neg \check{A} \cap \neg \check{C}) \cap (\neg \check{B}^c \cup \neg \check{C}^c) = (\neg \check{A} \cap \neg \check{C}) \cap \neg \check{B}^c$$

and $u \in \mathcal{U}$, we have

$$\check{T}(\neg e)(u) = \check{R}(\neg e)(u) = \max(\check{G}(\neg e)(u), \check{G}_3(\neg e)(u)).$$

$$\text{For } \neg e \in (\neg \check{B} \cap \neg \check{C}) - (\neg \check{A} \cap \neg \check{C}) = (\neg \check{B} \cap \neg \check{C}) \cap (\neg \check{A} \cap \neg \check{C})^c =$$

$$(\neg \check{B} \cap \neg \check{C}) \cap (\neg \check{A}^c \cup \neg \check{C}^c) = (\neg \check{B} \cap \neg \check{A}^c) \cap \neg \check{C}$$

and $u \in \mathcal{U}$, we have

$$\check{T}(\neg e)(u) = \check{Q}(\neg e)(u) = \max(\check{G}_1(\neg e)(u), \check{G}_3(\neg e)(u)).$$

$$\text{For } \neg e \in (\neg \check{A} \cap \neg \check{C}) \cap (\neg \check{B} \cap \neg \check{C}) = \neg \check{A} \cap \neg \check{B} \cap \neg \check{C}$$

and $u \in \mathcal{U}$, we have

$$\check{T}(\neg e)(u) = \min(\check{R}(\neg e)(u), \check{Q}(\neg e)(u))$$

$$= \min(\max(\check{G}(\neg e)(u), \check{G}_3(\neg e)(u)), \max(\check{G}_1(\neg e)(u), \check{G}_3(\neg e)(u))).$$

$$\text{This shows that } [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N_2)] \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3) = [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)] \cup_{\mathbb{E}}$$

$$[(\check{F}_1, \check{G}_1, \check{B}, N_2) \cap_{\mathbb{R}} (\check{F}_3, \check{G}_3, \check{C}, N_3)].$$

Similarly we can prove for other parts. $\square$

Lemma 4.1. Let $(\check{F}, \check{G}, \check{A}, N_1)$, $(\check{F}_1, \check{G}_1, \check{B}, N_2)$ and $(\check{F}_3, \check{G}_3, \check{C}, N_3)$ be any three N -Bipolar soft sets over a common universe $\mathcal{U}$. Then the following are true $\forall \gamma \in \{\cup_{\mathbb{E}}, \cap_{\mathbb{E}}, \cup_{\mathbb{R}}, \cap_{\mathbb{R}}\}$.

$$(i) [(\check{F}, \check{G}, \check{A}, N_1) \gamma (\check{F}_1, \check{G}_1, \check{B}, N_2)] \gamma (\check{F}_3, \check{G}_3, \check{C}, N_3) = (\check{F}, \check{G}, \check{A}, N_1) \gamma [(\check{F}_1, \check{G}_1, \check{B}, N_2) \gamma (\check{F}_3, \check{G}_3, \check{C}, N_3)].$$

$$(ii) (\check{F}, \check{G}, \check{A}, N_1) \gamma (\check{F}_1, \check{G}_1, \check{B}, N_2) = (\check{F}_1, \check{G}_1, \check{B}, N_2) \gamma (\check{F}, \check{G}, \check{A}, N_1).$$

$$(iii) (\check{F}, \check{G}, \check{A}, N_1) \gamma (\check{F}, \check{G}, \check{A}, N_1) = (\check{F}, \check{G}, \check{A}, N_1).$$

Proof. Straightforward. $\square$

Lemma 4.2. Let $(\check{\mathfrak{U}}, \check{\phi}, \check{A}, N_1)$, $(\check{\mathfrak{U}}, \check{\phi}, \phi, N_1)$ be the relative absolute N_1 -bipolar soft sets with parameters set $\check{A}$ and $\check{\phi}$, respectively. Let $(\check{\phi}, \check{\mathfrak{U}}, \check{A}, N_1)$, $(\check{\phi}, \check{\mathfrak{U}}, \phi, N_1)$ be the relative null N_1 -bipolar soft sets with parameters set $\check{A}$ and ϕ , respectively and $(\check{F}, \check{G}, \check{A}, N_1)$ and $(\check{F}_1, \check{G}_1, \check{A}, N_2)$ be two N -bipolar soft sets over the common universe $\mathcal{U}$. Then the following hold.

$$(i) [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{A}, N_2)] = [(\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{A}, N_2)].$$

$$(ii) [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{A}, N_2)] = [(\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{A}, N_2)].$$

$$(iii) (\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{\phi}, \check{\mathfrak{U}}, \check{A}, N_1) = (\check{F}, \check{G}, \check{A}, N_1) ; (\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{\phi}, \check{\mathfrak{U}}, \check{A}, N_1) = (\check{\phi}, \check{\mathfrak{U}}, \check{A}, N_1).$$

$$(iv) (\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{\mathfrak{U}}, \check{\phi}, \check{A}, N_1) = (\check{\mathfrak{U}}, \check{\phi}, \check{A}, N_1) ; (\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{\mathfrak{U}}, \check{\phi}, \check{A}, N_1) = (\check{F}, \check{G}, \check{A}, N_1).$$

$$(v) (\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{R}} (\check{F}^t, \check{G}^t, \check{A}, N_1) = (\check{\mathfrak{U}}, \check{\phi}, \check{A}, N_1).$$

$$(vi) (\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{R}} (\check{F}^b, \check{G}^b, \check{A}, N_1) = (\check{\phi}, \check{\mathfrak{U}}, \check{A}, N_1).$$

$$(vii) (\check{F}, \check{G}, \check{A}, N_1) \cup_{\mathbb{E}} (\check{\phi}, \check{\mathfrak{U}}, \phi, N_1) = (\check{F}, \check{G}, \check{A}, N_1)$$

$$(viii) (\check{F}, \check{G}, \check{A}, N_1) \cap_{\mathbb{E}} (\check{\mathfrak{U}}, \check{\phi}, \phi, N_1) = (\check{F}, \check{G}, \check{A}, N_1)$$

Proof. (i) Since the set of attributes for both the N -bipolar soft set is same so by the Definitions 4.1-4.4 it can be seen that the result is obvious.

(ii) As mentioned above.

(iii) It is obvious by the Definitions 4.1 - 4.4 and 4.7 - 4.9.

(iv) Obvious

- (v) Let $(u, r_e) \in (\tilde{F} \cup_{\mathbb{R}} \tilde{F}^t)(e)$ for all $u \in \mathcal{U}$ and $e \in \check{A}$
 $r_e = \max(r_e^1, r_e^{1'})$ such that $(u, r_e^1) \in \tilde{F}(e)$ and
 $(u, r_e^{1'}) \in \tilde{F}^t(e)$
Case 1 if $r_e^{1'} = N - 1 \iff r_e^1 < N - 1$
 $\iff r_e = N - 1$
Case 1 if $r_e^{1'} = 0 \iff r_e^1 = N - 1$
 $\iff r_e = N - 1$
 $\iff (u, r_e) \in \check{A}(e)$
Let $(u, r_{\neg e}) \in (\tilde{G} \cup_{\mathbb{R}} \tilde{G}^t)(\neg e)$ for all $u \in \mathcal{U}$ and $\neg e \in \neg \check{A}$
 $r_{\neg e} = \min(r_{\neg e}^1, r_{\neg e}^{1'})$ such that $(u, r_{\neg e}^1) \in \tilde{G}(\neg e)$ and
 $(u, r_{\neg e}^{1'}) \in \tilde{G}^t(\neg e)$
Case 1 if $r_{\neg e}^{1'} = 0 \iff r_{\neg e}^1 > 0$ or $r_{\neg e}^1 = 0$ when $r_e^1 < N - 1$.
 $\iff r_{\neg e} = 0$
Case 2 if $r_{\neg e}^{1'} = N - 1 \iff r_{\neg e}^1 = 0$
 $\iff r_{\neg e} = 0$
 $\iff (u, r_{\neg e}) \in \check{\phi}(\neg e)$.

The other parts can be proved easily. $\square$

Lemma 4.3. Let $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{F}_1, \tilde{G}_1, \check{B}, N)$ be two N -bipolar soft sets over $\mathcal{U}$.

- (i) $(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} [(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{F}, \tilde{G}, \check{A}, N)$
(ii) $(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{R}} [(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{F}, \tilde{G}, \check{A}, N)$
(iii) $(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} [(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{F}, \tilde{G}, \check{A}, N)$
(iv) $(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{E}} [(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{F}, \tilde{G}, \check{A}, N)$.

Proof. (i) Let $[(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{I}, \tilde{J}, \check{A} \cap \check{B}, N)$,
where for all $e \in \check{A} \cap \check{B}$ and $u \in \mathcal{U}$, we have

$$\tilde{I}(e)(u) = \min(\tilde{F}(e)(u), \tilde{F}_1(e)(u)) \quad (22)$$

$$\tilde{J}(\neg e)(u) = \max(\tilde{G}(\neg e)(u), \tilde{G}_1(\neg e)(u)) \quad (23)$$

Now, let $(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} [(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{K}, \tilde{L}, \check{A} \cup (\check{A} \cap \check{B}), N) = (\tilde{K}, \tilde{L}, \check{A}, N)$,
where for all $e \in \check{A} \cup (\check{A} \cap \check{B})$, we have

$$\tilde{K}(e)(u) = \begin{cases} \tilde{F}(e)(u) & \text{if } e \in \check{A} - (\check{A} \cap \check{B}) \\ \tilde{I}(e)(u) & \text{if } e \in \check{A} \cap \check{B} - \check{A} = \phi \\ \max(\tilde{F}(e)(u), \tilde{I}(e)(u)) & \text{if } e \in \check{A} \cap (\check{A} \cap \check{B}) \end{cases} \quad (24)$$

and

$$\tilde{L}(\neg e)(u) = \begin{cases} \tilde{G}(\neg e)(u) & \text{if } \neg e \in \neg \check{A} - (\neg \check{A} \cap \neg \check{B}) \\ \tilde{J}(\neg e)(u) & \text{if } \neg e \in (\neg \check{A} \cap \neg \check{B}) \\ \min(\tilde{G}(\neg e)(u), \tilde{J}(\neg e)(u)) & \text{if } \neg e \in \neg \check{A} \cap (\neg \check{A} \cap \neg \check{B}) \end{cases} \quad (25)$$

After putting values from (22) in (24) and (23) in (25), and solving for attribute sets we get

$$\tilde{K}(e)(u) = \begin{cases} \tilde{F}(e)(u) & \text{if } e \in \check{A} - (\check{A} \cap \check{B}) \\ \tilde{F}(e)(u) & \text{if } e \in \check{A} \cap \check{B} \end{cases} \quad (26)$$

and

$$\tilde{L}(\neg e)(u) = \begin{cases} \tilde{G}(\neg e)(u) & \text{if } \neg e \in \neg \check{A} - (\neg \check{A} \cap \neg \check{B}) \\ \tilde{G}(\neg e)(u) & \text{if } \neg e \in \neg \check{A} \cap \neg \check{B} \end{cases} \quad (27)$$

This shows that $(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} [(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)] = (\tilde{F}, \tilde{G}, \check{A}, N)$.

Similarly we can prove the remaining parts. $\square$

Lemma 4.4. If $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{F}_1, \tilde{G}_1, \check{B}, N)$ are the two N -bipolar soft sets over a common universe $\mathcal{U}$. Then the following are true.

- (i) $(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)$ is the smallest N -bipolar soft set over $\mathcal{U}$ which contains both $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{F}_1, \tilde{G}_1, \check{B}, N)$.
(ii) $(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)$ is the largest N -bipolar soft set over $\mathcal{U}$ which is contained in both $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{F}_1, \tilde{G}_1, \check{B}, N)$.

Proof. Straightforward. $\square$

Lemma 4.5. Let $(\tilde{F}, \tilde{G}, \check{A}, N)$ and $(\tilde{F}_1, \tilde{G}_1, \check{B}, N)$ be two N -bipolar soft sets over the common universe $\mathcal{U}$. Then the following are true.

- (i) $[(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^t = (\tilde{F}^t, \tilde{G}^t, \check{A}, N) \cap_{\mathbb{E}} (\tilde{F}_1^t, \tilde{G}_1^t, \check{B}, N)$
(ii) $[(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^t = (\tilde{F}^t, \tilde{G}^t, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1^t, \tilde{G}_1^t, \check{B}, N)$
(iii) $[(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^b = (\tilde{F}^b, \tilde{G}^b, \check{A}, N) \cap_{\mathbb{E}} (\tilde{F}_1^b, \tilde{G}_1^b, \check{B}, N)$
(iv) $[(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^b = (\tilde{F}^b, \tilde{G}^b, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1^b, \tilde{G}_1^b, \check{B}, N)$
(v) $[(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{R}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^t = (\tilde{F}^t, \tilde{G}^t, \check{A}, N) \cap_{\mathbb{R}} (\tilde{F}_1^t, \tilde{G}_1^t, \check{B}, N)$
(vi) $[(\tilde{F}, \tilde{G}, \check{A}, N) \cap_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^t = (\tilde{F}^t, \tilde{G}^t, \check{A}, N) \cup_{\mathbb{R}} (\tilde{F}_1^t, \tilde{G}_1^t, \check{B}, N)$

$$(vii) [(\check{F}, \check{G}, \check{A}, N) \cup_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N)]^b = (\check{F}^b, \check{G}^b, \check{A}, N) \cap_{\mathbb{R}} (\check{F}_1^b, \check{G}_1^b, \check{B}, N) \quad \text{and}$$

$$(viii) [(\check{F}, \check{G}, \check{A}, N) \cap_{\mathbb{R}} (\check{F}_1, \check{G}_1, \check{B}, N)]^b = (\check{F}^b, \check{G}^b, \check{A}, N) \cup_{\mathbb{R}} (\check{F}_1^b, \check{G}_1^b, \check{B}, N).$$

Proof. (i) Let $[(\check{F}, \check{G}, \check{A}, N) \cup_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N)] = (\check{I}, \check{J}, \check{A} \cup \check{B}, N)$. Where

$$\check{I}(e)(u) = \begin{cases} \check{F}(e)(u) & \text{if } e \in \check{A} - \check{B} \\ \check{F}_1(e)(u) & \text{if } e \in \check{B} - \check{A} \\ \max(\check{F}(e)(u), \check{F}_1(e)(u)) & \text{if } e \in \check{A} \cap \check{B} \end{cases} \quad (28)$$

and

$$\check{J}(\neg e)(u) = \begin{cases} \check{G}(\neg e)(u) & \text{if } \neg e \in \neg \check{A} - \neg \check{B} \\ \check{G}_1(\neg e)(u) & \text{if } \neg e \in \neg \check{B} - \neg \check{A} \\ \min(\check{G}(\neg e)(u), \check{G}_1(\neg e)(u)) & \text{if } \neg e \in \neg \check{A} \cap \neg \check{B} \end{cases} \quad (29)$$

Now $[(\check{F}, \check{G}, \check{A}, N) \cap_{\mathbb{E}} (\check{F}_1, \check{G}_1, \check{B}, N)]^t = (\check{I}, \check{J}, \check{A} \cup \check{B}, N)^t = (\check{I}^t, \check{J}^t, \check{A} \cup \check{B}, N)$

where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$, we have

$$\check{I}^t(e)(u) = \begin{cases} N-1 & \text{if } \check{I}(e)(u) < N-1 \\ 0 & \text{if } \check{I}(e)(u) = N-1 \end{cases} \quad (30)$$

and

$$\check{J}^t(\neg e)(u) = \begin{cases} 0 & \text{if } \check{J}(\neg e)(u) > 0 \\ 0 & \text{if } \check{J}(\neg e)(u) = 0 \text{ when } \check{I}(e)(u) < N-1 \\ N-1 & \text{if } \check{J}(\neg e)(u) = 0 \end{cases} \quad (31)$$

Thus

$$\check{I}^t(e)(u) = \begin{cases} N-1 & \text{if } \check{F}(e)(u) < N-1 \text{ and } e \in \check{A} - \check{B} \\ 0 & \text{if } \check{F}(e)(u) = N-1 \text{ and } e \in \check{A} - \check{B} \\ N-1 & \text{if } \check{F}_1(e)(u) < N-1 \text{ and } e \in \check{B} - \check{A} \\ 0 & \text{if } \check{F}_1(e)(u) = N-1 \text{ and } e \in \check{B} - \check{A} \\ N-1 & \text{if } \max(\check{F}(e)(u), \check{F}_1(e)(u)) < N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \\ 0 & \text{if } \max(\check{F}(e)(u), \check{F}_1(e)(u)) = N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \end{cases} \quad (32)$$

$$\check{J}^t(\neg e)(u) = \begin{cases} 0 & \text{if } \check{G}(\neg e)(u) > 0 \text{ and } \neg e \in \neg \check{A} - \neg \check{B} \\ 0 & \text{if } \check{G}(\neg e)(u) = 0 \text{ when } \check{F}(e)(u) < N-1 \\ & \text{and } \neg e \in \neg \check{A} - \neg \check{B} \\ N-1 & \text{if } \check{G}(\neg e)(u) = 0 \text{ and } \neg e \in \neg \check{A} - \neg \check{B} \\ 0 & \text{if } \check{G}_1(\neg e)(u) > 0 \text{ and } \neg e \in \neg \check{B} - \neg \check{A} \\ 0 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ with } \check{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg \check{B} - \neg \check{A} \\ N-1 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ and } \neg e \in \neg \check{B} - \neg \check{A} \\ 0 & \text{if } \min(\check{G}(\neg e)(u), \check{G}_1(\neg e)(u)) > 0 \\ & \text{and } \neg e \in \neg \check{A} \cap \neg \check{B} \\ 0 & \text{if } \min(\check{G}(\neg e)(u), \check{G}_1(\neg e)(u)) = 0 \\ & \text{when } \max(\check{F}(e)(u), \check{F}_1(e)(u)) < N-1 \\ & \text{and } \neg e \in \neg \check{A} \cap \neg \check{B} \\ N-1 & \text{if } \min(\check{G}(\neg e)(u), \check{G}_1(\neg e)(u)) = 0 \\ & \text{and } \neg e \in \neg \check{A} \cap \neg \check{B} \end{cases} \quad (33)$$

That is

$$\check{I}^t(e)(u) = \begin{cases} N-1 & \text{if } \check{F}(e)(u) < N-1 \text{ and } e \in \check{A} - \check{B} \\ 0 & \text{if } \check{F}(e)(u) = N-1 \text{ and } e \in \check{A} - \check{B} \\ N-1 & \text{if } \check{F}_1(e)(u) < N-1 \text{ and } e \in \check{B} - \check{A} \\ 0 & \text{if } \check{F}_1(e)(u) = N-1 \text{ and } e \in \check{B} - \check{A} \\ N-1 & \text{if } \check{F}(e)(u) < N-1 \text{ and } \check{F}_1(e)(u) < N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \\ 0 & \text{if } \check{F}(e)(u) = N-1 \text{ and } \check{F}_1(e)(u) = N-1, \\ & \text{or } \check{F}(e)(u) < N-1 \text{ and } \check{F}_1(e)(u) = N-1, \\ & \text{or } \check{F}(e)(u) = N-1 \text{ and } \check{F}_1(e)(u) < N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \end{cases} \quad (34)$$

and

$$\check{J}^t(\neg e)(u) = \begin{cases} 0 & \text{if } \check{G}(\neg e)(u) > 0 \text{ and } \neg e \in \neg\check{A} - \neg\check{B} \\ 0 & \text{if } \check{G}(\neg e)(u) = 0 \text{ with } \check{F}(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{A} - \neg\check{B} \\ N-1 & \text{if } \check{G}(\neg e)(u) = 0 \text{ and } \neg e \in \neg\check{A} - \neg\check{B} \\ 0 & \text{if } \check{G}_1(\neg e)(u) > 0 \text{ and } \neg e \in \neg\check{B} - \neg\check{A} \\ 0 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ with } \check{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{B} - \neg\check{A} \\ N-1 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ and } \neg e \in \neg\check{B} - \neg\check{A} \\ 0 & \text{if } \check{G}(\neg e)(u) > 0 \text{ and } \check{G}_1(\neg e)(u) > 0, \\ & \text{and } \neg e \in \neg\check{A} \cap \neg\check{B} \\ 0 & \text{if } \check{G}(\neg e)(u) = 0 \text{ when } \check{F}(e)(u) < N-1 \\ & \text{and } \check{G}_1(\neg e)(u) > 0, \text{ or} \\ & \check{G}(\neg e)(u) > 0 \text{ and } \check{G}_1(\neg e)(u) = 0 \\ & \text{when } \check{F}_1(e)(u) < N-1, \text{ or} \\ & \check{G}(\neg e)(u) = 0 \text{ when } \check{F}(e)(u) < N-1 \\ & \text{and } \check{G}_1(\neg e)(u) = 0 \\ & \text{when } \check{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{A} \cap \neg\check{B} \\ N-1 & \text{if } \check{G}(\neg e)(u) = 0 \text{ and } \check{G}_1(\neg e)(u) = 0, \\ & \text{or } \check{G}(\neg e)(u) = 0 \text{ or } \check{G}_1(\neg e)(u) = 0, \\ & \text{or } \check{G}_1(\neg e)(u) = 0 \text{ and } \check{G}(\neg e)(u) = 0 \\ & \text{when } \check{F}(e)(u) < N-1, \\ & \text{or } \check{G}(\neg e)(u) = 0 \text{ and } \check{G}_1(\neg e)(u) = 0 \\ & \text{when } \check{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{A} \cap \neg\check{B} \end{cases} \quad (35)$$

On the other hand $(\check{F}^t, \check{G}^t, \check{A}, N)$ for all $e \in \check{A}$ and $u \in \mathcal{U}$ is

$$\check{F}^t(e)(u) = \begin{cases} N-1 & \text{if } \check{F}(e)(u) < N-1 \\ 0 & \text{if } \check{F}(e)(u) = N-1 \end{cases} \quad (36)$$

and

$$\check{G}^t(\neg e)(u) = \begin{cases} 0 & \text{if } \check{G}(\neg e)(u) > 0 \\ 0 & \text{if } \check{G}(\neg e)(u) = 0 \text{ when } \check{F}(e)(u) < N-1 \\ N-1 & \text{if } \check{G}(\neg e)(u) = 0 \end{cases} \quad (37)$$

Also $(\check{F}_1^t, \check{G}_1^t, \check{B}, N)$ is for all $e \in \check{B}$ and $u \in \mathcal{U}$.

$$\check{F}_1^t(e)(u) = \begin{cases} N-1 & \text{if } \check{F}_1(e)(u) < N-1 \\ 0 & \text{if } \check{F}_1(e)(u) = N-1 \end{cases} \quad (38)$$

and

$$\check{G}_1^t(\neg e)(u) = \begin{cases} 0 & \text{if } \check{G}_1(\neg e)(u) > 0 \\ 0 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ when } \check{F}_1(e)(u) < N-1 \\ N-1 & \text{if } \check{G}_1(\neg e)(u) = 0 \end{cases} \quad (39)$$

Let $(\check{F}^t, \check{G}^t, \check{A}, N) \cap_{\mathbb{E}} (\check{F}_1^t, \check{G}_1^t, \check{B}, N) = (\check{K}, \check{L}, \check{A} \cup \check{B}, N)$ where for all $e \in \check{A} \cup \check{B}$ and $u \in \mathcal{U}$, we have

$$\check{K}(e)(u) = \begin{cases} \check{F}^t(e)(u) & \text{if } e \in \check{A} - \check{B} \\ \check{F}_1^t(e)(u) & \text{if } e \in \check{B} - \check{A} \\ \min(\check{F}^t(e)(u), \check{F}_1^t(e)(u)) & \text{if } e \in \check{A} \cap \check{B} \end{cases} \quad (40)$$

and

$$\check{L}(\neg e)(u) = \begin{cases} \check{G}_1^t(\neg e)(u) & \text{if } \neg e \in \neg\check{A} - \neg\check{B} \\ \check{G}_1^t(\neg e)(u) & \text{if } \neg e \in \neg\check{B} - \neg\check{A} \\ \max(\check{G}^t(\neg e)(u), \check{G}_1^t(\neg e)(u)) & \text{if } \neg e \in \neg\check{A} \cap \neg\check{B} \end{cases} \quad (41)$$

Then

$$\check{K}(e)(u) = \begin{cases} N-1 & \text{if } \check{F}(e)(u) < N-1 \text{ and } e \in \check{A} - \check{B} \\ 0 & \text{if } \check{F}(e)(u) = N-1 \text{ and } e \in \check{A} - \check{B} \\ N-1 & \text{if } \check{F}_1(e)(u) < N-1 \text{ and } e \in \check{B} - \check{A} \\ 0 & \text{if } \check{F}_1(e)(u) = N-1 \text{ and } e \in \check{B} - \check{A} \\ N-1 & \text{if } \min(\check{F}^t(e)(u), \check{F}_1^t(e)(u)) = N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \\ 0 & \text{if } \min(\check{F}^t(e)(u), \check{F}_1^t(e)(u)) = 0 \\ & \text{and } e \in \check{A} \cap \check{B} \end{cases} \quad (42)$$

and

$$\check{L}(\neg e)(u) = \begin{cases} 0 & \text{if } \check{G}(\neg e)(u) > 0 \text{ and } \neg e \in \neg\check{A} - \neg\check{B} \\ 0 & \text{if } \check{G}(\neg e)(u) = 0 \text{ with } \check{F}(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{A} - \neg\check{B} \\ N-1 & \text{if } \check{G}(\neg e)(u) = 0 \text{ and } \neg e \in \neg\check{A} - \neg\check{B} \\ 0 & \text{if } \check{G}_1(\neg e)(u) > 0 \text{ and } \neg e \in \neg\check{B} - \neg\check{A} \\ 0 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ with } \check{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{B} - \neg\check{A} \\ N-1 & \text{if } \check{G}_1(\neg e)(u) = 0 \text{ and } \neg e \in \neg\check{B} - \neg\check{A} \\ 0 & \text{if } \max(\check{G}^t(\neg e)(u), \check{G}_1^t(\neg e)(u)) = 0, \\ & \text{and } e \in \neg\check{A} \cap \neg\check{B} \\ N-1 & \text{if } \max(\check{G}^t(\neg e)(u), \check{G}_1^t(\neg e)(u)) = N-1 \\ & \text{and } \neg e \in \neg\check{A} \cap \neg\check{B} \end{cases} \quad (43)$$

That is

$$\tilde{K}(e)(u) = \begin{cases} N-1 & \text{if } \tilde{F}(e)(u) < N-1 \text{ and } e \in \check{A} - \check{B} \\ 0 & \text{if } \tilde{F}(e)(u) = N-1 \text{ and } e \in \check{A} - \check{B} \\ N-1 & \text{if } \tilde{F}_1(e)(u) < N-1 \text{ and } e \in \check{B} - \check{A} \\ 0 & \text{if } \tilde{F}_1(e)(u) = N-1 \text{ and } e \in \check{B} - \check{A} \\ N-1 & \text{if } \tilde{F}(e)(u) < N-1 \\ & \text{and } \tilde{F}_1(e)(u) < N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \\ 0 & \text{if } \tilde{F}(e)(u) = N-1 \\ & \text{and } \tilde{F}_1(e)(u) = N-1, \\ & \text{or } \tilde{F}(e)(u) < N-1 \\ & \text{and } \tilde{F}_1(e)(u) = N-1, \\ & \text{or } \tilde{F}(e)(u) = N-1 \\ & \text{and } \tilde{F}_1(e)(u) < N-1 \\ & \text{and } e \in \check{A} \cap \check{B} \end{cases} \quad (44)$$

and

$$\tilde{L}(\neg e)(u) = \begin{cases} 0 & \text{if } \tilde{G}(\neg e)(u) > 0 \text{ and } \neg e \in \neg\check{A} - \neg\check{B} \\ 0 & \text{if } \tilde{G}(\neg e)(u) = 0 \text{ with} \\ & \tilde{F}(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{A} - \neg\check{B} \\ N-1 & \text{if } \tilde{G}(\neg e)(u) = 0 \text{ and } \neg e \in \neg\check{A} - \neg\check{B} \\ 0 & \text{if } \tilde{G}_1(\neg e)(u) > 0 \text{ and } \neg e \in \neg\check{B} - \neg\check{A} \\ 0 & \text{if } \tilde{G}_1(\neg e)(u) = 0 \\ & \text{with } \tilde{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{B} - \neg\check{A} \\ N-1 & \text{if } \tilde{G}_1(\neg e)(u) = 0 \text{ and } \neg e \in \neg\check{B} - \neg\check{A} \\ 0 & \text{if } \tilde{G}(\neg e)(u) > 0 \text{ and } \tilde{G}_1(\neg e)(u) > 0, \\ & \text{and } \neg e \in \neg\check{A} \cap \neg\check{B} \\ 0 & \text{if } \tilde{G}(\neg e)(u) = 0 \text{ when} \\ & \tilde{F}(e)(u) < N-1 \text{ and } \tilde{G}_1(\neg e)(u) > 0, \\ & \text{or } \tilde{G}(\neg e)(u) > 0 \text{ and } \tilde{G}_1(\neg e)(u) = 0 \\ & \text{when } \tilde{F}_1(e)(u) < N-1, \text{ or} \\ & \tilde{G}(\neg e)(u) = 0 \\ & \text{when } \tilde{F}(e)(u) < N-1 \\ & \text{and } \tilde{G}_1(\neg e)(u) = 0 \text{ when} \\ & \tilde{F}_1(e)(u) < N-1 \text{ and} \\ & \neg e \in \neg\check{A} \cap \neg\check{B} \\ N-1 & \text{if } \tilde{G}(\neg e)(u) = 0 \text{ and } \tilde{G}_1(\neg e)(u) = 0, \\ & \text{or } \tilde{G}(\neg e)(u) = 0 \text{ or } \tilde{G}_1(\neg e)(u) = 0, \\ & \text{or } \tilde{G}_1(\neg e)(u) = 0 \text{ and } \tilde{G}(\neg e)(u) = 0 \\ & \text{when } \tilde{F}(e)(u) < N-1, \\ & \text{or } \tilde{G}(\neg e)(u) = 0 \text{ and } \tilde{G}_1(\neg e)(u) = 0 \\ & \text{when } \tilde{F}_1(e)(u) < N-1 \\ & \text{and } \neg e \in \neg\check{A} \cap \neg\check{B} \end{cases} \quad (45)$$

This implies that $[(\tilde{F}, \tilde{G}, \check{A}, N) \cup_{\mathbb{E}} (\tilde{F}_1, \tilde{G}_1, \check{B}, N)]^t = (\tilde{F}^t, \tilde{G}^t, \check{A}, N) \cap_{\mathbb{E}} (\tilde{F}_1^t, \tilde{G}_1^t, \check{B}, N)$.

Similarly we can prove the other parts. $\square$

We denote the set of all N -bipolar soft sets over $\mathcal{U}$ by $NBSS(\mathcal{U})_{\check{E}}$ with attribute set a subset of $\check{E}$ and for all finite values N , that is all $N_1 \leq N$. The sub collection of $NBSS(\mathcal{U})_{\check{E}}$ consisting of all N -bipolar soft sets over a common universe $\mathcal{U}$ with the set of attribute is equal to $\check{E}$, is denoted by $NBSS(\mathcal{U})^{\check{E}}$. The sub collection of $NBSS(\mathcal{U})_{\check{E}}$ with fixed value of $N = N_0$ is denoted by $N_0BSS(\mathcal{U})_{\check{E}}$. We denote by $N_0BSS(\mathcal{U})^{\check{E}}$, the sub collection of $NBSS(\mathcal{U})^{\check{E}}$ with a fixed $N = N_0$, that is it consist of all N -bipolar soft sets with the set of attributes equal to $\check{E}$ and $N = N_0$.

We denote the sub collection of $NBSS(\mathcal{U})_{\check{E}}$ with a set of attributes subsets of $\check{A}$ by $NBSS(\mathcal{U})_{\check{A}}$ and sub collection of $NBSS(\mathcal{U})_{\check{A}}$ with $N = N_0$ by $N_0BSS(\mathcal{U})_{\check{A}}$. The sub collection of $NBSS(\mathcal{U})_{\check{E}}$ with a fixed set of attributes equal to $\check{A}$ is denoted by $NBSS(\mathcal{U})^{\check{A}}$ and sub collection of $NBSS(\mathcal{U})^{\check{A}}$ with $N = N_0$ by $N_0BSS(\mathcal{U})^{\check{A}}$.

With these notations we have the following result.

Proposition 4.1. For $\gamma \in \{\cup_{\mathbb{E}}, \cap_{\mathbb{E}}, \cup_{\mathbb{R}}, \cap_{\mathbb{R}}\}$. The following are

- (i) $(NBSS(\mathcal{U})_{\check{E}}, \gamma)$ is a commutative semigroup.
- (ii) $(NBSS(\mathcal{U})^{\check{E}}, \gamma)$ is a commutative semigroup.
- (iii) $(NBSS(\mathcal{U})_{\check{A}}, \gamma)$ is a commutative semigroup.
- (iv) $(NBSS(\mathcal{U})^{\check{A}}, \gamma)$ is a commutative semigroup.
- (v) $(N_0BSS(\mathcal{U})_{\check{E}}, \gamma)$ is a commutative monoid.
- (vi) $(N_0BSS(\mathcal{U})^{\check{E}}, \gamma)$ is a commutative monoid.
- (vii) $(N_0BSS(\mathcal{U})_{\check{A}}, \gamma)$ is a commutative monoid.
- (viii) $(N_0BSS(\mathcal{U})^{\check{A}}, \gamma)$ is a commutative monoid.

Proof. It follows directly from Lemma 4.1 and 4.2. $\square$

All of the $N_0BSS(\mathcal{U})_{\check{E}}$, $N_0BSS(\mathcal{U})^{\check{E}}$, $N_0BSS(\mathcal{U})_{\check{A}}$ and $N_0BSS(\mathcal{U})^{\check{A}}$ are partially ordered by inclusion.

Proposition 4.2. $(N_0BSS(\mathcal{U})_{\check{E}}, \cap_{\mathbb{E}}, \cup_{\mathbb{R}})$ and $(N_0BSS(\mathcal{U})_{\check{E}}, \cup_{\mathbb{E}}, \cap_{\mathbb{R}})$ are distributive lattices. $(N_0BSS(\mathcal{U})_{\check{E}}, \cup_{\mathbb{R}}, \cap_{\mathbb{E}})$ and $(N_0BSS(\mathcal{U})_{\check{E}}, \cap_{\mathbb{R}}, \cup_{\mathbb{E}})$ are their duals respectively.

Proof. It directly follows from Lemma 4.1, 4.3 and Theorem 4.1. $\square$

Proposition 4.3. $(N_0BSS(\mathcal{U})_{\check{E}}, \cap_{\mathbb{R}}, \cup_{\mathbb{E}})$ is a bounded distributive lattice with greatest element $(\check{\mathcal{U}}, \check{\phi}, \check{E}, N_0)$ and least element $(\check{\phi}, \check{\mathcal{U}}, \check{\phi}, N_0)$, while $((N_0BSS(\mathcal{U})_{\check{E}}, \cup_{\mathbb{E}}, \cap_{\mathbb{R}}), (\check{\mathcal{U}}, \check{\phi}, \check{E}, N_0), (\check{\phi}, \check{\mathcal{U}}, \check{\phi}, N_0))$ is its dual.

Proof. The proof follows from Lemma 4.1, 4.2, 4.3 and Theorem 4.1. $\square$

Proposition 4.4. $(N_0BSS(\mathcal{U})_{\check{E}, \cap_{\mathbb{R}}, \cup_{\mathbb{E}}})$ is a bounded distributive lattice with greatest element $(\check{\mathcal{U}}, \check{\phi}, \check{E}, N_0)$ and least element $(\check{\phi}, \check{\mathcal{U}}, \check{E}, N_0)$, while $((N_0BSS(\mathcal{U})_{\check{E}, \cup_{\mathbb{R}}, \cap_{\mathbb{R}}}), (\check{\mathcal{U}}, \check{\phi}, \check{E}, N_0), (\check{\phi}, \check{\mathcal{U}}, \check{E}, N_0))$ is its dual.

Proof. The proof is similar to the proof of Proposition 4.3. $\square$

Proposition 4.5. $(N_0BSS(\mathcal{U})_{\check{A}, \cap_{\mathbb{E}}, \cup_{\mathbb{R}}})$ and $(N_0BSS(\mathcal{U})_{\check{A}, \cup_{\mathbb{E}}, \cap_{\mathbb{R}}})$ are distributive lattices. $(N_0BSS(\mathcal{U})_{\check{A}, \cup_{\mathbb{R}}, \cap_{\mathbb{E}}})$ and $(N_0BSS(\mathcal{U})_{\check{A}, \cap_{\mathbb{R}}, \cup_{\mathbb{E}}})$ are their duals respectively.

Proof. The proof is similar to the proof of Proposition 4.2. $\square$

Proposition 4.6. $(N_0BSS(\mathcal{U})_{\check{A}, \cap_{\mathbb{R}}, \cup_{\mathbb{E}}})$ is a bounded distributive lattice with greatest element $(\check{\mathcal{U}}, \check{\phi}, \check{A}, N_0)$ and least element $(\check{\phi}, \check{\mathcal{U}}, \check{A}, N_0)$, while $((N_0BSS(\mathcal{U})_{\check{A}, \cup_{\mathbb{R}}, \cap_{\mathbb{E}}}), (\check{\mathcal{U}}, \check{\phi}, \check{A}, N_0), (\check{\phi}, \check{\mathcal{U}}, \check{A}, N_0))$ is its dual.

Proof. Straightforward. $\square$

Proposition 4.7. $(N_0BSS(\mathcal{U})_{\check{A}, \cap_{\mathbb{R}}, \cup_{\mathbb{E}}}) = (N_0BSS(\mathcal{U})_{\check{A}, \cap_{\mathbb{E}}, \cup_{\mathbb{R}}})$ is a bounded distributive lattice with greatest element $(\check{\mathcal{U}}, \check{\phi}, \check{A}, N_0)$ and least element $(\check{\phi}, \check{\mathcal{U}}, \check{A}, N_0)$, while $((N_0BSS(\mathcal{U})_{\check{A}, \cup_{\mathbb{R}}, \cap_{\mathbb{E}}}), (\check{\mathcal{U}}, \check{\phi}, \check{A}, N_0), (\check{\phi}, \check{\mathcal{U}}, \check{A}, N_0))$ is its dual.

Proof. The proof is straightforward. $\square$

Proposition 4.8. $(N_0BSS(\mathcal{U})_{\check{A}, \cap_{\mathbb{R}}, \cup_{\mathbb{R}}, (\check{\phi}, \check{\mathcal{U}}, \check{A}, N_0), (\check{\mathcal{U}}, \check{\phi}, \check{A}, N_0)})$ is bounded distributive lattice.

Proof. The proof is straightforward. $\square$

Proposition 4.9. $N_0BSS(\mathcal{U})_{\check{E}}, N_0BSS(\mathcal{U})_{\check{A}}, N_0BSS(\mathcal{U})_{\check{E}}$ and $N_0BSS(\mathcal{U})_{\check{A}}$ satisfy De-Morgan's law with respect to top compliment and bottom compliment.

Proof. It follows directly from Lemma 4.5. $\square$

5 Decision making procedure for N -Bipolar soft set

Bipolar soft sets have been used in many applications. Here we devise some decision making algorithms for N -Bipolar soft sets. Algorithm 1 gives the feasible solution depending only on grades assigned to the objects of universe with respect to each attribute. Algorithm 2 can be used when some of the attributes are given weightage by the decision maker upon which one wants to decide. In other words, the decision is made depending on the weightage of the attributes. When the decision maker wants to consider those objects as solution that have grades above a certain limit (or number from the assessment space) specified by him, then by using Algorithm 3 it will be easy for him to decide. The three algorithms give some sort of different results when applied to the same example. As these three algorithms also consider the negative aspects of attributes, so they give beneficial result encountering the presence of undesirable attributes. This shows

the extent to which these methods are adaptable to any situation is by the need of the practitioner.

Algorithm 1

- 1: Input $\mathcal{U} = \{u_1, u_2, \dots, u_q\}$ and $\check{A} = \{e_1, e_2, \dots, e_s\}$.
- 2: Input N -bipolar soft set $(\check{F}, \check{G}, \check{A}, N)$ with $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$, $N \in \{2, 3, \dots\}$ so that for all $u_i \in \mathcal{U}$, $e_j \in \check{A} \exists r_{i,j_{e_j}} \in \check{R}$ and also $\neg e_j \in \neg \check{A} \exists r_{i,j_{\neg e_j}} \in \check{R}$ for each u_i .
- 3: Compute $\lambda_i = \sum_{j=1}^s r_{i,j_{e_j}}$ and $\lambda'_i = \sum_{j=1}^s r_{i,j_{\neg e_j}}$, for each u_i .
- 4: Then evaluate $\lambda_i - \lambda'_i$ for each of the corresponding cell.
- 5: Find the index i such that $\nu_i = \max(\lambda_i - \lambda'_i)$.
- 6: The result is any one of u_i which satisfies the step 5.

Algorithm 1 is applied to Example 3.1 in Tables 35 and 36.

Table 35: 7-soft set $(\check{F}, \check{A}, 7)$

$(\check{F}, \check{A}, 7)$	e_1	e_2	e_3	e_4	e_5	e_6	λ_i
u_1	3	3	4	5	4	2	21
u_2	4	0	3	6	6	3	22
u_3	1	1	2	1	3	1	9
u_4	6	2	1	2	1	0	12
u_5	1	3	4	3	1	1	13

Table 36: 7-soft set $(\check{G}, \neg \check{A}, 7)$

$(\check{G}, \neg \check{A}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$	λ'_i
u_1	3	2	2	1	2	4	14
u_2	2	6	3	0	0	3	14
u_3	3	5	4	5	3	5	25
u_4	0	0	5	4	5	6	20
u_5	3	1	2	3	5	5	19

From Tables 35 and 36 we can calculate the following values;

$$\lambda_1 - \lambda'_1 = 7, \lambda_2 - \lambda'_2 = 8, \lambda_3 - \lambda'_3 = -16, \lambda_4 - \lambda'_4 = -8, \text{ and } \lambda_5 - \lambda'_5 = -6.$$

This shows that by Step 5 we get $\nu_i = u_2$

Algorithm 2 For weighted choice values.

- 1: Input $\mathcal{U} = \{u_1, u_2, \dots, u_q\}$ and $\check{A} = \{e_1, e_2, \dots, e_s\}$ and a weight $\pi_j \in [0, 1]$ for each $j \in \{1, 2, \dots, s\}$.
 - 2: Input N -bipolar soft set $(\check{F}, \check{G}, \check{A}, N)$ with $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$, $N \in \{2, 3, \dots\}$ so that for all $u_i \in \mathcal{U}$, $e_j \in \check{A}$, $\exists r_{i,j_{e_j}} \in \check{R}$ and also $\neg e_j \in \neg \check{A} \exists r_{i,j_{\neg e_j}} \in \check{R}$ for each u_i .
 - 3: Compute $\lambda_i^w = \sum_{j=1}^s r_{i,j_{e_j}} \pi_j$ and $\lambda_i'^w = \sum_{j=1}^s r_{i,j_{\neg e_j}} (1 - \pi_j)$, for each u_i
 - 4: Then evaluate $\lambda_i^w - \lambda_i'^w$ for each of the corresponding cell.
 - 5: Find the index i such that $\nu_i^w = \max(\lambda_i^w - \lambda_i'^w)$ for each $i \in \{1, 2, \dots, q\}$
 - 6: The result is any one of u_i which satisfies the step 5.
-

We applied Algorithm 2 to Example 3.1 given in Tables 37 and 38. From Table 37 and 38 we can calculate the following values for the weights $\pi_1 = 0.6$, $\pi_2 = 0.1$, $\pi_3 = 0.4$, $\pi_4 = 0.6$, $\pi_5 = 0.5$, $\pi_6 = 0.1$ and $1 - \pi_1 = 0.4$, $1 - \pi_2 = 0.9$, $1 - \pi_3 = 0.6$, $1 - \pi_4 = 0.4$, $1 - \pi_5 = 0.5$, $1 - \pi_6 = 0.9$; $\lambda_1^w - \lambda_1'^w = -0.3$, $\lambda_2^w - \lambda_2'^w = -0.2$, $\lambda_3^w - \lambda_3'^w = -12.4$, $\lambda_4^w - \lambda_4'^w = -6.6$, and $\lambda_5^w - \lambda_5'^w = -6.6$. This shows that by Step 5 we get $\nu_i^w = u_2$.

Algorithm 3 For T -choice value.

- 1: Input $\mathcal{U} = \{u_1, u_2, \dots, u_q\}$ and $\check{A} = \{e_1, e_2, \dots, e_s\}$.
 - 2: Input N -bipolar soft set $(\check{F}, \check{G}, \check{A}, N)$ with $\check{R} = \{0, 1, 2, 3, \dots, N-1\}$, $N \in \{2, 3, \dots\}$ so that for all $u_i \in \mathcal{U}$, $e_j \in \check{A} \exists r_{i,j_{e_j}} \in \check{R}$ and also $\neg e_j \in \neg \check{A} \exists r_{i,j_{\neg e_j}} \in \check{R}$ for each u_i .
 - 3: Compute
$$r_{i,j_e}^T = \begin{cases} 1 & \text{if } r_{i,j_{e_j}} \geq T, \\ 0 & \text{otherwise.} \end{cases} \quad (46)$$

$$r_{i,j_{\neg e}}^T = \begin{cases} 1 & \text{if } r_{i,j_{\neg e_j}} \geq N - T, \\ 0 & \text{otherwise.} \end{cases} \quad (47)$$
 - 4: Compute $\lambda_i^T = \sum_{j=1}^s r_{i,j_{e_j}}^T$ and $\lambda_i^{T'} = \sum_{j=1}^s r_{i,j_{\neg e_j}}^T$ for each u_i
 - 5: Then evaluate $\lambda_i^T - \lambda_i^{T'}$ for each of the corresponding cell.
 - 6: Find the index i such that $\nu_i^T = \max(\lambda_i^T - \lambda_i^{T'})$.
 - 7: The result is any one of u_i which satisfies the step 6.
-

We applied Algorithm 3 to Example 3.1 given in Tables 39 and 40.

Table 39: extended 2-Bipolar $T = 3$ choice values

$(\check{F}^T, \check{A})$	e_1	e_2	e_3	e_4	e_5	e_6	λ_i^T
u_1	1	1	1	1	1	0	5
u_2	1	0	1	1	1	1	5
u_3	0	0	0	0	1	0	1
u_4	1	0	0	0	0	0	1
u_5	0	1	1	1	0	0	3

Table 40

$(\check{G}^T, \neg \check{A})$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$	$\lambda_i^{T'}$
u_1	0	0	0	0	0	1	1
u_2	0	1	0	0	0	0	1
u_3	0	1	1	1	0	1	4
u_4	0	0	1	1	1	1	4
u_5	0	0	0	0	1	1	2

From Table 39 and 40 we can calculate the following values; $\lambda_1^T - \lambda_1^{T'} = 4$, $\lambda_2^T - \lambda_2^{T'} = 4$, $\lambda_3^T - \lambda_3^{T'} = -3$, $\lambda_4^T - \lambda_4^{T'} = -3$, and $\lambda_5^T - \lambda_5^{T'} = 1$. This shows that by Step 5 we get $\nu_i^T = u_2$ and $\nu_i^{T'} = u_1$.

6 Application to Conflict Analysis

N -bipolar soft sets can be applied in real life problems. One of its most distinguished application is in conflict analysis. Conflict arises when there is a disagreement on finding a feasible solution. We use the Algorithm of Ali et al. [4] and applied on various N -bipolar soft sets. We found some examples where we attained the actions that seem infeasible. So to study the degree of disagreement we use the concept of N -bipolar soft sets which also studies the negative aspect of attributes. The problem under our concern is of multi-attribute and involved multi-decision makers, and we are required to find the most feasible action agreed by all decision makers having the lowest degree of disagreement. Even if the algorithm [4] gives just one feasible solution we would also be able to calculate the least disagreeable action, which helps a lot to decide.

6.1 Ali et al. [4] conflict model

Qian et al. [33],[34] introduced two different optimistic and pessimistic multi-granulation rough sets. Ali et al. [4] developed the dominance based optimistic and pessimistic multi-granulation rough set. Ali et al. [4] considered the decision as an information system $\mathcal{S} = (A, C, D, E)$ is the multi-decision and multi-attribute information system, where A is finite set of actions $a_i, i = 1, 2, 3, \dots, n$, C is a finite set of conditional attributes $C_j, j = 1, 2, 3, \dots, |C|$, D is a finite set of decisional attributes $D_k, k = 1, 2, 3, \dots, |D|$ and E is a finite set of the domain for

Table 37: extended 7-Bipolar weighted choice values

$(\check{F}, \check{A}, 7)$	e_1	e_2	e_3	e_4	e_5	e_6	λ_i^w
	$\pi_1 = 0.6$	$\pi_2 = 0.1$	$\pi_3 = 0.4$	$\pi_4 = 0.6$	$\pi_5 = 0.5$	$\pi_6 = 0.1$	
u_1	1.8	0.3	1.6	3.0	2.0	0.2	8.9
u_2	2.4	0	1.2	3.6	3.0	0.3	10.5
u_3	0.6	0.1	0.8	0.6	1.5	0.1	3.7
u_4	3.6	0.2	0.4	1.2	0.5	0	5.9
u_5	0.6	0.3	1.6	1.8	0.5	0.1	4.9

Table 38

$(\check{G}, \neg\check{A}, 7)$	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$	λ_i^w
	$1 - \pi_1$	$1 - \pi_2$	$1 - \pi_3$	$1 - \pi_4$	$1 - \pi_5$	$1 - \pi_6$	
u_1	1.2	1.8	1.2	0.4	1.0	3.6	9.2
u_2	0.8	5.4	1.8	0	0	2.7	10.7
u_3	1.2	4.5	2.4	2.0	1.5	4.5	16.1
u_4	0	0	3.0	1.6	2.5	5.4	12.5
u_5	1.2	0.9	1.2	1.2	2.5	4.5	11.5

the information functions $f(a_i, C_j)$ and $g(a_i, D_k)$. The values of information functions $f(a_i, C_j)$ and $g(a_i, D_k)$ are integers.

Definition 6.1. [4] Let $\mathcal{S} = (A, C, D, E)$ be a multiple attribute and multi-decision information system and $X \subseteq A$. The dominance relation defined by conditional attribute C denoted by $\mathcal{P}_C$ and dominance relation defined by decision attribute D is denoted by $\mathcal{P}_D$. Let

$$[a_i]_{\mathcal{P}_C}^+ = \{a_j \in A : f(a_j, C) \geq f(a_i, C)\}$$

$$[a_i]_{\mathcal{P}_C}^- = \{a_j \in A : f(a_j, C) \leq f(a_i, C)\}$$

represents the C -dominating and C -dominated set with respect to a_i over the conditional attribute C of $\mathcal{S}$. Similarly

$$[a_i]_{\mathcal{P}_D}^+ = \{a_j \in A : g(a_j, D) \geq g(a_i, D)\}$$

$$[a_i]_{\mathcal{P}_D}^- = \{a_j \in A : g(a_j, D) \leq g(a_i, D)\}$$

represents the D -dominating and D -dominated set with respect to a_i over the decision attribute D of $\mathcal{S}$.

6.1.1 Dominance optimistic MGRS [4]

The dominance optimistic multi-granulation lower and upper approximation of X are defined by

$$\underline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O = \{a \in A : [a]_{\mathcal{P}_C}^+ \subseteq X \vee [a]_{\mathcal{P}_D}^+ \subseteq X\}$$

$$\overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O = \{a \in A : [a]_{\mathcal{P}_C}^- \cap X \neq \emptyset \wedge [a]_{\mathcal{P}_D}^- \cap X \neq \emptyset\}.$$

Then $((\underline{(X)})_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O, \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O)$ are the dominating optimistic

multi-granulation rough set if $\underline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O \neq \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O$. The optimistic multi-granulation boundary region of X is

$$BN^O(X) = (\overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O - \underline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O).$$

Similarly

$$\underline{(X)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O = \{a \in A : [a]_{\mathcal{P}_C}^- \subseteq X \vee [a]_{\mathcal{P}_D}^- \subseteq X\}$$

$$\overline{(X)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O = \{a \in A : [a]_{\mathcal{P}_C}^+ \cap X \neq \emptyset \wedge [a]_{\mathcal{P}_D}^+ \cap X \neq \emptyset\}.$$

Then $((\underline{(X)})_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O, \overline{(X)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O)$ is called the dominated optimistic multi-granulation rough set if $\underline{(X)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O \neq \overline{(X)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O$.

Ali et al. [4] Proposed algorithm

Input: Information system $\mathcal{S} = (A, C, D, E)$.

Step 1 Construct $\underline{(X_k)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O$ and $\underline{(X_k)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O$, where $X_k = \{a \in A : g(a, D_k) \neq 0\}$.

Step 2: If $\underline{(X_k)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O = \underline{(X_k)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O$

then $\delta_k = \underline{(X_k)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O = \underline{(X_k)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O$

otherwise $\delta_k = \underline{(X_k)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^O - \underline{(X_k)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^O$.

Step 3 ; If $\delta = \bigcap_{k=1}^{|D|} \delta_k \neq \emptyset$;

go to output.

otherwise $|D| = |D| - 1$; go back to step 3.

6.1.2 Dominance pessimistic MGRS [4]

The dominance pessimistic multi-granulation lower and upper approximation of X are defined by

$$\underline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P = \{a \in A : [a]_{\mathcal{P}_C}^+ \subseteq X \wedge [a]_{\mathcal{P}_D}^+ \subseteq X\}$$

$$\overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P = \{a \in A : [a]_{\mathcal{P}_C}^- \cap X \neq \emptyset \vee [a]_{\mathcal{P}_D}^+ \cap X \neq \emptyset\}.$$

Then $((\underline{(X)})_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P, \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P)$ are dominating pessimistic multi-

granulation rough set if $\underline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P \neq \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P$. The pessimistic multi-granulation boundary region of X is

$$BN^P(X) = \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P - \underline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P.$$

Similarly

$$\underline{(X)}_{\mathcal{P}_C^- + \mathcal{P}_D^-}^P = \{a \in A : [a]_{\mathcal{P}_C}^- \subseteq X \wedge [a]_{\mathcal{P}_D}^- \subseteq X\}$$

$$\overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^-}^P = \{a \in A : [a]_{\mathcal{P}_C}^+ \cap X \neq \emptyset \vee [a]_{\mathcal{P}_D}^+ \cap X \neq \emptyset\}.$$

Then $((X)_{\mathcal{P}_C^+ + \mathcal{P}_D^-}^P, \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^-}^P)$ is called the dominated pessimistic multi-granulation rough set if $(X)_{\mathcal{P}_C^+ + \mathcal{P}_D^-}^P \neq \overline{(X)}_{\mathcal{P}_C^+ + \mathcal{P}_D^-}^P$.

Ali et al. [4] Proposed algorithm

Input: Information system $S = (A, C, D, E)$.

Step 1 Construct $(X_k)_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P$ and $(X_k)_{\mathcal{P}_C^- + \mathcal{P}_D^-}^P$, where $X_k = \{a \in A : g(a, D_k) \neq 0\}$.

Step 2: If $(X_k)_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P = (X_k)_{\mathcal{P}_C^- + \mathcal{P}_D^-}^P$ then $\delta_k = (X_k)_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P = (X_k)_{\mathcal{P}_C^- + \mathcal{P}_D^-}^P$ otherwise $\delta_k = (X_k)_{\mathcal{P}_C^+ + \mathcal{P}_D^+}^P - (X_k)_{\mathcal{P}_C^- + \mathcal{P}_D^-}^P$.

Step 3 ; If $\delta = \bigcap_{k=1}^{|D|} \delta_k \neq \emptyset$; go to output otherwise $|D| = |D| - 1$; go back to step 3.

6.2 Problem statement

Our problem is that when the Algorithm of Ali et al. [4] is applied to positive set of N -bipolar soft set it also gives those objects of universe as solution which were not certainly feasible. So we introduce a slight different system by the involvement of least disagreeable actions. So despite of the system taken in [4], we divide it into two N -Bipolar soft sets for the conditional attributes and decisional attributes respectively. Consider the problem as $S = \{\mathcal{U}, (\tilde{F}, \tilde{G}, \tilde{C}, N_1), (\tilde{\gamma}, \tilde{\eta}, \tilde{D}, N_2)\}$ over a common universe $\mathcal{U}$, where $\mathcal{U}$ contains all the possible actions. $(\tilde{F}, \tilde{G}, \tilde{C}, N_1)$ is the N -Bipolar soft sets for the conditional attributes $\tilde{C}$ over $\mathcal{U}$ and $(\tilde{\gamma}, \tilde{\eta}, \tilde{D}, N_2)$ is the N -Bipolar soft set over $\mathcal{U}$ for the decisional attributes. We call S as multi-attribute and multi decisional information system.

6.3 Multi-granulation N -bipolar rough sets

We use the dominance relations used by Ali et al. [4] and mould it into our work as follows; $u_i \tilde{\succ}_{\tilde{Q}} u_j$ if $\tilde{\alpha}(\tilde{Q})(u_j) \geq \tilde{\alpha}(\tilde{Q})(u_i)$ means that $u_i \tilde{\succ}_{\tilde{Q}_m} u_j$ if $\tilde{\alpha}(\tilde{Q}_m)(u_j) \geq \tilde{\alpha}(\tilde{Q}_m)(u_i)$ for all $\tilde{Q}_i \in \tilde{Q}$ that is u_i dominates u_j with respect to all attributes $\tilde{Q}_m$ in $\tilde{Q}$, where $\tilde{\alpha} \in \{\tilde{F}, \tilde{G}, \tilde{\gamma}, \tilde{\eta}\}$ and $\tilde{Q} \in \{\tilde{C}, \neg\tilde{C}, \tilde{D}, \neg\tilde{D}\}$.

The dominance classes to be used in N -bipolar soft sets is denoted and defined by

$$\hat{P}_{\tilde{C}}^{\tilde{F}+}(u_i) = \{u_j \in \mathcal{U} : \tilde{F}(\tilde{C})(u_j) \geq \tilde{F}(\tilde{C})(u_i)\}$$

$$\hat{P}_{\tilde{C}}^{\tilde{F}-}(u_i) = \{u_j \in \mathcal{U} : \tilde{F}(\tilde{C})(u_j) \leq \tilde{F}(\tilde{C})(u_i)\}$$

are $\tilde{C}$ -dominating and $\tilde{C}$ -dominated sets with respect to u_i over the conditional attribute $\tilde{C}$

$$\hat{P}_{\neg\tilde{C}}^{\tilde{G}+}(u_i) = \{u_j \in \mathcal{U} : \tilde{G}(\neg\tilde{C})(u_j) \geq \tilde{G}(\neg\tilde{C})(u_i)\}$$

$$\hat{P}_{\neg\tilde{C}}^{\tilde{G}-}(u_i) = \{u_j \in \mathcal{U} : \tilde{G}(\neg\tilde{C})(u_j) \leq \tilde{G}(\neg\tilde{C})(u_i)\}$$

are $\neg\tilde{C}$ -dominating and $\neg\tilde{C}$ -dominated sets with respect to u_i over the conditional attribute $\neg\tilde{C}$

$$\hat{P}_{\tilde{D}}^{\tilde{\gamma}+}(u_i) = \{u_j \in \mathcal{U} : \tilde{\gamma}(\tilde{D})(u_j) \geq \tilde{\gamma}(\tilde{D})(u_i)\}$$

$$\hat{P}_{\tilde{D}}^{\tilde{\gamma}-}(u_i) = \{u_j \in \mathcal{U} : \tilde{\gamma}(\tilde{D})(u_j) \leq \tilde{\gamma}(\tilde{D})(u_i)\}$$

are $\tilde{D}$ -dominating and $\tilde{D}$ -dominated sets with respect to u_i over

the decisional attribute $\tilde{D}$

$$\hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}(u_i) = \{u_j \in \mathcal{U} : \tilde{\eta}(\neg\tilde{D})(u_j) \geq \tilde{\eta}(\neg\tilde{D})(u_i)\}$$

$$\hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}(u_i) = \{u_j \in \mathcal{U} : \tilde{\eta}(\neg\tilde{D})(u_j) \leq \tilde{\eta}(\neg\tilde{D})(u_i)\}$$

are $\neg\tilde{D}$ -dominating and $\neg\tilde{D}$ -dominated sets with respect to u_i over the decisional attribute $\neg\tilde{D}$.

The dominance optimistic multi-granulation lower and upper approximations are as follows.

$$\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O = \{u \in \mathcal{U} : \hat{P}_{\tilde{C}}^{\tilde{F}+}(u) \subseteq \chi_k \vee \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}(u) \subseteq \chi_k\}$$

$$\overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O = \{u \in \mathcal{U} : \hat{P}_{\tilde{C}}^{\tilde{F}+}(u) \cap \chi_k \neq \emptyset \wedge \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}(u) \cap \chi_k \neq \emptyset\}$$

$$\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O = \{u \in \mathcal{U} : \hat{P}_{\neg\tilde{C}}^{\tilde{G}+}(u) \subseteq \chi_k \vee \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}(u) \subseteq \chi_k\}$$

$$\overline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O = \{u \in \mathcal{U} : \hat{P}_{\neg\tilde{C}}^{\tilde{G}+}(u) \cap \chi_k \neq \emptyset \wedge \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}(u) \cap \chi_k \neq \emptyset\}.$$

Then the set $\{(\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O, \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O), (\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O, \overline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O)\}$ is called dominating optimistic multi-granulation N -Bipolar rough set if $\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O \neq \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O$

$$\text{and } \underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O \neq \overline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O.$$

Similarly

$$\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O = \{u \in \mathcal{U} : \hat{P}_{\tilde{C}}^{\tilde{F}-}(u) \subseteq \chi_k \vee \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}(u) \subseteq \chi_k\}$$

$$\overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O = \{u \in \mathcal{U} : \hat{P}_{\tilde{C}}^{\tilde{F}-}(u) \cap \chi_k \neq \emptyset \wedge \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}(u) \cap \chi_k \neq \emptyset\}$$

$$\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}-} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}}^O = \{u \in \mathcal{U} : \hat{P}_{\neg\tilde{C}}^{\tilde{G}-}(u) \subseteq \chi_k \vee \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}(u) \subseteq \chi_k\}$$

$$\overline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}-} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}}^O = \{u \in \mathcal{U} : \hat{P}_{\neg\tilde{C}}^{\tilde{G}-}(u) \cap \chi_k \neq \emptyset \wedge \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}(u) \cap \chi_k \neq \emptyset\}$$

$$\{(\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O, \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O), (\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}-} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}}^O, \overline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}-} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}}^O)\}$$

is the dominated optimistic multi-granulation N -Bipolar rough set if $\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O \neq \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O$ and $\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}-} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}}^O \neq \overline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}-} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}-}}^O$, where $\chi_k = \{u \in \mathcal{U} : \tilde{\gamma}(\tilde{D}_k)(u) \neq 0\}$ and $\chi_k^- = \{u \in \mathcal{U} : \tilde{\eta}(\neg\tilde{D}_k)(u) \neq 0\}$.

6.4 Proposed algorithm

Steps for the positive sets are identical to the algorithm of Ali et al. [4] for the positive set under our proposed notations of dominance relation. **Input** Information system $S = \{\mathcal{U}, (\tilde{F}, \tilde{G}, \tilde{C}, N_1), (\tilde{\gamma}, \tilde{\eta}, \tilde{D}, N_2)\}$.

Step 1(a): Construct $\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O$ and $\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O$, where $\chi_k = \{u \in \mathcal{U} : \tilde{\gamma}(\tilde{D}_k)(u) \neq 0\}$.

Step 2(a): If $\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O = \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O$ then $\Omega_k = \underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O = \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O$

otherwise $\Omega_k = \underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O - \overline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}+} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}+}}^O$.

Step 3(a): If $\Omega = \bigcap_{k=1}^{|D|} \Omega_k \neq \emptyset$ go to the output, otherwise $|\tilde{D}| = |\tilde{D}| - 1$ go to the step 3(a)

For the not set we have the following algorithm.

Step 1(b): Construct $\underline{\chi}_{\hat{P}_{\neg\tilde{C}}^{\tilde{G}+} + \hat{P}_{\neg\tilde{D}}^{\tilde{\eta}+}}^O$ and $\underline{\chi}_{\hat{P}_{\tilde{C}}^{\tilde{F}-} + \hat{P}_{\tilde{D}}^{\tilde{\gamma}-}}^O$ where $\chi_k^- = \{u \in \mathcal{U} : \tilde{\eta}(\neg\tilde{D}_k)(u) \neq 0\}$.

Step 2(b): If $\chi_{k, \check{P}_{\check{C}}^+ + \check{P}_{\check{D}}^+}^{-O} = \chi_{k, \check{P}_{\check{C}}^- + \check{P}_{\check{D}}^-}^{-O}$ then $\Omega_k^- = \chi_{k, \check{P}_{\check{C}}^+ + \check{P}_{\check{D}}^+}^{-O} = \chi_{k, \check{P}_{\check{C}}^- + \check{P}_{\check{D}}^-}^{-O}$ otherwise $\Omega_k^- = \chi_{k, \check{P}_{\check{C}}^- + \check{P}_{\check{D}}^-}^{-O} - \chi_{k, \check{P}_{\check{C}}^+ + \check{P}_{\check{D}}^+}^{-O}$.

Step 3(b): If $\Omega^- = \bigcap_{k=1}^{|\check{D}|} \Omega_k^- \neq \phi$ go to the output otherwise $|\neg \check{D}| = |\check{D}| - 1$ go to the step 3(b).

Step 4 Evaluate $\Omega \cap \Omega^-$.

The intersection will give that action as a solution which is agreed by all decision makers and has least degree of disagreement.

The steps 1(a) to 3(a) give those feasible actions which are agreed by all decision makers. The output of the steps 1(b) to 3(b) gives those action which have least degree of disagreement. The benefit for attaining the least disagreed action is because sometimes output of steps 1(a) to 3(a) give many actions which seems infeasible so it help us to take those actions which are agreed by all the decision maker and has least disagreement as grading assigned to them. The algorithm is explained using Example 6.1 and the calculations are made in Tables 45-52.

Example 6.1. Let the board of directors of a company wants to invest in a prestigious company. They have a list of 7 companies from which they have to select suitable one. Let $\mathcal{U}$ be the universe of companies to be invested $\mathcal{U} = \{u_1, u_2, \dots, u_7\}$ and $\check{C} = \{\check{C}_1, \check{C}_2, \check{C}_3, \check{C}_4\} = \{\text{management, wide economic moat, stable earnings, efficient operations}\}$ be the conditional attributes by which they have to decide and $\neg \check{C} = \{\neg \check{C}_1, \neg \check{C}_2, \neg \check{C}_3, \neg \check{C}_4\} = \{\text{no management, narrow economic moat, unstable earnings, inefficient operations}\}$. $\check{D} = \{\check{D}_1, \check{D}_2, \check{D}_3, \check{D}_4\}$ and $\neg \check{D} = \{\neg \check{D}_1, \neg \check{D}_2, \neg \check{D}_3, \neg \check{D}_4\}$ are the decisional attributes which are based on the opinion of decision makers depending on the ranking of the conditional attributes.

Table 41: 4-soft set on conditional attributes

$(\check{F}, \check{C}, 4)$	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$
u_1	3	2	1	3
u_2	2	3	2	2
u_3	1	1	2	1
u_4	3	3	2	3
u_5	2	2	2	1
u_6	1	2	1	2
u_7	1	1	0	1

Table 42: 4-soft set on conditional attributes

$(\check{G}, \neg \check{C}, 4)$	$\neg \check{C}_1$	$\neg \check{C}_2$	$\neg \check{C}_3$	$\neg \check{C}_4$
u_1	0	1	2	0
u_2	1	0	1	1
u_3	2	2	1	2
u_4	0	0	1	0
u_5	1	1	1	2
u_6	2	1	2	1
u_7	2	2	3	2

Table 43: 5-soft set on decisional attributes

$(\check{\gamma}, \check{D}, 5)$	$\check{D}_1$	$\check{D}_2$	$\check{D}_3$	$\check{D}_4$
u_1	3	2	4	2
u_2	3	4	3	3
u_3	2	1	2	3
u_4	4	4	4	3
u_5	2	3	0	3
u_6	1	3	3	2
u_7	0	0	2	1

Table 44: 5-soft set on decisional attributes

$(\check{\eta}, \neg \check{D}, 5)$	$\neg \check{D}_1$	$\neg \check{D}_2$	$\neg \check{D}_3$	$\neg \check{D}_4$
u_1	1	2	0	2
u_2	1	0	1	1
u_3	2	3	2	1
u_4	0	0	0	1
u_5	2	1	4	1
u_6	3	1	1	2
u_7	4	4	2	3

Table 45

$\mathcal{U}$	$\hat{P}_{\check{C}}^{F+}(u_i)$	$\hat{P}_{\check{C}}^{F-}(u_i)$
u_1	$\{u_1, u_4\}$	$\{u_1, u_6, u_7\}$
u_2	$\{u_2, u_4\}$	$\{u_2, u_3, u_5, u_6, u_7\}$
u_3	$\{u_2, u_3, u_4, u_5\}$	$\{u_3, u_7\}$
u_4	$\{u_4\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$
u_5	$\{u_2, u_4, u_5\}$	$\{u_3, u_5, u_7\}$
u_6	$\{u_1, u_2, u_4, u_6\}$	$\{u_6, u_7\}$
u_7	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_7\}$

Table 46

$\mathcal{U}$	$\hat{P}_{\hat{D}}^{\gamma+}(u_i)$	$\hat{P}_{\hat{D}}^{\gamma-}(u_i)$
u_1	$\{u_1, u_4\}$	$\{u_1, u_7\}$
u_2	$\{u_2, u_4\}$	$\{u_2, u_3, u_5, u_6, u_7\}$
u_3	$\{u_2, u_3, u_4\}$	$\{u_3, u_7\}$
u_4	$\{u_4\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$
u_5	$\{u_2, u_4, u_5\}$	$\{u_5\}$
u_6	$\{u_2, u_4, u_6\}$	$\{u_6, u_7\}$
u_7	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_7\}$

Table 47

χ_k	$\underline{\chi}_k^O \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+}$	$\underline{\chi}_k^O \hat{P}_{\hat{C}}^{\hat{F}-} + \hat{P}_{\hat{D}}^{\gamma-}$
χ_1	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$	$\{u_5\}$
χ_2	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$	$\{u_5\}$
χ_3	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_3, u_6, u_7\}$
χ_4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

Table 48

Ω_k	$\underline{\chi}_k^O \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+} - \underline{\chi}_k^O \hat{P}_{\hat{C}}^{\hat{F}-} + \hat{P}_{\hat{D}}^{\gamma-}$
Ω_1	$\{u_1, u_2, u_3, u_4, u_6\}$
Ω_2	$\{u_1, u_2, u_3, u_4, u_6\}$
Ω_3	$\{u_2, u_4\}$
Ω_4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

By step 3(a) we get $\Omega = \{u_2, u_4\}$.

Now for not set we follow the steps 1(b) to 3(b).

Table 49

$\mathcal{U}$	$\hat{P}_{-\hat{C}}^{\hat{G}+}(u_i)$	$\hat{P}_{-\hat{C}}^{\hat{G}-}(u_i)$
u_1	$\{u_1, u_6, u_7\}$	$\{u_1, u_4\}$
u_2	$\{u_2, u_3, u_5, u_6, u_7\}$	$\{u_2, u_4\}$
u_3	$\{u_3, u_7\}$	$\{u_2, u_3, u_4, u_5\}$
u_4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_4\}$
u_5	$\{u_3, u_5, u_7\}$	$\{u_2, u_4, u_5\}$
u_6	$\{u_6, u_7\}$	$\{u_1, u_2, u_4, u_6\}$
u_7	$\{u_7\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

Table 50

$\mathcal{U}$	$\hat{P}_{-\hat{D}}^{\eta+}(u_i)$	$\hat{P}_{-\hat{D}}^{\eta-}(u_i)$
u_1	$\{u_1, u_7\}$	$\{u_1, u_4\}$
u_2	$\{u_2, u_3, u_5, u_6, u_7\}$	$\{u_2, u_4\}$
u_3	$\{u_3, u_7\}$	$\{u_2, u_3, u_4\}$
u_4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_4\}$
u_5	$\{u_5\}$	$\{u_2, u_4, u_5\}$
u_6	$\{u_6, u_7\}$	$\{u_2, u_4, u_6\}$
u_7	$\{u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$

Table 51

χ_k	$\underline{\chi}_k^O \hat{P}_{-\hat{C}}^{\hat{G}-} + \hat{P}_{-\hat{D}}^{\eta-}$	$\underline{\chi}_k^O \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+}$
χ_1^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_2, u_3, u_5, u_6, u_7\}$
χ_2^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_3, u_5, u_6, u_7\}$
χ_3^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_2, u_3, u_5, u_6, u_7\}$
χ_4^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

Table 52

Ω_k	$\underline{\chi}_k^O \hat{P}_{-\hat{C}}^{\hat{G}-} + \hat{P}_{-\hat{D}}^{\eta-} - \underline{\chi}_k^O \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+}$
Ω_1^-	$\{u_4\}$
Ω_2^-	$\{u_2, u_4\}$
Ω_3^-	$\{u_1, u_4\}$
Ω_4^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

By step 3(b) $\Omega^- = \{u_4\}$. Thus by step 4 $\{u_4\}$ is the only one feasible company which has least degree of disagreement and agreed by all decision makers.

For the pessimistic case we have the following definitions

$$\underline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+} = \{u \in \mathcal{U}; \hat{P}_{\hat{C}}^{\hat{F}+}(u) \subseteq \chi_k \wedge \hat{P}_{\hat{D}}^{\gamma+}(u) \subseteq \chi_k\}$$

$$\overline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+} = \{u \in \mathcal{U}; \hat{P}_{\hat{C}}^{\hat{F}+}(u) \cap \chi_k \neq \phi \vee \hat{P}_{\hat{D}}^{\gamma+}(u) \cap \chi_k \neq \phi\}$$

$$\underline{\chi}_k^P \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+} = \{u \in \mathcal{U}; \hat{P}_{-\hat{C}}^{\hat{G}+}(u) \subseteq \chi_k^- \wedge \hat{P}_{-\hat{D}}^{\eta+}(u) \subseteq \chi_k^-\}$$

$$\overline{\chi}_k^P \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+} = \{u \in \mathcal{U}; \hat{P}_{-\hat{C}}^{\hat{G}+}(u) \cap \chi_k^- \neq \phi \vee \hat{P}_{-\hat{D}}^{\eta+}(u) \cap \chi_k^- \neq \phi\}.$$

Then the set $\{(\underline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+}, \overline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+}), (\underline{\chi}_k^P \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+}, \overline{\chi}_k^P \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+})\}$ is called dominating pessimistic multi-granulation N -Bipolar rough set if $\underline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+} \neq \overline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}+} + \hat{P}_{\hat{D}}^{\gamma+}$

$$\text{and } \underline{\chi}_k^P \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+} \neq \overline{\chi}_k^P \hat{P}_{-\hat{C}}^{\hat{G}+} + \hat{P}_{-\hat{D}}^{\eta+}.$$

Similarly

$$\underline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}-} + \hat{P}_{\hat{D}}^{\gamma-} = \{u \in \mathcal{U}; \hat{P}_{\hat{C}}^{\hat{F}-}(u) \subseteq \chi_k \wedge \hat{P}_{\hat{D}}^{\gamma-}(u) \subseteq \chi_k\}$$

$$\overline{\chi}_k^P \hat{P}_{\hat{C}}^{\hat{F}-} + \hat{P}_{\hat{D}}^{\gamma-} = \{u \in \mathcal{U}; \hat{P}_{\hat{C}}^{\hat{F}-}(u) \cap \chi_k \neq \phi \vee \hat{P}_{\hat{D}}^{\gamma-}(u) \cap \chi_k \neq \phi\}$$

$\overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} = \{u \in \mathcal{U}; \hat{P}_{\check{C}}^{\check{G}}(u) \subseteq \chi_k^- \wedge \hat{P}_{\check{D}}^{\check{\eta}}(u) \subseteq \chi_k^-\}$
 $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} = \{u \in \mathcal{U}; \hat{P}_{\check{C}}^{\check{G}}(u) \cap \chi_k^- \neq \emptyset \vee \hat{P}_{\check{D}}^{\check{\eta}}(u) \cap \chi_k^- \neq \emptyset\}$
 $\{(\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}, \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}), (\overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}, \chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}})\}$
 is the dominated pessimistic multi-granular N-Bipolar rough
 set if $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} \neq \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$ and $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} \neq \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$,
 where $\chi_k = \{u \in \mathcal{U} : \check{\gamma}(\check{D}_k)(u) \neq 0\}$ and
 $\chi_k^- = \{u \in \mathcal{U} : \check{\eta}(\neg \check{D}_k)(u) \neq 0\}$.

Proposed algorithm Step 1(a) to 3(a) are identical to the algorithm of Ali et al. [4] for the positive set under our defined notations of dominance relation, we name those steps as 1(a) - 3(a).

Input Information system $\mathbf{S} = \{\mathcal{U}, (\check{F}, \check{G}, \check{C}, N_1), (\check{\gamma}, \check{\eta}, \check{D}, N_2)\}$.

Step1 (a): Construct $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$ and $\overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$, where
 $\chi_k = \{u \in \mathcal{U} : \check{\gamma}(\check{D}_k)(u) \neq 0\}$.

Step2 (a): If $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} = \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$

then $\Omega_k = \chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} = \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$

otherwise $\Omega_k = \chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} - \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$.

Step3(a): If $\Omega = \bigcap_{k=1}^{|\check{D}|} \Omega_k \neq \emptyset$ go to the output, otherwise
 $|\check{D}| = |\check{D}| - 1$ go to the step 3(a)

For the not set we have the following algorithm.

Step1 (b): Construct $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$ and $\overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$ where

$\chi_k^- = \{u \in \mathcal{U} : \check{\eta}(\neg \check{D}_k)(u) \neq 0\}$.

Step2(b): If $\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} = \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$ then $\Omega_k^- =$

$\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} = \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$ otherwise $\Omega_k^- = \chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} - \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$

$\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$.

Step3(b): If $\Omega^- = \bigcap_{k=1}^{|\neg \check{D}|} \Omega_k^- \neq \emptyset$ go to the output otherwise
 $|\neg \check{D}| = |\neg \check{D}| - 1$ go to the step 3(b).

Step 4 Evaluate $\Omega \cap \Omega^-$

the intersection will give that action as a solution which is agreed by all and has least degree of disagreement. We applied this algorithm to Example 6.1 and calculations are in Tables 53 - 56.

Table 53

χ_k	$\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$	$\overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$
χ_1	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$	$\emptyset$
χ_2	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$	$\emptyset$
χ_3	$\{u_1, u_2, u_4, u_6\}$	$\{u_1, u_3, u_6, u_7\}$
χ_4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

Table 54

Ω_k	$\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} - \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$
Ω_1	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$
Ω_2	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$
Ω_3	$\{u_2, u_4\}$
Ω_4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

By step 3(a) we get $\Omega = \{u_2, u_4\}$.

Now for not set we follow the steps 1(b) to 3(b).

Table 55

χ_k^-	$\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$	$\overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$
χ_1^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_2, u_3, u_5, u_6, u_7\}$
χ_2^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_3, u_5, u_6, u_7\}$
χ_3^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_2, u_3, u_5, u_6, u_7\}$
χ_4^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

Table 56

Ω_k^-	$\chi_k^P{}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}} - \overline{\chi_k^P}^{\hat{P}_{\check{C}}^{\check{G}} + \hat{P}_{\check{D}}^{\check{\eta}}}$
Ω_1^-	$\{u_4\}$
Ω_2^-	$\{u_2, u_4\}$
Ω_3^-	$\{u_1, u_4\}$
Ω_4^-	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$

By Step 3 and 4 we have u_4 it means that it is the most appropriate and optimal consensus.

Thus the pessimistic and optimistic both approaches give the same result.

7 Comparison Analysis

In this section we will compare the Bipolar N - soft sets and Bipolar soft sets with our model N -bipolar soft sets. First we will compare Bipolar N -soft set with N -bipolar soft set. The main difference is that Bipolar N -soft sets are the bipolar extension of N -soft sets, while our model is the generalization of bipolar soft sets, that is why the set of attributes in [23] is same for positive and negative grading, but we have positive and negative attribute sets in our introduced concept. In [23] they have bipolarity over grades but in ours we have bipolarity over attribute set. In [23], the model introduced by them does not give the cause or reason behind giving positive or negative rankings, the example of restaurant in [23] needs reasoning remarks for grading, that is the positive and negative ticks are followed by remarks, but on the other hand, the use of bipolar attribute set

clarifies the grading at very first sight and is more understandable to rank for example in our Example 3.2 the dish prepared by three chefs were graded on the basis of positive and negative attributes which gave a sound perception for hiring a chef. Considering the example of positive and negative mentions of customer of a restaurant [23], the attributes given by them seem unable to interpret the cause of grading by ticks, lacking the sufficient information to grade (that is which thing is being considered while grading about the particular thing), so it could be better if we use the following attribute set: $\check{A} = \{\text{quick service, high quality food, expensive}\}$ and for $\neg\check{A} = \{\text{late service, low quality food, cheap}\}$. If user has to decide between alternatives then this model is ineffective, while our model fits best for such decisions as we did in Example 3.1, where Mr. X wants to buy a house, we used green surroundings and as its alternative we had commercial area. There is no condition imposed in definition to construct bipolar N -soft set [23], also the parameter which is given a positive grade 3 is also given the negative grade -4 for the bipolar 6-soft set.

As far as Bipolar soft sets are concerned, N -bipolar soft sets are extension of bipolar soft sets. Bipolar soft sets are a particular case of N -bipolar soft sets when $N = 2$ which we discussed in Section 3 in a Remark 3.1 and discussion below it. The bipolar soft set does not provide the information about the degree of presence of positive and negative attribute. They only give binary system for grading, which only presents yes or no for the members of universe having the positivity or negativity of the particular attribute, our concept generalizes it to measure the extent to which the presence or absence of positive and negative aspect in the object. In the first step, we explained how to grade negative attributes and what are the suitable negative attributes that can be taken against a particular positive attribute. This we explain as follows, in the example taken by Shabir and Naz [36] for bipolar mood charts. They take the attributes $\check{E} = \{\text{Severe Mania, Sever depression, Anxiety, Medication, Side effects}\}$ and the set $\neg\check{E} = \{\text{Mild Mania, Mild Depression, No Anxiety, No Medication, No Side effects}\}$. We observed that severe mania and mild mania are the degrees of Mania, similar is the case for depression. Since to rank any attribute is to distinguish the degree of that particular attribute present in any member of universe, so we define the attributes as $\check{E} = \{\text{Mania, depression, Anxiety, Medication, Side effects}\}$ and the set $\neg\check{E} = \{\text{No Mania, No Depression, No Anxiety, No Medication, No Side effects}\}$. Our introduced concept of bipolarity can be implemented in daily life decisions, which makes it so empirical and valuable, to show this, examples discussed in this paper are from our daily life.

8 Conclusion

We have developed a new theory (N -bipolar soft set) based on ranking or grading approach with the consideration of negative aspects. This theory is non binary as well as non-continuous (fuzziness). Each concept introduced in this paper is explained with clear examples. The emphasis is made on the importance

of negative sides of attributes and their usefulness in decision making. The algebraic structures of N -bipolar soft sets are also explained here which may be of keen interest of researchers. We have given three algorithms for decision making. These algorithms reveal their flexible and versatile design which makes them adaptable to users need. These algorithms provide users, feasible and optimal consensus that not only has highest positivity but also gives lowest negativity. The importance of conflict analysis is not only limited to real world but also providing researchers a sound ground for research. We introduced the impact of degree of disagreement and its effect towards decision of multi-attribute and multi decisional problems. This concept opened up many avenues for researchers. We can also go beyond this concept by developing theory about incomplete N -bipolar soft sets. It is also possible to combine this concept with fuzziness. We expect that this paper gives an idea for the beginning of new study.

Compliance with ethical standards

Conflict of interest

All authors declare that they have no conflict of interest.

Ethical Approval

This article does not contain any studies with human participants performed by any of the authors.

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