Internal Resonance of Hyperelastic Thin-Walled Cylindrical Shells under Harmonic Axial Excitation and Time-Varying Temperature Field

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Research Article

Keywords: Hyperelastic cylindrical shell, axisymmetric nonlinear vibration, harmonic balance method, internal resonance, isolated bubble

Posted Date: August 17th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-761448/v1>

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Version of Record: A version of this preprint was published at Thin-Walled Structures on June 1st, 2022. See the published version at <https://doi.org/10.1016/j.tws.2022.109256>.

Internal resonance of hyperelastic thin-walled cylindrical shells under harmonic axial excitation and time-varying temperature field

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Abstract In this paper, the internal resonance characteristics of hyperelastic cylindrical shells under the time-varying temperature field are investigated for the first time, and the evolution of the isolated bubble is carried out. Through the analysis of the influences of temperature on material parameters, the hyperelastic strain energy density function in the unsteady temperature field is presented. The governing equations describing the axisymmetric nonlinear vibration are derived from the nonlinear thin shell theory and the variational principle. With the harmonic balance method and the arc length method, the steady state solutions of shells are obtained, and their stabilities are determined. The influences of the discrete mode number, structural and temperature parameters on the nonlinear behaviors are examined. The role of the parameter variation in evolution behaviors of isolated bubble responses is revealed under the condition of 3:1 internal resonance. The results manifest that both structural and temperature parameters can affect the resonance range of the response curve, and the perturbed temperature has a more significant effect on the stable region of the solution.

Keywords Hyperelastic cylindrical shell · axisymmetric nonlinear vibration · harmonic balance method · internal resonance · isolated bubble

1 Introduction

Due to the light weight, high strength and stiffness ratio, cylindrical shells are widely applied in many important engineering fields, such as the energy transportation, shipbuilding and aerospace. The pipeline conveying fluid and the spacecraft or missile, both can be modeled as the cylindrical shell for simplifying analysis^[1-3]. In addition, the mechanical behavior of biological tissues such as bones and blood vessels can also be simulated with the cylindrical shell. However, it should be noted that for those structures, hyperelastic material models should be adopted to describe the nonlinear mechanical behaviors^[4, 5]. Generally, cylindrical shells are usually subjected to different loads (such as the axial force and temperature), which are prone to induce the large amplitude vibration, and then destroy the structural safety. Therefore, it is essential and necessary to investigate the nonlinear vibration characteristics of hyperelastic thin-walled cylindrical shells subjected to external loads for the optimization of design and improvement of safety.

The thin-walled shell is usually the best and only choice for those aerospace and navigation structures, as well as engineering structures such as vehicle and architecture building. Due to the complexity of the vibration behavior of the continuum structure, the discretization method is widely applied to simplify the continuum into a system with finite degree of freedom. This method is also very common in the vibration analysis of beams, and there is a solid theoretical foundation for the related analysis methods. However, shells especially exhibit certain effects that are not presented in beams or even plates, and cannot be interpreted by engineers who are only familiar with the beam-type vibration theory^[6]. Thus, it is helpful to improve the understanding of basic phenomena in vibrations of plates and shells, and it will be useful in explaining experimental measurements or the results of the finite element programs. Radwanska et al.^[7] presented a comprehensive introduction to the elastic plate and shell theory, formulations and solutions of fundamental mechanical problems (statics, stability and free vibrations) using the exact approaches and approximate computational methods, and also emphasized the modern capabilities of the finite element technology. With the load increases, the shell behavior will show some essential geometric nonlinearity who plays a key role in the structural safety. Therefore, the

1 influence of nonlinearity has also been paid more and more attention in the
2 dynamic analysis of thin-walled shells. Meanwhile, with the development of
3 material science, the combination between shells and new materials is getting
4 more and more inseparably associated, especially, for plates and shells composed
5 of composite materials. The monograph contributed by Reddy ^[8] systematically
6 introduced various aspects related to laminated composite structures, including the
7 virtual work principle, variational method, anisotropic elasticity, plate and shell
8 theory (including classical theory, first-order and third-order shear deformation
9 theory), geometric nonlinearity and finite element analysis. Shen ^[9, 10] dedicated to
10 the investigation of the geometrically nonlinear problems of inhomogeneous
11 isotropic and functionally graded plates and shells, which includes the large
12 deflection, post-buckling and nonlinear vibration. Amabili ^[11] studied the plates
13 and shells composed of composite materials, soft materials and biological
14 materials, and researched the hyperelastic, viscoelasticity and nonlinear damping,
15 which are pioneering works considered both material and geometric
16 nonlinearities.

17 The researches on nonlinear dynamic behaviors of cylindrical shells have
18 attracted much attention from a large number of scholars. Ye and Wang ^[12]
19 analyzed the nonlinear forced vibration of the functionally graded graphene
20 platelet-reinforced metal foam thin-walled cylindrical shell, and found that the
21 inclusion of graphene platelets in the shells weakens the nonlinear coupling effect.
22 Based on the von Kármán geometric nonlinear strain-displacement relationship
23 and the first-order shear deformation theory, Zhang et al. ^[13] studied the nonlinear
24 radial breathing vibration of a carbon fiber reinforced polymer laminated
25 cylindrical shell with different temperatures under both axial pressure and radial
26 line load. Yang et al. ^[14] studied the nonlinear vibration of a carbon fiber
27 reinforced polymer laminated cylindrical shell with 1:2 internal resonance,
28 primary parametric resonance and 1/2 subharmonic resonance. Based on Reddy's
29 third order shear deformation theory and the Galerkin method, Vuong and Duc ^[15]
30 examined the nonlinear vibration of the functionally graded moderately thick
31 toroidal shell segments resting on the Pasternak type elastic foundation. Liu et al.
32 ^[2] investigated the nonlinear breathing vibration of an eccentric rotating
33 composite laminated cylindrical shell subjected to the lateral and temperature

1 excitations. Parvez et al. ^[16] presented the nonlinear dynamic response of
2 laminated composite cylindrical shells under periodic external forces, and
3 explored the parameters influencing the transition between the softening and
4 hardening nonlinear behaviors of cylindrical shells. Shen et al. ^[17] investigated the
5 nonlinear flexural vibrations of carbon nanotube-reinforced composite laminated
6 cylindrical shells with negative Poisson's ratios in thermal environments. Wang et
7 al. ^[18] investigated the strongly nonlinear traveling waves in a thermo-hyperelastic
8 cylindrical shell.

9 Compared with those structures composed of composite materials or other new
10 materials, there are relatively few researches on the dynamics of hyperelastic
11 structures, and most of them are also focused on relatively simple structures, such
12 as the membrane ^[19-21] or the beam ^[22-24] contributed by Soares, Gonçalves, Zhao,
13 Chen and Wang. Dong et al. ^[25] proposed a novel approach to tune the resonance
14 frequency of circular hyperelastic membrane-based energy harvesters via different
15 stretch ratios applied to membranes, which provided an alternative tuning strategy
16 to enable energy harvesting from different ambient vibration sources in various
17 environments. Wang and Zhu ^[26] investigated the nonlinear vibrations of a
18 hyperelastic beam under time-varying axial loading are derived via the extended
19 Hamilton's principle. Iglesias et al. ^[27] studied the large-amplitude axisymmetric
20 free vibration of an incompressible hyperelastic orthotropic cylinder, and analyzed
21 the influence of initial conditions, structural and material parameters on the
22 dynamic behavior. With the Gent-Gent hyperelastic model, Alibakhshi and
23 Heidari ^[28] researched the nonlinear vibration of a dielectric elastomer balloon
24 considering the strain hardening and the second invariant of the Cauchy-Green
25 deformation tensor, and found that the second invariant parameter could suppress
26 the chaotic motion of the system. Due to the existence of geometric and material
27 nonlinearities, it is difficult to investigate the vibrations of hyperelastic plate and
28 shell structures, however, there are still a few related works. Breslavsky et al. ^[29]
29 presented the static and dynamic responses of circular cylindrical shells composed
30 of hyperelastic arterial materials based on the nonlinear high-order shear
31 deformation theory, and pointed out that the resonant regime with both driven and
32 companion modes active should possess more complicated nonlinear dynamics.
33 Amabili et al. ^[30] conducted the numerical analysis and experimental research on

1 the hyperelastic behavior of a thin square silicone rubber plate, obtained a good
2 agreement between the first four natural modes and frequencies numerically and
3 experimentally.

4 It is not difficult to find that there are numerous researches on the nonlinear
5 dynamics of cylindrical shells, however, few researchers pay attention to the
6 dynamic behaviors of hyperelastic cylindrical shells considering the material
7 nonlinearity, and fewer works considering the temperature effect. Therefore, this
8 paper studies the nonlinear dynamic behaviors of the hyperelastic thin-walled
9 cylindrical shells under harmonic axial excitation and time-varying temperature
10 field. The remainder of this paper is organized as follows: Section 2 introduces the
11 thermo-hyperelastic constitutive relationship; Section 3 gives the strain-
12 displacement relationship of the thin-walled cylindrical shell, and derives the
13 nonlinear governing differential equations; Section 4 analyzes the natural
14 frequency characteristics of the shells and calculates the corresponding internal
15 resonance parameters; Section 5 presents the solution of nonlinear equations and
16 the stability of periodic solutions; Section 6 discusses the influences of the
17 discrete mode number, structural and temperature parameters via the numerical
18 simulation; Section 7 draws the conclusions.

19 **2. Thermo-hyperelasticity and strain-displacement** 20 **relationship**

21 Consider a thin-walled cylindrical shell composed of a class of incompressible
22 thermo-hyperelastic materials in a time-varying temperature field, which is
23 subjected to the axial load at both ends. As shown in Fig. 1, the length, diameter
24 of the middle surface and thickness of the cylindrical shell are represented by H ,
25 D and h , respectively. The cylindrical coordinate system (x, θ, z) is
26 established on the mid-surface of the shell, where x , θ , z are the axial,
27 circumferential and radial directions. Let u , v and w be the displacements of
28 a point at the mid-surface, and u_1 , u_2 and u_3 be the displacements of an
29 arbitrary point of the shell accordingly.

Fig. 1 Cylindrical shell configuration under periodic axial loads

The Lagrange strain tensor and the right Cauchy-Green deformation tensor which characterize the deformation of the thin-walled cylindrical shell are expressed as follows

$$\mathbf{E} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2\varepsilon_{11}+1 & \varepsilon_{12} & 0 \\ \varepsilon_{12} & 2\varepsilon_{22}+1 & 0 \\ 0 & 0 & 2\varepsilon_{33}+1 \end{bmatrix} \quad (1)$$

where ε_{ij} ($i, j=1 \sim 3$) is the Green strain. The principal invariants of the right Cauchy-Green deformation tensor are

$$I_1 = \text{tr}(\mathbf{C}) = 2(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 3,$$

$$I_2 = \frac{1}{2} \left[(\text{tr} \mathbf{C})^2 - \text{tr}(\mathbf{C}^2) \right] = 4(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33}) - \varepsilon_{12}^2 + 3, \quad (2)$$

$$I_3 = J^2 = \det(\mathbf{C}) = (2\varepsilon_{33}+1) \left[(2\varepsilon_{11}+1)(2\varepsilon_{22}+1) - \varepsilon_{12}^2 \right]$$

Using with the incompressible constraint $J=1$ yields

$$\varepsilon_{33} = \frac{1}{2} \left\{ \left[(2\varepsilon_{11}+1)(2\varepsilon_{22}+1) - \varepsilon_{12}^2 \right]^{-1} - 1 \right\} \quad (3)$$

For the case of small strains, expanding Eq. (3) about all the strains in it up to the fourth order gives

$$\begin{aligned} \varepsilon_{33} = & -(\varepsilon_{11} + \varepsilon_{22}) + 2(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2) \\ & - 4(\varepsilon_{11} + \varepsilon_{22}) \left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \frac{1}{2}\varepsilon_{12}^2 \right) + 8 \left(\varepsilon_{11}^4 + \varepsilon_{22}^4 + \frac{1}{16}\varepsilon_{12}^4 \right) \\ & + 8\varepsilon_{11}\varepsilon_{22}(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2) + 6\varepsilon_{12}^2(\varepsilon_{11}^2 + \varepsilon_{22}^2) + \dots \end{aligned} \quad (4)$$

Based on the Kirchhoff-love hypothesis (that is, the stress components perpendicular to the mid-surface are negligible, which is a very good approximation for thin-walled shells) and the assumption of axisymmetric vibrations (the circumferential displacement is zero), the displacements of a generic point on the cylindrical shell are given as follows

$$u_1 = u(x, t) - \frac{\partial w(x, t)}{\partial x} z, u_2 = 0, u_3 = w(x, t) \quad (5)$$

Additionally, the geometrical nonlinearity of the shell yields the following strain-displacement relationships [8]

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] - \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) z + \frac{1}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 z^2, \\ \varepsilon_{22} &= \frac{w}{R} + \frac{1}{2} \left(\frac{w}{R} \right)^2 \end{aligned} \quad (6)$$

For incompressible thermo-hyperelastic materials, the thermal effect is mainly reflected in their strain energy functions. In this paper, the 8-chain non-Gaussian network model, i.e. the Arruda-Boyce model [31] is adopted, as follows,

$$\Phi_{T_i} = \mu_{T_i} N_{T_i} \left(\frac{\mathcal{G}}{\lambda_m} \sqrt{\frac{I_1}{3}} + \ln \frac{\mathcal{G}}{\sinh \mathcal{G}} \right) \quad (7)$$

$$\mu_{T_i} = nkT_i = \frac{T_i}{T_0 (1 + \alpha_T \Delta T_i)^3} \mu_0, n_{T_i} = \frac{n_0}{(1 + \alpha_T \Delta T_i)^3}, \quad (8)$$

$$n_0 N_0 = n_{T_i} N_{T_i}, N_{T_i} = (1 + \alpha_T \Delta T_i)^3 N_0, \Delta T_i = T_i - T_0$$

where α_T is the linear thermal expansion coefficient, T_i is the environmental temperature, μ_{T_i} is the shear modulus at the temperature of T_i , and $\Delta T = T_i - T_0$ is the temperature difference (T_0 is room temperature, $(\cdot)_0$ is the value of a parameter at the temperature of T_0 , $(\cdot)_{T_i}$ is the value of a parameter at the temperature of T_i), $\lambda_m = \sqrt{N_{T_i}}$ is the limited stretch ratio of chains, N_{T_i} is the average number of segments in a chain, $\mathcal{G} = L^{-1} \left(\sqrt{I_1 / (3N_{T_i})} \right)$ is the inverse Langevin function. For the periodic uniform environmental temperature, the temperature T_M of the thin-walled shell is assumed to be periodic, then T_M is given by

$$T_M = T_A + \Delta T_p \cos(\omega_T t) \quad (9)$$

where T_A is the average temperature part of the environmental temperature and ΔT_p is the amplitude of the perturbed temperature part (α_T is a very small parameter, and $\alpha_T \Delta T_p$ is also a small quantity). In this case,

$$\mu_M = \frac{T_A + \Delta T_p \cos(\omega_T t)}{T_A (1 + \alpha_T \Delta T_p \cos(\omega_T t))} \mu_{T_A}, N_M = [1 + \alpha_T \Delta T_p \cos(\omega_T t)]^3 N_{T_A} \quad (10)$$

$$\mu_M \approx [1 + P_\mu \cos(\omega_T t)] \mu_{T_A}, N_M \approx [1 + P_N \cos(\omega_T t)] N_{T_A} \quad (11)$$

$$\frac{\mu_M}{N_M^n} = \frac{\mu_{T_A}}{N_{T_A}^n} \frac{1 + P_\mu \cos(\omega_T t)}{[1 + P_N \cos(\omega_T t)]^n} \approx \frac{\mu_{T_A}}{N_{T_A}^n} [1 + (P_\mu - n P_N) \cos(\omega_T t)] \quad (12)$$

$$P_\mu = \left(\frac{1}{T_A} - 3\alpha_T \right) \Delta T_p, P_N = 3\alpha_T \Delta T_p \quad (13)$$

Moreover, the series expansion form for Eq. (7) (keeping the first five terms) is often adopted as follows,

$$\Phi_M = \mu_M \sum_{i=1}^5 \frac{C_i}{N_M^{i-1}} (I_1^i - 3^i) \quad (14)$$

$$\begin{aligned} \Phi_M &= \sum_{i=1}^5 \frac{\mu_{T_A}}{N_{T_A}^{i-1}} \{1 + [P_\mu - (i-1)P_N] \cos(\omega_T t)\} C_i (I_1^i - 3^i) \\ &= \mu_{T_A} \sum_{i=1}^5 \frac{C_i}{N_{T_A}^{i-1}} (I_1^i - 3^i) + \cos(\omega_T t) \mu_{T_A} \sum_{i=1}^5 \frac{C_i}{N_{T_A}^{i-1}} [P_\mu - (i-1)P_N] (I_1^i - 3^i) \end{aligned} \quad (15)$$

The material parameters required for the calculation are listed in Table 1.

Table 1. Material parameters ^[31, 32]

Material parameters	Valves
μ_0 (MPa)	0.2853
N_0	26.54
α_T (K ⁻¹)	3.6×10^{-4}
T_0 (K)	297
C_1	1/2
C_2	1/20
C_3	11/1050
C_4	19/7000
C_5	519/673750

Combined with Eqs.(2), (4) and (14), the specific expression of the strain energy function is obtained.

3. System of governing equation

The expressions of the kinetic energy and potential energy can be expressed as follows

$$D_p = \frac{1}{2} \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^H \int_0^{2\pi} (\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2) R d\theta dx dz, \quad (16)$$

$$E_p = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^H \int_0^{2\pi} \Phi R d\theta dx dz$$

where ρ is the mass density, t is the time, Φ is the strain energy function, and the dot denotes the derivative with respect to t .

The Ritz method is employed to approximate the axial displacement u and the deflection w . The boundary conditions should be satisfied at the shell ends with the assumed modes which are axisymmetric. Then, the displacements are expanded by using the eigenmodes as follows

$$\begin{cases} u(x, t) = \sum_{m=1}^M u_{m0}(t) \cos \frac{m\pi x}{H}, \\ w(x, t) = \sum_{m=1}^M w_{m0}(t) \sin \frac{m\pi x}{H} \end{cases} \quad (17)$$

where m is the longitudinal half wave number, and M is the truncation order. It should be noted that under the assumption of axisymmetric motion, the circumferential wave numbers is 0, $u_{m0}(t)$ and $w_{m0}(t)$ are generalized coordinates related to time t .

In presence of axial harmonic loads acting on the shell, the virtual work of the external force is done by a time-varying axial load $p_x(t)$ which is applied at both ends of the shell. It is positive in the x direction, that is, $-p_x(t)$ is applied at $x=0$ and $p_x(t)$ is applied at $x=H$. Thus, the uniform axial load $q_x(t)$ is expressed as follows

$$q_x(x, \theta, z, t) = p_x(t) [-\delta(x) + \delta(x-H)], \quad F_x(t) = 2\pi R h p_x(t) \quad (18)$$

where δ is the Dirac delta function, and $F_x(t)$ is the net force at the end. Therefore, the virtual work done by the external force can be expressed as

$$W_{ex} = \int_0^H \int_0^{2\pi} \int_{-h/2}^{h/2} (q_x u + q_\theta v + q_r w) dx R d\theta dz = -F_x(t) \sum_{m=1}^M u_{m0}(t) [-1 + (-1)^m] \quad (19)$$

1 The other part of the virtual work is done by the non-conservative damping force.
 2 Assuming that the non-conservative damping force is viscous type, Rayleigh's
 3 dissipation function is adopted to obtain the virtual work

$$\begin{aligned}
 W_d &= \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^H \int_0^{2\pi} (c_{1j} \dot{u}^2 + c_{2j} \dot{v}^2 + c_{3j} \dot{w}^2) R d\theta dx dz \\
 &= \pi h R \int_0^H (c_{1j} \dot{u}^2 + c_{2j} \dot{v}^2 + c_{3j} \dot{w}^2) dx
 \end{aligned}
 \tag{20}$$

5 where c_{ij} is damping parameter related to the natural frequency of the shell and
 6 can be evaluated from experiments. The Lagrange equation describing the
 7 axisymmetric motion for a hyperelastic thin-walled cylindrical shell is given by

$$\frac{d}{dt} \frac{\partial (D_p - E_p)}{\partial \dot{q}_i} - \frac{\partial (D_p - E_p)}{\partial q_i} = - \frac{\partial W_d}{\partial \dot{q}_i} + \frac{\partial W_e}{\partial q_i}
 \tag{21}$$

9 where q_i is the generalized displacement and i is the mode number. The form of
 10 external load is expressed as follows

$$F = -F_0 \cos(\Omega t + \theta_0)
 \tag{22}$$

12 where F_0 , Ω and θ_0 are the excitation amplitude, excitation frequency and
 13 phase difference between the axial excitation and the temperature. With the aid of
 14 related equations, a system of nonlinear differential equations describing the
 15 axisymmetric motion of thin-walled cylindrical shells can be written in the
 16 following matrix form

$$\begin{aligned}
 \mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + [\mathbf{E} + \delta_1 \cos(\omega_T t)] \mathbf{K}_1 \mathbf{q} + [\mathbf{E} + \delta_2 \cos(\omega_T t)] \mathbf{K}_2(\mathbf{q}) \mathbf{q} \\
 + [\mathbf{E} + \delta_3 \cos(\omega_T t)] \mathbf{K}_3(\mathbf{q}, \mathbf{q}) \mathbf{q} = \mathbf{F} \cos(\Omega t + \theta_0)
 \end{aligned}
 \tag{23}$$

18 where $\mathbf{E}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}_1$, $\mathbf{q}$ and $\mathbf{P}$ are the identity matrix, mass matrix,
 19 damping matrix, linear stiffness matrix, generalized displacement column vector
 20 and load amplitude column vector, respectively. $\mathbf{K}_2$ and $\mathbf{K}_3$ are quadratic
 21 nonlinear stiffness matrix and cubic nonlinear stiffness matrix, which give the
 22 quadratic and cubic non-linear stiffness terms associated with displacements. δ_1 ,
 23 δ_2 and δ_3 are disturbance parts of stiffness matrix caused by temperature
 24 variation. The related matrix dimension is $2M \times 2M$, where M is the
 25 truncation order.

26 Furthermore, it is convenient to introduce the following dimensionless
 27 notations,

$$\begin{aligned} \tau = \omega_0 t, \eta = \frac{\Omega}{\omega_0}, \eta_T = \frac{\omega_T}{\omega_0}, \bar{\mathbf{q}} = \frac{1}{h} \mathbf{q}, \bar{\mathbf{F}} = \frac{1}{\omega_0^2 h} \mathbf{M}^{-1} \mathbf{F}, \\ \bar{\mathbf{C}} = \frac{1}{\omega_0} \mathbf{M}^{-1} \mathbf{C}, \bar{\mathbf{K}} = \frac{1}{\omega_0^2} \mathbf{M}^{-1} \mathbf{K}_1, \bar{\mathbf{K}}_2 = \frac{h}{\omega_0^2} \mathbf{M}^{-1} \mathbf{K}_2, \bar{\mathbf{K}}_3 = \frac{h^2}{\omega_0^2} \mathbf{M}^{-1} \mathbf{K}_3 \end{aligned} \quad (24)$$

where ω_0 is the fundamental frequency. Hence, the system (23) is rewritten as

$$\begin{aligned} \bar{\mathbf{q}}'' + \bar{\mathbf{C}}\bar{\mathbf{q}}' + \bar{\mathbf{K}}_1\bar{\mathbf{q}} + \bar{\mathbf{K}}_2(\bar{\mathbf{q}})\bar{\mathbf{q}} + \bar{\mathbf{K}}_3(\bar{\mathbf{q}}, \bar{\mathbf{q}})\bar{\mathbf{q}} = \bar{\mathbf{F}} \cos(\eta\tau) \\ - [\delta_1 \bar{\mathbf{K}}_1\bar{\mathbf{q}} + \delta_2 \bar{\mathbf{K}}_2(\bar{\mathbf{q}})\bar{\mathbf{q}} + \delta_3 \bar{\mathbf{K}}_3(\bar{\mathbf{q}}, \bar{\mathbf{q}})\bar{\mathbf{q}}] \cos(\eta_T\tau - \theta_0) \end{aligned} \quad (25)$$

In summary, the axisymmetric nonlinear differential equations describing the motion of the thin-walled cylindrical shell in the time-varying temperature field subjected to the axial excitation have been derived. It is not difficult to find that the system of nonlinear equations present the influences of the parametric excitation and external excitation.

4. Natural frequency analysis

In this paper, the nonlinear vibration of thin-walled shell is considered. The thickness-diameter ratio, the radius and the material density are taken as $h/R = 0.02$, $R = 100\text{mm}$ and $\rho = 1100\text{kg} \cdot \text{m}^{-3}$.

4.1 Influence of structural parameters on frequency

For thin-walled cylindrical shells, the most important structural parameter is the length-diameter ratio. The influence of this parameter on the natural frequency is analyzed in the following, and the temperature is taken as the room temperature (i.e., 297K).

1 **Fig. 2** Relation between the natural frequency and length-diameter ratio (solid line: modes with
2 $m=1\sim5$; dash-dotted line: the same order mode with $m=1\sim5$)

3 Refer to the existing Ref. [33], with the assumption of axisymmetric motion,
4 for the given longitudinal half wave number m , there are two modes which can
5 be obtained from the system (23), and denoted the one with lower frequency as
6 the m -th order mode, while the other higher as the same order mode of the m -th
7 order mode. The solid line in Fig. 2 shows the 1~5th modes from bottom to top. It
8 is not difficult to find that the corresponding low-order mode frequency for $m=1$
9 possesses the fundamental frequency ω_0 of the shell (the red solid line). The
10 dash-dotted lines with the same color as the solid lines are the same order modes
11 with the higher natural frequency and the same longitudinal half wave number.
12 For the convenience of later description, the m -order mode is denoted as $(m, 1)$,
13 the corresponding same mode is denoted as $(m, 2)$, the corresponding natural
14 frequencies are denoted as ω_{m1} and ω_{m2} . For the fundamental frequency, denote
15 that $\omega_0 = \omega_{11}$. Moreover, the 3:1 internal resonance is mainly considered in this
16 paper, then the structural parameters satisfying the internal resonance conditions
17 are determined. The intersections of black dashed line and other lines in Fig. 2 are
18 those structural parameters with 3:1 internal resonance, which can be calculated
19 by the dichotomy. As shown in Fig. 2, the natural frequency of the shell is quite
20 sensitive to the variation of structural parameters when the length-diameter ratio is
21 small. That is, in this situation, a small change in the structural parameters will
22 lead to a significant change in the natural frequency, which should result to
23 extremely complex and unexplained internal resonance behavior [33]. In order to
24 avoid this problem, this paper will only discuss the larger length-diameter ratio
25 which satisfies the internal resonance condition. There are four structural
26 parameters satisfying the condition and those parameters are greater than 3, which
27 are listed in Table 2.

28 The internal resonance behavior is very common in various structures, such as
29 typical suspended cables, arches, beams, plates and shells [12, 14, 34-36]. Meanwhile,
30 the internal resonance leads to mode interactions, an energy exchange or a
31 coupling among the modes [37], thus numerous researchers applied it to the energy
32 harvest, and revealed that the working frequency range can be effectively
33 broadened [38-42].

Table2. Critical structural parameters for 3:1 internal resonance

Frequency ratios between modes	Valves of α_i
$\omega_{51} / \omega_{11} = 3$	4.952848653250840
$\omega_{41} / \omega_{11} = 3$	5.321955514024012
$\omega_{12} / \omega_{11} = 3$	3.902195478263820
$\omega_{22} / \omega_{11} = 3$	3.303421113840159

4.2 Effect of temperature on frequency ratio

For the internal resonance behavior of structures, the frequency ratio between the natural frequencies plays a key role. In this paper, the influence of average environmental temperature on the frequency ratio of the shell is examined. The structural parameters are taken as $\alpha = H / D = 3.9022$, $\varepsilon = h / R = 0.02$.

Fig 3. Relation between natural frequency and variation of average environmental temperature

Fig. 4 Relation between natural frequency ratio and variation of average environmental temperature

1 As shown in Fig. 3, the natural frequency increases with the increasing average
 2 temperature. The reason is that the constitutive model adopted in this paper
 3 gradually hardens as the temperature rises. Moreover, it can be inferred from Eq.
 4 (8) that the shear modulus of the material is proportional to the temperature, that
 5 is, the higher the environmental temperature, the greater the stiffness, which leads
 6 to a greater natural frequency. Meanwhile, it is identified that the variation of
 7 average environmental temperature has no effect on the natural frequency ratio, as
 8 shown in Fig. 4. Additionally, the internal resonance behavior is very strongly
 9 dependent on the natural frequency ratio. Therefore, a preliminarily conclusion
 10 can be drawn that the average environmental temperature may not have a
 11 significant effect on the axisymmetric nonlinear vibration of the cylindrical shell.

12 **5. Solution**

13 From the foregoing analysis, when the length-diameter ratio equals to the
 14 parameters given in Table 2, the mode frequencies of the shell are commensurable
 15 or nearly commensurable, which may lead to the internal resonance. Therefore,
 16 this article will consider the nonlinear vibration behavior in a periodic uniform
 17 temperature field when the parameter is approximately or exactly taken as the
 18 values in Table 2.

19 There are three characteristic frequency parameters in Eq. (25), i.e., the
 20 fundamental frequency $\omega_0 = \omega_{11}$, the external excitation frequency Ω , and the
 21 periodic uniform temperature frequency ω_T . Equation (24) gives the frequency
 22 ratios $\eta = \Omega / \omega_0$ and $\eta_T = \omega_T / \omega_0$. For the length-diameter ratio parameters
 23 listed in Table 2, it is patently obvious that $3\omega_0 = \omega_i$, where ω_i is the natural
 24 frequency of other discrete mode. Since this paper only focuses on the low-
 25 frequency resonance, the frequency should satisfy the requirement that $\Omega \approx \omega_0$,
 26 i.e., $\eta \approx 1$.

27 It is not difficult to find that Eq. (25) is a system of nonlinear differential
 28 equations with quadratic and cubic nonlinear terms. For this kind of equations, the
 29 analytical method usually fails, and the common strategy is to adopt the numerical

1 integration method or the approximate analytical method. Moreover, in order to
 2 reveal the influence of structural parameters, the variation should result to some
 3 stiff equations which leads to extremely high computational effort or low
 4 accuracy. Fortunately, the approximate analytical methods overcome these
 5 shortcomings. For the case that only the steady-state periodic solution is required,
 6 the harmonic balance method is an extremely excellent choice. In the field of the
 7 research on the nonlinear dynamic system, Nayfeh and Mook ^[43] are the pioneers
 8 who adopt the harmonic balance method, and a more systematic introduction to
 9 this method can be found in the works of Krack and Gross ^[44].

10 Employing the harmonic balance method to solve Eq. (25), it is necessary to
 11 assume that the solutions of the Eq. (25) can be approximated by the following
 12 Fourier series

$$13 \quad \begin{cases} U_{k0} = A_{01}^{(k)} + \sum_{j=1}^N [B_{j1}^{(k)}(t) \cos(j\eta\tau) + C_{j1}^{(k)}(t) \sin(j\eta\tau)], \\ W_{k0} = A_{02}^{(k)} + \sum_{j=1}^N [B_{j2}^{(k)}(t) \cos(j\eta\tau) + C_{j2}^{(k)}(t) \sin(j\eta\tau)] \end{cases} \quad (26)$$

$$14 \quad A_{0i}^{(k)} = A_{0i}^{(k)}, A_{ji}^{(k)} = \sqrt{B_{ji}^{(k)2} + C_{ji}^{(k)2}}, i = 1, 2; j = 1, 2, \dots, N \quad (27)$$

15 where k is the longitudinal half wave number, j is the order of harmonic.
 16 $i=1$ represents the axial direction, while $i=2$ represents the radial direction.

17 For the investigation of the resonance, the frequencies of external excitation
 18 and temperature are both close to the fundamental frequency, i.e., $\omega_0 \approx \Omega = \omega_T$,
 19 and $\eta = \eta_T \approx 1$. Moreover, it can be inferred from Eqs. (8)~(15) that when the
 20 perturbed temperature ΔT_p equals to zero, the parametric excitation part of Eq.
 21 (25) will also be zero. and the parametrically excited vibration degenerates into
 22 nonlinear forced vibration, namely

$$23 \quad \ddot{\mathbf{q}} + \bar{\mathbf{C}}\dot{\mathbf{q}} + \bar{\mathbf{K}}_1\mathbf{q} + \bar{\mathbf{K}}_2(\mathbf{q})\mathbf{q} + \bar{\mathbf{K}}_3(\mathbf{q}, \mathbf{q})\mathbf{q} = \bar{\mathbf{F}} \cos(\eta\tau) \quad (28)$$

24 Substituting Eq. (26) into the corresponding nonlinear differential equations,
 25 sorting out the coefficient matrix of different harmonics, and then solving it by the
 26 arc length method ^[33], the periodic solution of the equation can be obtained.

6. Analysis and discussion

The discrete mode method is used to approximate the nonlinear vibration of the shell, in general, it is necessary to discuss the influence of the discrete mode number on the convergence of the solution.

6.1 Influence of discrete mode number

Before analyzing the influence of the discrete mode number, this subsection first analyzes the different influence between odd and even order modes.

(a)

(b)

(c)

(d)

(e)

Fig. 5 Frequency amplitude relationship of 1-order harmonics of different modes for

$$\alpha = 3.9022, \quad \Omega = \omega_0 = \omega_{12}/3$$

As shown in Fig. 5, when the axial excitation acting on both ends of the cylindrical shell is symmetrical, the axisymmetric vibration modes with even number would not be excited. It can be clearly seen that when the longitudinal half wave number m is even, the amplitude values given by the frequency amplitude response curve are all zero (as shown in Figs. 5(b) and 5(d)). Thus, the parameters α_2 and α_4 given in Table 2 which are the frequency ratio of even order mode to fundamental mode, will not be discussed and analyzed further.

Since the internal resonance condition is satisfied or nearly satisfied, it can be observed that there are many nonlinear peaks in the response curve, which reveals the complex energy transfer behavior, and more details in the parameter analysis will be discussed later. In addition, it is not difficult to find that the amplitude magnitude of the high-order mode is far less than that of low-order mode, which indicates that most energy of the shell vibration converges in the lower frequency region. Therefore, for the analysis of shells, it is essential to focus on the low order modes which possess smaller longitudinal half wave number. The case that the parameter equals to α_5 would not be analyzed further, which means this paper mainly focuses on the case that the structural parameter is somewhere around α_3 . In this situation, the frequency ratio of the fundamental mode to its same order mode satisfies the 3:1 internal resonance condition.

Fig. 6 Frequency amplitude relationship of 1-order harmonics of different modes with

$$\alpha = 3.9022 \quad \text{and} \quad \Omega = \omega_{12} = 3\omega_0$$

As shown in Fig. 5, it can be found that when the axial excitation frequency is close to the fundamental frequency, the low-frequency mode is directly excited, while the high-frequency mode is hardly to be excited. Figure 5(a) clearly shows that the low-order mode which has the largest axial displacement is dominant, while the high-order modes present a very significant difference in the amplitude magnitude. Interestingly, when the external excitation frequency is close to the high-frequency mode, as shown in Fig. 6 (a), it can be found that the response of the low-order mode is no longer dominated by the axial displacement, and the multiple peak phenomenon no longer appears. There is only a left-bent peak which shows the strong softening behavior. The reason is that when the length-diameter ratio is large, the radial displacement of the high-order mode is dominant [33], which means the most energy is distributed in the radial direction, and the energy of the high-order mode is easily transferred to the low-order mode. Figure 6(a) clearly shows that the radial displacement of low-order mode is largest, which is different from Fig. 5(a).

The previous analysis manifests that, for the same computational complexity, to keep only odd order modes in discrete modes is more efficient. Additionally, the influence of the discrete mode number on the structural response is analyzed in the following part.

(a)

(b)

(c)

(d)

(e)

(f)

Fig. 7 Influence of discrete mode number on frequency amplitude relationship of different harmonics with $\alpha = 3.9022$ and $m = 1$

It is not difficult to find that the dynamic response of the shell can be accurately described by using two odd order discrete modes, while only one odd order discrete mode should make the response curve tend to be the hardening behavior. Figure 7 also shows that when the condition of 3:1 internal resonance is satisfied approximately, the axial and radial motions of the shell are both dominated by the first and third harmonics which possess similar curve shapes in two directions and the different amplitude magnitude. Additionally, another distinction is that the

second harmonics possess different curve shapes in two directions, and the softening behavior is more obvious for the radial direction. Moreover, the energy distribution between two directions is different, i.e., the most energy of axial direction converges in the first harmonic which is the low frequency region, while the more energy of radial direction converges in the third harmonic which is the high frequency region.

6.2 Influence of structural parameters

In this subsection, the value of the length-diameter ratio which satisfy the 3:1 internal resonance is taken as the critical parameter, i.e., $\alpha = \alpha_3$, and the influence of the ratio on the response curve is discussed from the two cases that the ratio gradually decreases or increases. Meanwhile, the stability of the solution is determined, which is based on the eigenvalues of the Jacobian matrix, that is, if the real parts of all eigenvalues are negative, the solution is stable. If there is an eigenvalue with the positive real part, then the solution is unstable [33]. As shown in Figs. 8 and 9, the length-diameter ratio is slightly smaller or greater than the critical parameter α_3 , where the dash-dotted lines represent the unstable periodic solution, while the solid line represent the stable periodic solution. Moreover, it is similar to the previous discussion in Fig. 7, i.e., the shell displacement is dominated by the first and third harmonics, while the second harmonics are minor.

(a)

(b)

(c)

(d)

(e)

(f)

Fig. 8 Influence of decreasing structural parameters on frequency amplitude relationship

Figure 8 mainly shows the case that the structural parameters are slightly less than the critical parameter α_3 . In this situation, the shell response has three peaks named as the left peak, the middle peak and the right peak respectively, which is more clearly distinguished from Figs. 8(e) and 8(f) (i.e., the third harmonic). Obviously, the left is a typical softening peak with both stable and unstable parts, and there will be a jumping near the frequency where the stability changes. Based on this feature, it is not difficult to infer that for the first and second harmonics, the positions of the left and the middle are exchanged. The middle is a completely unstable peak, which is similar to the case of 2:1 internal resonance^[45], while the right is a completely stable peak. With the decreasing ratio, the stable region on the right side of the left peak will gradually decrease until it disappears, and the middle peak will gradually shift to the left until merge with the left peak, which will generate a new left peak, however, it is completely unstable. As the parameter decreases further, the new left peak will shrink between the tip and the valley, and generate an isolated bubble response which is completely unstable. As the ratio decreases further, the isolated bubble will gradually shrink until it disappears

1 completely, and so does the left peak. For this situation, there will be only a
 2 completely stable right peak.

3 Additionally, when the length-diameter ratio is in a certain frequency range,
 4 there is no stable solution (such as $\alpha = 3.900740$, the corresponding frequency
 5 range is about $[0.9995, 0.9998]$), which indicates that the chaotic response may
 6 occur in this range. As the ratio decreases, the parameter condition of internal
 7 resonance is no longer satisfied gradually, then the frequency domain of response
 8 will decrease gradually. It is necessary to state that, during the process of the
 9 parameter evolution, the stability of the right peak remains, but the amplitude
 10 changes.

(a)

(b)

(c)

(d)

(e)

(f)

Fig. 9 Influence of increasing structural parameters on frequency amplitude relationship

Figure 9 mainly shows the case that the structural parameters are slightly larger than the critical parameter α_3 . In this situation, the stable region of the left peak will gradually increase with the increasing ratio, and the unstable middle peak will gradually shift to the right and merge with the right peak. As the ratio increases, the middle peak and the right peak will generate an isolated triangular bubble response, where the bottom of the triangle is composed of the unstable middle peak, and most of the others are composed of the stable right peak. Obviously, the evolution process of the 3-order harmonic demonstrates this statement. Meanwhile, since there are stable periodic solutions in the isolated response, there is a phenomenon of coexistence of stable solutions in the corresponding frequency range, that is, one excitation frequency should correspond to two stable periodic solutions, and there are also some jumping phenomena. It is also worth pointed out that for the 1-order and 2-order harmonics, the isolated triangular bubble response may not be obvious due to the existence of the intersection points, however, those triangle bubbles cannot be traced by the arc length method directly, which indicates those bubbles are isolated with the main curves. In order to obtain the isolated bubbles, it is necessary to adopt the strategy mixing the perturbation method and arc length method. As the ratio increases further, the isolated triangular bubble will shrink until it disappears completely. Finally, only the left peak which characterizes the softening remains.

In summary, with the increasing ratio, the parameter condition of internal resonance is no longer satisfied gradually, however, it is different from the case of Fig. 8, due to the dominant left peak remains, the frequency domain of response would not decrease significantly.

6.3 Influence of temperature parameters

According to Eq. (3), there are three parameters related to temperature, namely the average temperature T_A , the perturbed temperature ΔT_p and the phase difference θ_0 between temperature and excitation. The influence of these parameters will be discussed in this subsection. The discussion above mentioned

1 manifests that the responses are dominated by the first and third order harmonics
 2 for the 3:1 internal resonance, and the shapes of the axial and radial response
 3 curves are similar, while the amplitude magnitudes are different. Therefore, this
 4 subsection only presents the first and third order harmonics of the radial
 5 displacement, and the length-diameter ratio is taken as $\alpha = 3.902195$.

8 **Fig. 10** Influence of average temperature on frequency amplitude relationship for $\Delta T_p = 0K$
 9 and $\theta_0 = 0$

10 As shown in Fig. 10, the influence of average temperature on frequency
 11 amplitude relationship is presented. Obviously, the higher the average
 12 temperature, the smaller the response amplitude. The reason is that the shear
 13 modulus is directly proportional to temperature which is shown in Eq. (8). Thus,
 14 the higher the temperature, the greater the shear modulus, which means the greater
 15 the stiffness of the shell, and the stronger the resistible capacity to the
 16 deformation, i.e., the smaller the response amplitude.

19 **Fig. 11** Influence of perturbed temperature on frequency amplitude relationship for $T_A = 297K$,
 20 $\theta_0 = 0$

Figure 11 shows the influence of the perturbed temperature on the vibration response. With the increasing perturbed temperature, the middle peak will gradually distorted and merges with the left peak, which is kind of similar to the situation with gradually decreasing length-diameter ratio (as shown in Fig. 8). However, there is no bubble response during the evolution process, and the right peak will change more significantly. Meanwhile, the softening characteristics of the cylindrical shell will be suppressed gradually with the increasing perturbed temperature, while the hardening characteristics will be enhanced significantly, which is more obvious for the first harmonic. When the perturbed temperature is large enough (for example, $\Delta T_p = 25K$, the black line in Fig. 11), the hyperelastic thin-walled cylindrical shell will present a typical hardening peak.

Fig. 12 Influence of phase difference on frequency amplitude relationship for $\Delta T_p = 0K$,

$$T_A = 297K$$

Figure 12 mainly shows the influence of the phase difference between the perturbed temperature and the external excitation on the shell response. Obviously, when the phase difference is π , the response presents the largest amplitude (the magenta lines), and the amplitude of the three peaks are enhanced significantly. That is, the shell response can be adjusted by controlling the phase difference between the excitation and the temperature.

In this subsection, it can be found that the influence of temperature parameters on the response is similar to that of the length-diameter ratio. However, the hardening characteristic caused by the increasing temperature makes some difference. The frequency amplitude response curve evolves more smoothly with the structural parameter compared with the temperature parameter. Thus, the

1 isolated bubbles only occur during the evolution process of the structural
2 parameter. Comparing Fig. 8(a) and Fig. 11(a), the evolution behavior of
3 frequency amplitude curve manifests that the temperature (especially the
4 perturbed temperature) will distort the response curve and lead to a more irregular
5 result. Meanwhile, the distortion will enhance the hardening significantly.

6 Briefly, the variation of the structural parameter and the temperature both play
7 an important role in the shell response, they bear relations as well as distinctions.
8 Therefore, it is necessary and essential to conduct the comprehensive analysis,
9 which leads to a better understanding and prediction for the nonlinear dynamic
10 behavior of the hyperelastic thin-walled cylindrical shell.

11 **7. Conclusions**

12 In this paper, the axisymmetric nonlinear vibration of the hyperelastic thin-walled
13 shell is investigated by the harmonic balance method and the arc length method,
14 and the 3:1 internal resonance of the shell is analyzed. Firstly, the system of
15 nonlinear governing differential equations with time-varying parameters are
16 derived based on the variational method. Then the periodic solution is calculated
17 via using the harmonic balance method and the arc length method, and the
18 stability is determined. Finally, the influences of structure and temperature
19 parameters on the evolution of nonlinear vibration are discussed. The mainly
20 conclusions are drawn as follows:

21 (1) For axisymmetric nonlinear vibrations of hyperelastic thin-walled cylindrical
22 shells, it is inaccurate to carry out the analysis with a single mode. Meanwhile,
23 since the even order modes will not be excited, it is more effective to discretize
24 the model with only the odd order mode.

25 (2) For hyperelastic thin-walled cylindrical shells with 3:1 internal resonance, the
26 length-diameter ratio has an extremely important impact on the shell resonance.
27 The reason is that this parameter plays a critical role in the frequency ratio which
28 is the necessary condition for internal resonance. Near the internal resonance
29 parameters, there are abundant nonlinear dynamic behaviors, including the

1 jumping, softening and hardening, and isolated bubble response. Specially, the left
2 peak will gradually shrink, and the isolated bubble occurs when the decreasing
3 length-diameter ratio is smaller than the critical parameter; while the right peak
4 will gradually shrink, and the isolated bubble occurs when the increasing length-
5 diameter ratio is larger than the critical parameter.

6 (3) The influence of temperature parameters on the nonlinear dynamic behavior of
7 the shell is similar to that of the length-diameter ratio. However, the bubble will
8 not occurs during the temperature parameter varies. In addition, the perturbed
9 temperature has a more obvious effect on the stability of the solution.

10 **Acknowledgements**

11 This work was supported by the National Natural Science Foundation of China (Nos. 11672069,
12 11672062, 11872145, 11902068), the Natural Science Foundation of Liaoning Province (grant
13 numbers 2020-BS-077), and the Programme of Introducing Talents of Discipline to Universities
14 Project (No. B08014).

15 **Compliance with ethical standards**

16 **Conflict of interest**

17 The authors declare that they have no conflict of interests.

18 **Data availability statement**

19 All data included in this study are available upon request by contact with the corresponding author.

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