New Private Types for the Cubic-Quartic Optical Solitons in Birefringent Fibers in Its Four Forms of Nonlinear Refractive Index

Emad H.M. Zahran

Benha University

Ahmet Bekir (✉ bekirahmet@gmail.com)

Eskisehir <https://orcid.org/0000-0001-9394-4681>

Research Article

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Posted Date: August 13th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-762672/v1>

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Version of Record: A version of this preprint was published at Optical and Quantum Electronics on November 2nd, 2021. See the published version at <https://doi.org/10.1007/s11082-021-03330-y>.

New private types for the cubic-quartic optical solitons in birefringent fibers in its four forms of nonlinear refractive index

Emad H.M. Zahran ¹, Ahmet Bekir ^{2,*}

¹Department of Mathematical and Physical Engineering, Benha University, Faculty of Engineering, Shubra, Egypt

²Neighbourhood of Akcaglan, Imarli Street, Number: 28/4, 26030, Eskisehir, Turkey

*Corresponding author E-mail: bekirahmet@gmail.com ; e_h_zahran@hotmail.com

ABSTRACT

From the point power of view of the extended simple equation method, multiple new private distinct types for the cubic-quartic optical soliton birefringent fibers with four forms nonlinear refractive index which have and play a vital effective effect in all modern telecommunications process have been extracted. The suggested method which continuously gives surprise accurate results for most nonlinear science problem has been implemented to extract multiple new accurate cubic-quartic soliton for the different forms of the NLSE which are, the cubic-quartic in polarization-preserving fibers with the kerr-low nonlinearity, quadratic-cubic law nonlinearity, parabolic law nonlinearity and non-local law nonlinearity. Actual comparison between the achieved results and that demonstrated by other authors has been established.

Keywords: The cubic-quartic optical solitons in birefringent fibers, the extended simple equation method; the optical soliton solutions.

1-Introduction

Recently, it is important to study the transition of light through optical fibers which play principal rule in all wireless communication, modern telecommunications, audio waves and asocial media process. The birefringence is often quantified as the maximum difference between refractive indices exhibited by the material in general Crystals with non-cubic crystal structures are often birefringent, as are plastics under mechanical stress. In related subject Birefringence is used in many optical devices. Liquid-crystal displays, which are used utilized in medical diagnostics. In fact the double refraction phenomenon depends on Birefringence, because when a light ray incident upon a birefringent material it will split into two rays which have slightly various paths under the polarization property. In the literature this effect has been observed in calcite, a crystal having one of the strongest birefringence by the Danish scientist Rasmus Bartholin [1]. Moreover, Augustin-Jean Fresnel at 19th described the phenomenon in terms of polarization he can understand that the light can be considered as a wave with field components in transverse polarization. In addition the birefringence for which the optical property of a material having a refractive index that depends on the polarization and propagation direction of light has been discussed by [2, 3]. Some studies have been demonstrated to investigate various forms of the NLSE which describes important nonlinear problems in different branches of sciences [4-14]. Recently, big number of authors discussed various forms of the cubic-quartic NLSE through their efforts to extract the soliton solutions

via different techniques [15-25]. In the last few decades little studies through good efforts of some authors to various special forms of NLSE for the cubic-quartic of the birefringence for which the optical property of a material having a refractive index that depends on the polarization and propagation direction of light have been investigated through [26-28].

This article focused on applying the ESEM [8-9], [14] to achieve new types of optical solitons to cubic-quartic NLSE with its special forms which mentioned in the abstract. This article is organized as follow, the introduction section which gives short survey of the forms NLSE as well as the methods that used to solve it, in section two we will give description of the ESEM and its application to find the soliton solution of the suggested model. In section three the application the ESEM schema to extract optical soliton of the cubic-quartic for the Kerr-Low NLSE in polarization-preserving fibers. In section four, the application the ESEM schema to extract optical soliton of the cubic-quartic for the quadratic-low NLSE in polarization-preserving fibers. In section five the application the ESEM schema to extract optical soliton of the cubic-quartic for the parabolic-low NLSE in polarization-preserving fibers. In section six the application the ESEM schema to extract optical soliton of the cubic-quartic for the Non-local-low NLSE in polarization-preserving fibers. In section seven brief conclusion of our work has been established.

2. Description of the ESEM

In overview, any nonlinear evolution equation can be written in the form,

$$R(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{tt}, \dots) = 0 \quad (1)$$

Where R is a function of $\psi(x, t)$ and its partial derivatives that involves the highest order derivatives and nonlinear terms, according to the transformation $\psi(x, t) = \psi(\zeta)$, $\zeta = x - C_0 t$ equation (1) can be reduced to the following ODE:

$$S(\psi, \psi', \psi'', \dots) = 0 \quad (2)$$

Where S is a function in $\psi(\zeta)$ and its total derivatives, while $' = \frac{d}{d\zeta}$

The constructed solution according to this method is:

$$\psi(\zeta) = \sum_{i=-N}^N A_i \phi^i(\zeta). \quad (3)$$

Where the positive integer N in Eq. (3) can be located by balancing the highest order derivative term and the nonlinear term, while the arbitrary constants A_i could be calculated later, the function $\phi(\zeta)$ satisfies the following new ansatz equation

$$\phi'(\zeta) = a_0 + a_1 \phi + a_2 \phi^2 \quad (4)$$

Where a_0, a_1 and a_2 other arbitrary constants which admit these two cases;

(1) If $a_1 = a_3 = 0$ it will transform to the Riccati equation [18–19], which has the following solutions;

$$\phi(\zeta) = \frac{\sqrt{a_0 a_2}}{a_2} \tan(\sqrt{a_0 a_2} (\zeta + \zeta_0)), a_0 a_2 \neq 0 \quad (5)$$

$$\phi(\zeta) = \frac{\sqrt{-a_0 a_2}}{a_2} \tanh(\sqrt{-a_0 a_2} \zeta - \frac{\rho \ln \zeta_0}{2}), a_0 a_2 \neq 0, \zeta \neq 0, \rho = \pm 1 \quad (6)$$

(2) If $a_0 = a_3 = 0$, it will transform the Bernoulli equation [19], which has the following solutions;

$$\varphi(\zeta) = \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0 \quad (7)$$

$$\varphi(\zeta) = \frac{-a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 + a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0 \quad (8)$$

And the general solution to ansatz equation (4) is as follows:

$$\varphi(\zeta) = -\frac{1}{a_2} \left(a_1 - \sqrt{4a_1a_2 - a_1^2} \tan \left(\frac{\sqrt{4a_1a_2 - a_1^2}}{2} (\zeta + \zeta_0) \right) \right), 4a_1a_2 \neq a_1^2, a_2 \neq 0, \quad (9)$$

$$\varphi(\zeta) = \frac{1}{a_2} \left(a_1 + \sqrt{4a_1a_2 - a_1^2} \tanh \left(\frac{\sqrt{4a_1a_2 - a_1^2}}{2} (\zeta + \zeta_0) \right) \right), 4a_1a_2 \neq a_1^2, a_2 \neq 0. \quad (10)$$

Where ζ_0 denotes to integration constancy.

Finally, via inserting equation (3) into equation (4) and equating the coefficients of different powers of φ^i to zero, we get a system of algebraic equations, by solving it we obtain the values of the unknown variable mentioned in these equations. Moreover, inserting these achieved variables into equation (4) then the aimed solutions have been extracted.

3. The Cubic-Quartic NLSE for the Kerr-Low nonlinearity using the ESEM

In the framework of [27-28] the cubic-quartic for the Kerr-low NLSE in polarization-preserving fibers can be proposed in the form,

$$iQ_t + iaQ_{xxx} + bQ_{xxxx} + c|Q|^{2m}Q = 0 \quad (11)$$

Where a, b and c denote respectively to the 3-th order dispersion index, the 4-th order dispersion index, Kerr-Low refractive coefficient while $i = \sqrt{-1}$.

The cubic-quartic NLSE in birefringent fiber for Kerr-Low are,

$$\begin{aligned} iu_t + ia_1u_{xxx} + b_1u_{xxxx} + [c_1|u|^2 + d_1|v|^2]u &= 0 \\ iv_t + ia_2v_{xxx} + b_2v_{xxxx} + [c_2|v|^2 + d_2|u|^2]v &= 0 \end{aligned} \quad (12)$$

In the framework of the ESEM the solution is,

$$u(x, t) = \psi_1(\zeta) e^{i\eta_1(x, t)}, \zeta = k_1x + w_1t, \eta = q_1x + \delta_1t \quad (13)$$

$$v(x, t) = \psi_2(\zeta) e^{i\eta_2(x, t)}, \zeta = k_2x + w_2t, \eta = q_2x + \delta_2t \quad (14)$$

Hence,

$$u_t = i\delta_1\psi_1 e^{i\eta_1} + w_1\psi_1' e^{i\eta_1} \quad (15)$$

$$u_x = iq_1\psi_1 e^{i\eta_1} + k_1\psi_1' e^{i\eta_1} \quad (16)$$

$$u_{xx} = -q_1^2\psi_1 e^{i\eta_1} + 2ik_1q_1\psi_1' e^{i\eta_1} + k_1^2\psi_1'' e^{i\eta_1} \quad (17)$$

$$u_{xxx} = -iq_1^3\psi_1 e^{i\eta_1} - 3k_1q_1^2\psi_1' e^{i\eta_1} + 3iq_1k_1^2\psi_1'' e^{i\eta_1} + k_1^3\psi_1''' e^{i\eta_1} \quad (18)$$

$$u_{xxxx} = q_1^4 \psi_1 e^{i\eta_1} - 4ik_1 q_1^3 \psi_1' e^{i\eta_1} - 6k_1^2 q_1^2 \psi_1'' e^{i\eta_1} + 4iq_1 k_1^3 \psi_1''' e^{i\eta_1} + k_1^4 \psi_1'''' e^{i\eta_1} \quad (19)$$

$$|u^2| = \psi_1^2 \quad (20)$$

$$v_t = i\delta_2 \psi_2 e^{i\eta_2} + w_2 \psi_2' e^{i\eta_2} \quad (21)$$

$$v_x = iq_2 \psi_2 e^{i\eta_2} + k_2 \psi_2' e^{i\eta_2} \quad (22)$$

$$v_{xx} = -q_2^2 \psi_2 e^{i\eta_2} + 2ik_2 q_2 \psi_2' e^{i\eta_2} + k_2^2 \psi_2'' e^{i\eta_2} \quad (23)$$

$$v_{xxx} = -iq_2^3 \psi_2 e^{i\eta_2} - 3k_2 q_2^2 \psi_2' e^{i\eta_2} + 3iq_2 k_2^2 \psi_2'' e^{i\eta_2} + k_2^3 \psi_2''' e^{i\eta_2} \quad (24)$$

$$v_{xxxx} = q_2^4 \psi_2 e^{i\eta_2} - 4ik_2 q_2^3 \psi_2' e^{i\eta_2} - 6k_2^2 q_2^2 \psi_2'' e^{i\eta_2} + 4iq_2 k_2^3 \psi_2''' e^{i\eta_2} + k_2^4 \psi_2'''' e^{i\eta_2} \quad (25)$$

$$|v^2| = \psi_2^2 \quad (26)$$

Via inserting the relations (13) and (15-20) into the first part of Eq. (12) and the relations (14) and (21-26) into the second part of Eq. (12), the following real and imaginary parts must be emerged respectively,

$$\text{Re} \Rightarrow b_j k_j^4 \psi_j'''' - (3a_j q_j k_j^2 + 6b_j q_j^2 k_j^2) \psi_j'' + (b_j q_j^4 + 6a_j q_j^3 - \delta_j) \psi_j + (c_j + d_j) \psi_j^3 = 0. \quad (27)$$

$$\text{Im} \Rightarrow (a_j k_j^3 + 4b_j q_j k_j^3) \psi_j''' - (3k_j a_j q_j^2 + 4k_j b_j q_j^3 - w_j) \psi_j' = 0. \quad (28)$$

Equation (28) implies $a_j = -4b_j q_j$, $w_j = 3k_j a_j q_j^2 + 4k_j b_j q_j^3$, hence under this constrain Eq.

(27) will be converted to,

$$b_j k_j^4 \psi_j'''' + 6b_j q_j^2 k_j^2 \psi_j'' - (3b_j q_j^4 + \delta_j) \psi_j + (c_j + d_j) \psi_j^3 = 0. \quad (29)$$

Now, via integrating Eq. (29) with respect to ζ it becomes,

$$b_j k_j^4 \psi_j''' + 6b_j q_j^2 k_j^2 \psi_j' - \frac{1}{2} (3b_j q_j^4 + \delta_j) \psi_j^2 + \frac{1}{4} (c_j + d_j) \psi_j^4 = 0. \quad (30)$$

Equation (30) represents either the first or the second parts of equation (12) which are the same, for similarity we will implement the solution for the first part $j = 1$ which is,

$$b_1 k_1^4 \psi_1''' + 6b_1 q_1^2 k_1^2 \psi_1' - \frac{1}{2} (3b_1 q_1^4 + \delta_1) \psi_1^2 + \frac{1}{4} (c_1 + d_1) \psi_1^4 = 0. \quad (31)$$

Via balancing ψ_1''', ψ_1^4 at Eq. (31) leads to $4N = N + 3$ which implies that $N = 1$, hence the solution is,

$$\psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi \quad (32)$$

Where $\varphi' = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3$

Case 1: The first family, in which $a_1 = a_3 = 0 \Rightarrow \varphi' = a_0 + a_2 \varphi^2$, consequently

$$\psi_1' = -\frac{a_0 A_{-1}}{\varphi^2} + A_1 a_0 + A_1 a_2 \varphi^2 - a_2 A_{-1} \quad (33)$$

$$\psi_1''' = \frac{-6a_0^3 A_{-1}}{\varphi^4} - \frac{8a_0^2 a_2 A_{-1}}{\varphi^2} - 2A_{-1} a_0 a_2^2 + 2A_1 a_2 a_0^2 + 8A_1 a_0 a_2^2 \varphi^2 + 6A_1 a_2^3 \varphi^4 \quad (34)$$

Via inserting $\psi_1''', \psi_1', \psi_1^2$ and ψ_1^4 into equation (31), collecting and equating the coefficients of various powers of φ^i to zero leads to system of equations, by solving it we get these results,

$$a_0 = \frac{-\sqrt{3A_1} q_1}{2k_1 \sqrt{A_1}}, a_2 = \frac{\sqrt{3A_1} q_1}{2k_1 \sqrt{A_1}}, \delta_1 = \frac{3 \left(\frac{-6\sqrt{3A_1} b_1 k_1 q_1^3}{\sqrt{A_1}} - A_1 b_1 q_1^4 \right)}{A_1}, c_1 = \frac{-d_1 A_1^3 - \left(\frac{9\sqrt{3} A_1^{\frac{3}{2}} b_1 k_1 q_1^3}{A_1^{\frac{3}{2}}} \right)}{A_1^3}, A_0 = 0 \quad (35)$$

$$a_0 = \frac{\sqrt{3A_1} q_1}{2k_1 \sqrt{A_1}}, a_2 = \frac{-\sqrt{3A_1} q_1}{2k_1 \sqrt{A_1}}, \delta_1 = \frac{3 \left(\frac{6\sqrt{3A_1} b_1 k_1 q_1^3}{\sqrt{A_1}} - A_1 b_1 q_1^4 \right)}{A_1}, c_1 = \frac{-d_1 A_1^3 + \left(\frac{9\sqrt{3} A_1^{\frac{3}{2}} b_1 k_1 q_1^3}{A_1^{\frac{3}{2}}} \right)}{A_1^3}, A_0 = 0 \quad (36)$$

$$a_0 = \frac{-3q_1^2}{a_2 k_1^2}, \delta_1 = \frac{-3(2a_2 b_1 k_1^2 q_1^2 + A_1 b_1 q_1^2)}{A_1}, c_1 = \frac{-d_1 A_1^3 - 24a_2^3 b_1 q_1^4}{A_1^3}, A_{-1} = 0, A_0 = 0 \quad (37)$$

$$a_0 = \frac{-3q_1^2}{a_2 k_1^2}, \delta_1 = \frac{-3(-2a_2 b_1 k_1^2 q_1^2 + A_{-1} b_1 q_1^2)}{A_{-1}}, c_1 = \frac{-d_1 A_{-1}^3 + 24a_0^3 b_1 q_1^4}{A_{-1}^3}, A_1 = 0, A_0 = 0 \quad (38)$$

These 4-differents results will achieve 4-various types of solutions, from which we will plot the extracted solutions for the first and the third results.

Firstly, for the first result Eq. (35):

$$a_0 = \frac{-\sqrt{3A_1} q_1}{2k_1 \sqrt{A_1}}, a_2 = \frac{\sqrt{3A_1} q_1}{2k_1 \sqrt{A_1}}, \delta_1 = \frac{3 \left(\frac{-6\sqrt{3A_1} b_1 k_1 q_1^3}{\sqrt{A_1}} - A_1 b_1 q_1^4 \right)}{A_1}, c_1 = \frac{-d_1 A_1^3 - \left(\frac{9\sqrt{3} A_1^{\frac{3}{2}} b_1 k_1 q_1^3}{A_1^{\frac{3}{2}}} \right)}{A_1^3}, A_0 = 0$$

This result can be simplified to be,

$$A_1 = 1, A_0 = 0, A_1 = 1, a_0 = -0.9, a_2 = 0.9, k_1 = b_1 = d_1 = q_1 = C_0 = 1, c_1 = -16.6, \delta_1 = -34.2, \zeta_0 = 2, \rho = -1$$

From the point of view of the proposed method the solution is,

$$\begin{aligned} \varphi(\zeta) &= \frac{\sqrt{-a_0 a_2}}{a_2} \tanh(\sqrt{-a_0 a_2} \zeta - \frac{\rho \ln \zeta_0}{2}), a_0 a_2 \neq 0, \zeta \neq 0, \rho = \pm 1 \\ \varphi(\zeta) &= \tanh(0.9\zeta + 0.3) \\ \varphi(\zeta) &= \tanh(0.9x - 7.2t + 0.3) \end{aligned} \quad (39)$$

$$\begin{aligned} \psi_1(\zeta) &= \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi \\ \psi_1(\zeta) &= \coth(0.9x - 7.2t + 0.3) + \tanh(0.9x - 7.2t + 0.3) \end{aligned} \quad (40)$$

$$\begin{aligned} u(x, t) &= \psi_1(\zeta) e^{i\eta_1(x, t)}, \zeta = k_1 x + w_1 t, \eta_1 = q_1 x + \delta_1 t \\ u(x, t) &= (\coth[0.9x - 7.2t + 0.3] + \tanh[0.9x - 7.2t + 0.3]) e^{i(x - 34.2t)} \end{aligned} \quad (41)$$

$$\text{Re } u = (\coth[0.9x - 7.2t + 0.3] + \tanh[0.9x - 7.2t + 0.3]) \times \cos(x - 34.2t) \quad (42)$$

$$\text{Im } u = (\coth[0.9x - 7.2t + 0.3] + \tanh[0.9x - 7.2t + 0.3]) \times \sin(x - 34.2t) \quad (43)$$

Fig.1 The cubic-quartic soliton of the Re. part of kerr-low Eq.(42) in **2D** and **3D** with values: $w=-8$
 $A_{-1}=1, A_0=0, A_1=1, a_0=-0.9, a_2=0.9, k_1=b_1=d_1=q_1=C_0=1, c_1=-16.6, \delta_1=-34.2, \zeta_0=2, \rho=-1, t=0.1$ for 2D

Fig.2 The cubic-quartic soliton of the Im. part of kerr-low Eq.(43) in **2D** and **3D** with values: $w=-8$
 $A_{-1}=1, A_0=0, A_1=1, a_0=-0.9, a_2=0.9, k_1=b_1=d_1=q_1=C_0=1, c_1=-16.6, \delta_1=-34.2, \zeta_0=2, \rho=-1, t=0.1$ for 2D

Secondly, for the third result Eq. (36):

$$a_0 = \frac{-3q_1^2}{a_2k_1^3}, \delta_1 = \frac{-3k_1b_1q_1^2(-12a_2 + A_1q_1^2)}{A_1}, c_1 = \frac{-d_1A_1^3 - 24a_2^3b_1q_1^4}{A_1^3}, A_{-1} = 0, A_0 = 0$$

This result can be simplified to be,

$$A_{-1}=A_0=0, A_1=1, a_0=0.9+0.13i, a_2=0.3+0.01i, k_1=b_1=d_1=q_1=c_1=C_0=1, \delta_1=-4.8-0.6i, \zeta_0=2, \rho=-1$$

From the point of view of the proposed method the solution is,

$$\varphi(\zeta) = \frac{\sqrt{a_0a_2}}{a_2} \tan(\sqrt{a_0a_2}(\zeta + \zeta_0)), a_0a_2 \neq 0$$

$$\varphi(\zeta) = 5.7 \tan[1.7x - 13.6t + 1.7] \quad (44)$$

$$\psi(\zeta) = 5.7 \tan[1.7x - 13.6t + 1.7] \quad (45)$$

$$u(x, t) = 5.7 \tan[1.7x - 13.6t + 1.7] \times e^{i[x - (4.8 + 0.6i)t]} \quad (46)$$

$$\text{Re } u = 5.7 \tan[1.7x - 13.6t + 1.7] \times e^{0.6t} \times \cos(x - 4.8t) \quad (47)$$

$$\text{Im } u = 5.7 \tan[1.7x - 13.6t + 1.7] \times e^{0.6t} \times \sin(x - 4.8t) \quad (48)$$

Fig.3 The cubic-quartic soliton of the Re. part of kerr-low Eq.(47) in **2D** and **3D** with values: $w_l = -8$
 $A_{-1} = A_0 = 0, A_1 = 1, |a_0| = 9.6, |a_2| = 0.3, k_1 = b_1 = d_1 = q_1 = \zeta_0 = C_0 = 1, c_1 = -25, \delta_1 = -4.8 - 0.6i, t = 0.1$ for **2D**

Fig.4 The cubic-quartic soliton of the Im. part of kerr-low Eq.(48) in **2D** and **3D** with values: $w_l = -8$
 $A_{-1} = A_0 = 0, A_1 = 1, |a_0| = 9.6, |a_2| = 0.3, k_1 = b_1 = d_1 = q_1 = \zeta_0 = C_0 = 1, c_1 = -25, \delta_1 = -4.8 - 0.6i, t = 0.1$ for **2D**

Via the same manner, we can plot the other two cases.

Case 2: The second family, in which $a_0 = a_3 = 0 \Rightarrow \varphi' = a_1\varphi + a_2\varphi^2$, consequently

$$\psi_1' = A_1 a_2 \varphi^2 + a_1 A_1 \varphi - \frac{A_{-1} a_1}{\varphi} - A_{-1} a_2, \quad (49)$$

$$\psi_1''' = 6A_1 a_2^3 \varphi^4 + 12a_1 A_1 a_2^2 \varphi^3 + 7A_1 a_1^2 a_2 \varphi^2 + A_1 a_1^3 \varphi - A_{-1} a_2 a_1^2 - \frac{a_1^3 A_{-1}}{\varphi}, \quad (50)$$

Via inserting $\psi_1''', \psi_1' \psi_1^2$ and ψ_1^4 into equation (31), collecting and equating the coefficients of various powers of φ^i to zero implies a system of equations, by solving it many solutions will be achieved from which only 8-results are suitable while the remaining are refused,

$$a_1 = \frac{2\sqrt{3}q_1}{k_1}, a_2 = \frac{\sqrt{3}A_1 q_1}{A_0 k_1}, \delta_1 = -3 \left(\frac{12\sqrt{3}b_1 k_1 q_1^3 + A_0 b_1 q_1^4}{A_0} \right), c_1 = -d_1 - \frac{72b_1 q_1^3 \sqrt{3}}{\sqrt{k_1} A_0^3}, A_{-1} = 0 \quad (51)$$

$$a_1 = \frac{-2\sqrt{3}q_1}{k_1}, a_2 = \frac{-\sqrt{3}A_1 q_1}{A_0 k_1}, \delta_1 = -3 \left(\frac{-12\sqrt{3}b_1 k_1 q_1^3 + A_0 b_1 q_1^4}{A_0} \right), c_1 = -d_1 + \frac{72b_1 q_1^3 \sqrt{3}}{\sqrt{k_1} A_0^3}, A_{-1} = 0 \quad (52)$$

$$a_1 = \frac{i q_1 \sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1 q_1^4, c_1 = -d_1 \quad (53)$$

$$a_1 = \frac{-i q_1 \sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1 q_1^4, c_1 = -d_1 \quad (54)$$

$$a_1 = \frac{-i q_1 \sqrt{6}}{k_1}, \delta_1 = -3b_1 q_1^4, c_1 = -d_1, A_1 = 0 \quad (55)$$

$$a_1 = \frac{iq_1\sqrt{6}}{k_1}, \delta_1 = -3b_1q_1^4, c_1 = -d_1, A_1 = 0 \quad (56)$$

$$a_1 = \frac{iq_1\sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1q_1^4, c_1 = -d_1, A_{-1} = 0 \quad (57)$$

$$a_1 = \frac{-iq_1\sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1q_1^4, c_1 = -d_1, A_{-1} = 0 \quad (58)$$

From which we can get 8-various solutions, we will plot some cases say, the first, the fifth and the eight results.

For the 1-st result which is,

$$a_1 = \frac{2\sqrt{3}q_1}{k_1}, a_2 = \frac{\sqrt{3}A_1q_1}{A_0k_1}, \delta_1 = -3\left(\frac{12\sqrt{3}b_1k_1q_1^3 + A_0b_1q_1^4}{A_0}\right), c_1 = -d_1 - \frac{72b_1q_1^3\sqrt{3}}{\sqrt{k_1}A_0^3}, A_{-1} = 0$$

This result can be simplified to be,

$$A_{-1} = 0, A_0 = 3, A_1 = 1, a_1 = 3.5, a_2 = 0.6, k_1 = q_1 = d_1 = \zeta_0 = 1, c_1 = -5.6, \delta_1 = -23.8$$

From the point of view of the suggested method the solution is,

$$\begin{aligned} \varphi(\zeta) &= \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0 \\ \varphi(\zeta) &= \frac{3.5 \text{Exp}[3.5(x - 8t + 1)]}{1 - 0.6 \text{Exp}[3.5(x - 8t + 1)]} \end{aligned} \quad (59)$$

$$\psi_1(\zeta) = 3 + \frac{3.5 \text{Exp}[3.5(x - 8t + 1)]}{1 - 0.6 \text{Exp}[3.5(x - 8t + 1)]} \quad (60)$$

$$\begin{aligned} u(x, t) &= \psi_1(\zeta) e^{i\eta_1(x, t)}, \zeta = k_1x + w_1t, \eta_1 = q_1x + \delta_1t \\ u(x, t) &= \left(3 + \frac{3.5 \text{Exp}[3.5(x - 8t + 1)]}{1 - 0.6 \text{Exp}[3.5(x - 8t + 1)]} \right) e^{i(x - 23.8t)} \end{aligned} \quad (61)$$

$$\text{Re } u = \left(3 + \frac{3.5 \text{Exp}[3.5(x - 8t + 1)]}{1 - 0.6 \text{Exp}[3.5(x - 8t + 1)]} \right) \times \cos(x - 23.8t) \quad (62)$$

$$\text{Im } u = \left(3 + \frac{3.5 \text{Exp}[3.5(x - 8t + 1)]}{1 - 0.6 \text{Exp}[3.5(x - 8t + 1)]} \right) \times \sin(x - 23.8t) \quad (63)$$

Fig.5 The cubic-quartic soliton of the Re. part of kerr-low Eq.(62) in **2D** and **3D** with values: $w_1=8$

$$A_{-1}=0, A_0=3, A_1=1, a_1=3.5, a_2=0.6, k_1=q_1=d_1=\zeta_0=1, c_1=-5.6, \delta_1=-23.8, t=0.1 \text{ for } 2D$$

Fig.6 The cubic-quartic soliton of the Im. part of kerr-low Eq.(63) in **2D** and **3D** with values: $w_1=8$

$$A_{-1}=0, A_0=3, A_1=1, a_1=3.5, a_2=0.6, k_1=q_1=d_1=\zeta_0=1, c_1=-5.6, \delta_1=-23.8, t=0.1 \text{ for } 2D$$

For the fifth result which is, $a_1 = \frac{-iq_1\sqrt{6}}{k_1}, \delta_1 = -3b_1q_1^4, c_1 = -d_1, A_1 = 0,$

This result can be simplified to be,

$$A_{-1}=1, A_0=1, A_1=0, a_1=-2.4i, a_2=0, k_1=w_1=q_1=d_1=\zeta_0=c_0=1, c_1=-1, \delta_1=-3$$

$$\varphi(\zeta) = \frac{-a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 + a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0$$

$$\varphi(\zeta) = -2.4i \text{Exp}[-2.4i(x - 8t + 1)] \quad (64)$$

$$\psi_1(\zeta) = \frac{1}{-2.4i \text{Exp}[-2.4i(x - 8t + 1)]} + 1 \quad (65)$$

$$\psi_1(\zeta) = 0.4i \text{Exp}[2.4i(x - 8t + 1)] + 1 \quad (66)$$

$$u(x, t) = \psi_1(\zeta) e^{i\eta_1(x, t)}, \zeta = k_1x + w_1t, \eta = q_1x + \delta_1t$$

$$u(x, t) = ([1 - 0.4 \sin(2.4x - 19.2t + 2.4)] + 0.4i \cos(2.4x - 19.2t + 2.4)) e^{i(x - 3t)} \quad (67)$$

$$\text{Re } u = ((1 - 0.4 \sin[2.4x - 19.2t + 2.4]) \cos[x - 3t] - 0.4 \cos[2.4x - 19.2t + 2.4] \sin[x - 3t]) \quad (68)$$

$$\text{Im } u = ((1 - 0.4 \sin[2.4x - 19.2t + 2.4]) \sin[x - 3t] + 0.4 \cos[2.4x - 19.2t + 2.4] \cos[x - 3t]) \quad (69)$$

Fig.7 The cubic-quartic soliton of the Re. part of kerr-low Eq.(68) in **2D** and **3D** with values: $w_1=8$

$$A_{-1}=1, A_0=1, A_1=0, a_1=-2.4i, a_2=0, k_1=q_1=d_1=\zeta_0=1, c_1=-1, \delta_1=-3, t=0.1 \text{ for } 2D$$

Fig.8 The cubic-quartic soliton of the Im. part of kerr-low Eq.(69) in **2D** and **3D** with values: $w_1=-8$

$$A_{-1}=1, A_0=1, A_1=0, a_1=-2.4i, a_2=0, k_1=q_1=d_1=\zeta_0=1, c_1=-1, \delta_1=-3, t=0.1 \text{ for } 2D$$

For the 8-th result which is, $a_1 = \frac{-iq_1\sqrt{6}}{k_1}, a_2=0, \delta_1=-3b_1q_1^4, c_1=-d_1, A_{-1}=0,$

This result can be simplified to be,

$$A_{-1}=0, A_0=1, A_1=1, a_1=2.4i, a_2=0, k_1=q_1=d_1=\zeta_0=1, c_1=-1, \delta_1=-3$$

$$\varphi(\zeta) = \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0$$

$$\varphi(\zeta) = 2.4i \text{Exp}[2.4i(x - 8t + 1)] \quad (70)$$

$$\psi_1(\zeta) = [1 - 2.4 \sin(2.4x - 19.2t + 2.4)] + 2.4i \cos(2.4x - 19.2t + 2.4) \quad (71)$$

$$u(x, t) = ([1 - 2.4 \sin(2.4x - 19.2t + 2.4)] + 2.4i \cos(2.4x - 19.2t + 2.4)) e^{i(x-3t)} \quad (72)$$

$$\text{Re } u = ([1 - 2.4 \sin(2.4x - 19.2t + 2.4)] \times \cos(x - 3t) - 2.4 \cos(2.4x - 19.2t + 2.4) \times \sin(x - 3t)) \quad (73)$$

$$\text{Im } u = (2.4 \cos(2.4x - 19.2t + 2.4) \times \cos(x - 3t) + 2.4([1 - 2.4 \sin(2.4x - 19.2t + 2.4)] \times \sin(x - 3t))) \quad (74)$$

Fig.9 The cubic-quartic soliton of the Re. part of kerr-low Eq.(73) in **2D** and **3D** with values: $w_1=-8$

$$A_{-1}=0, A_0=1, A_1=1, a_1=2.4i, a_2=0, k_1=q_1=d_1=\zeta_0=1, c_1=-1, \delta_1=-3, t=0.1 \text{ for } 2D$$

Fig.10 The cubic-quartic soliton of the Im. part of kerr-low Eq.(74) in **2D** and **3D** with values: $w_1=-8$

values: $A_{-1}=0, A_0=1, A_1=1, a_1=2.4i, a_2=0, k_1=q_1=d_1=\zeta_0=1, c_1=-1, \delta_1=-3, t=0.1 \text{ for } 2D$

In the same manner, we can plot the other cases.

4. The Cubic-Quartic NLSE for the Quadratic-Low nonlinearity using the ESEM

The cubic-quartic NLSE in polarization-preserving fibers with quadratic-cubic law can be proposed in the form,

$$iQ_t + iaQ_{xxx} + bQ_{xxxx} = (c|Q| + d|Q|^2)Q \quad (75)$$

The cubic-quartic NLSE in birefringent fiber for quadratic-cubic law is,

$$iu_t + ia_1u_{xxx} + b_1u_{xxxx} + c_1u\sqrt{|u|^2 + |v|^2 + uv^* + vu^*} + (d_1|u|^2 + e_1|v|^2)u = 0 \quad (76)$$

$$iv_t + ia_2v_{xxx} + b_2v_{xxxx} + c_2v\sqrt{|v|^2 + |u|^2 + uv^* + vu^*} + (d_2|v|^2 + e_2|u|^2)v = 0 \quad (77)$$

Via inserting the relations (13-26) into equation (76) and (77) will transform them to the following real and imaginary equations respectively:

$$\text{Re} \Rightarrow b_j k_j^4 \psi_j''' - (3a_j q_j k_j^2 + 6k_j^2 q_j^2) \psi_j'' + (b_j q_j^4 + a_j q_j^3 - \delta_j) \psi_j + 2c_j \psi_j^2 + (d_j + e_j) \psi_j^3 = 0 \quad (78)$$

$$\text{Im} \Rightarrow k_j^3 (a_j + 4b_j q_j) \psi_j''' + (w_j - 3a_j k_j q_j^2 - 4k_j q_j^3) \psi_j' = 0 \quad (79)$$

Eq. (79) implies $a_j = -4b_j q_j$, $w_j = 3a_j q_j^2 + 4b_j q_j^3$, hence under this constrain Eq.(78) will be converted to,

$$b_j k_j^4 \psi_j''' + 6b_j k_j^2 q_j^2 \psi_j'' - (3b_j q_j^4 + \delta_j) \psi_j + 2c_j \psi_j^2 + (d_j + e_j) \psi_j^3 = 0 \quad (80)$$

Equation (80) represents either the first part Eq.(76) or the second part Eq.(77) which is the same, for similarity we will implement the solution for the first part $j = 1$ which is,

$$b_1 k_1^4 \psi_1''' + 6b_1 k_1^2 q_1^2 \psi_1'' - (3b_1 q_1^4 + \delta_1) \psi_1 + 2c_1 \psi_1^2 + (d_1 + e_1) \psi_1^3 = 0 \quad (81)$$

Now, via integrating Eq. (81) with respect to ζ it becomes,

$$b_1 k_1^4 \psi_1''' + 6b_1 k_1^2 q_1^2 \psi_1' - \frac{1}{2} (3b_1 q_1^4 + \delta_1) \psi_1^2 + \frac{2}{3} c_1 \psi_1^3 + \frac{1}{4} (d_1 + e_1) \psi_1^4 = 0 \quad (82)$$

Via balancing ψ_1''', ψ_1^4 at Eq.(82) leads to $4N = N + 3$ which implies that $N = 1$, hence the solution is:

$$\psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi \quad (83)$$

Where $\varphi' = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3$

Case 1: The first family, in which $a_1 = a_3 = 0 \Rightarrow \varphi' = a_0 + a_2 \varphi^2$, consequently

$$\psi_1' = -\frac{a_0 A_{-1}}{\varphi^2} + A_1 a_0 + A_1 a_2 \varphi^2 - a_2 A_{-1} \quad (84)$$

$$\psi_1''' = \frac{-6a_0^3 A_{-1}}{\varphi^4} - \frac{8a_0^2 a_2 A_{-1}}{\varphi^2} - 2A_{-1} a_0 a_2^2 + 2A_1 a_2 a_0^2 + 8A_1 a_0 a_2^2 \varphi^2 + 6A_1 a_2^3 \varphi^4 \quad (85)$$

Via inserting $\psi_1''', \psi_1', \psi_1^2$ and ψ_1^4 into equation (82), collecting and equating the coefficients of various powers of φ^i to zero leads to system of equations from which we construct that there are no solutions achieved because either $a_0 = 0$ or $a_2 = 0$ or both or $A_{-1} = A_1 = 0$.

Case 2: The second family, in which $a_0 = a_3 = 0 \Rightarrow \varphi' = a_1 \varphi + a_2 \varphi^2$, consequently

$$\psi_1' = A_1 a_2 \varphi^2 + a_1 A_1 \varphi - \frac{A_{-1} a_1}{\varphi} - A_{-1} a_2 \quad (86)$$

$$\psi_1''' = 6A_1 a_2^3 \varphi^4 + 12a_1 A_1 a_2^2 \varphi^3 + 7A_1 a_1^2 a_2 \varphi^2 + A_1 a_1^3 \varphi - A_{-1} a_2 a_1^2 - \frac{a_1^3 A_{-1}}{\varphi}$$

Via inserting $\psi_1''', \psi_1', \psi_1^2, \psi_1^3$ and ψ_1^4 into equation (82), collecting and equating the coefficients of various powers of φ^i to zero leads to system of equations which proposed big number of very complicated solutions from which only 12-solutions are valued and the other achieved solutions are refused because either $a_1 = 0$ or $A_{-1} = A_1 = 0$, we will choose only 8-solutions of them where the remaining are very long which are,

$$A_{-1} = 0, a_1 = \frac{i\sqrt{6} q_1}{k_1}, a_2 = \frac{i\sqrt{6} A_1 q_1}{A_0 k_1}, b_1 = \frac{-iA_0^3 (d_1 + e_1)}{144\sqrt{6} k_1 q_1^3}, c_1 = -\frac{3}{4} A_0 (d_1 + e_1), \quad (87)$$

$$\delta_1 = \frac{i[144iA_0^2 k_1 (d_1 + e_1) + \sqrt{6} A_0^3 q_1 (d_1 + e_1)]}{288k_1}.$$

$$A_{-1} = 0, a_1 = \frac{-i\sqrt{6} q_1}{k_1}, a_2 = \frac{-i\sqrt{6} A_1 q_1}{A_0 k_1}, b_1 = \frac{-iA_0^3 (d_1 + e_1)}{144\sqrt{6} k_1 q_1^3}, c_1 = -\frac{3}{4} A_0 (d_1 + e_1), \quad (88)$$

$$\delta_1 = \frac{i[-144iA_0^2 k_1 (d_1 + e_1) + \sqrt{6} A_0^3 q_1 (d_1 + e_1)]}{288k_1}.$$

$$A_{-1} = 0, A_0 = 0, a_1 = \frac{i\sqrt{6} q_1}{k_1}, a_2 = \frac{(-1)^{\frac{1}{3}} A_1 (d_1 + e_1)^{\frac{1}{3}}}{2(b_1)^{\frac{1}{3}} (k_1)^{\frac{4}{3}}}, c_1 = \frac{(-1)^{\frac{2}{3}} (3b_1 k_1)^{\frac{1}{3}} [3i\sqrt{6} q_1 (d_1 + e_1)]}{2(d_1 + e_1)^{\frac{1}{3}}}, \quad (89)$$

$$\delta_1 = \frac{3[-4(-1)^{\frac{1}{3}} (3b_1 k_1)^{\frac{2}{3}} d_1^2 q_1^2 - 8(-1)^{\frac{1}{3}} (3b_1 k_1)^{\frac{1}{3}} d_1 e_1 q_1^2 - 4(-1)^{\frac{1}{3}} (3b_1 k_1)^{\frac{2}{3}} e_1^2 q_1^2] - b_1 q_1^4}{(d_1 + e_1)^{\frac{5}{3}}}$$

$$A_{-1} = 0, A_0 = 0, a_1 = \frac{-i\sqrt{6} q_1}{k_1}, a_2 = \frac{(-1)^{\frac{1}{3}} A_1 (d_1 + e_1)^{\frac{1}{3}}}{2(b_1)^{\frac{1}{3}} (k_1)^{\frac{4}{3}}}, c_1 = \frac{(-1)^{\frac{2}{3}} (3b_1 k_1)^{\frac{1}{3}} [3i\sqrt{6} q_1 (d_1 + e_1)]}{2(d_1 + e_1)^{\frac{1}{3}}}, \quad (90)$$

$$\delta_1 = \frac{3[-4(-1)^{\frac{1}{3}} (3b_1 k_1)^{\frac{2}{3}} d_1^2 q_1^2 - 8(-1)^{\frac{1}{3}} (3b_1 k_1)^{\frac{1}{3}} d_1 e_1 q_1^2 - 4(-1)^{\frac{1}{3}} (3b_1 k_1)^{\frac{2}{3}} e_1^2 q_1^2] - b_1 q_1^4}{(d_1 + e_1)^{\frac{5}{3}}}$$

$$A_{-1} = 0, A_0 = 0, a_1 = \frac{i\sqrt{6} q_1}{k_1}, a_2 = \frac{-A_1 (d_1 + e_1)^{\frac{1}{3}}}{2(b_1)^{\frac{1}{3}} (k_1)^{\frac{4}{3}}}, c_1 = \frac{(3b_1 k_1)^{\frac{1}{3}} [3i\sqrt{6} q_1 (d_1 + e_1)]}{2(d_1 + e_1)^{\frac{1}{3}}}, \quad (91)$$

$$\delta_1 = \frac{3[4(3b_1 k_1)^{\frac{2}{3}} d_1^2 q_1^2 + 8(3b_1 k_1)^{\frac{1}{3}} d_1 e_1 q_1^2 + 4(3b_1 k_1)^{\frac{2}{3}} e_1^2 q_1^2] - b_1 q_1^4}{(d_1 + e_1)^{\frac{5}{3}}}$$

$$A_{-1}=0, A_0=0, a_1 = \frac{-i\sqrt{6}q_1}{k_1}, a_2 = \frac{-A_1(d_1+e_1)^{\frac{1}{3}}}{2(b_1)^{\frac{1}{3}}(k_1)^{\frac{4}{3}}}, c_1 = \frac{(3b_1k_1)^{\frac{1}{3}}[3i\sqrt{6}q_1(d_1+e_1)]}{2(d_1+e_1)^{\frac{1}{3}}}, \quad (92)$$

$$\delta_1 = \frac{3[4(3b_1k_1)^{\frac{2}{3}}d_1^2q_1^2 + 8(3b_1k_1)^{\frac{1}{3}}d_1e_1q_1^2 + 4(3b_1k_1)^{\frac{2}{3}}e_1^2q_1^2] - b_1q_1^4}{(d_1+e_1)^{\frac{5}{3}}}$$

$$A_{-1}=0, A_0=0, a_1 = \frac{i\sqrt{6}q_1}{k_1}, a_2 = \frac{-A_1(d_1+e_1)^{\frac{1}{3}}}{2(b_1)^{\frac{1}{3}}(k_1)^{\frac{4}{3}}}, c_1 = \frac{(-3b_1k_1)^{\frac{1}{3}}[3i\sqrt{6}q_1(d_1+e_1)]}{2(d_1+e_1)^{\frac{1}{3}}}, \quad (93)$$

$$\delta_1 = \frac{3[4(-3b_1k_1)^{\frac{2}{3}}d_1^2q_1^2 + 8(-3b_1k_1)^{\frac{1}{3}}d_1e_1q_1^2 + 4(-3b_1k_1)^{\frac{2}{3}}e_1^2q_1^2] - b_1q_1^4}{(d_1+e_1)^{\frac{5}{3}}}$$

$$A_{-1}=0, A_0=0, a_1 = \frac{-i\sqrt{6}q_1}{k_1}, a_2 = \frac{-A_1(d_1+e_1)^{\frac{1}{3}}}{2(b_1)^{\frac{1}{3}}(k_1)^{\frac{4}{3}}}, c_1 = \frac{(-3b_1k_1)^{\frac{1}{3}}[3i\sqrt{6}q_1(d_1+e_1)]}{2(d_1+e_1)^{\frac{1}{3}}}, \quad (94)$$

$$\delta_1 = \frac{3[4(-3b_1k_1)^{\frac{2}{3}}d_1^2q_1^2 + 8(-3b_1k_1)^{\frac{1}{3}}d_1e_1q_1^2 + 4(-3b_1k_1)^{\frac{2}{3}}e_1^2q_1^2] - b_1q_1^4}{(d_1+e_1)^{\frac{5}{3}}}$$

For simplicity we will choose only one of these results say the first one, to extract its own solution and plotted it.

For the 1-th result which is,

$$A_{-1}=0, a_1 = \frac{i\sqrt{6}q_1}{k_1}, a_2 = \frac{i\sqrt{6}A_1q_1}{A_0k_1}, b_1 = \frac{-iA_0^3(d_1+e_1)}{144\sqrt{6}k_1q_1^3}, c_1 = -\frac{3}{4}A_0(d_1+e_1),$$

$$\delta_1 = \frac{-A_0^2(d_1+e_1)}{2} + i\frac{\sqrt{6}A_0^3q_1(d_1+e_1)}{288k_1}.$$

This result will simplified to be,

$$A_{-1}=0, A_0=4, A_1=1, a_1=0.6i, a_2=0.6i, b_1=-0.4i, e_1=d_1=k_1=q_1=\zeta_0=1, c_1=-6, \delta_1=-16+1.1i$$

Via these values of constants the solution in the framework of the proposed method is,

$$\varphi(\zeta) = \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0$$

$$\varphi(\zeta) = \frac{i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]}{1 - i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]} \quad (95)$$

$$\psi_1(\zeta) = 4 + \frac{i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]}{1 - i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]}$$

$$\psi_1(\zeta) = \frac{4 - 3i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]}{1 - i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]} \quad (96)$$

$$u(x,t) = \left(\frac{4 - 3i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]}{1 - i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]} \right) \times e^{i(x+1.1t-16t)}$$

$$u(x,t) = e^{-1.1t} \left(\frac{4 - 3i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]}{1 - i\sqrt{6} \text{Exp}[i\sqrt{6}(x+t+1)]} \right) \times e^{i(x-16t)} \quad (97)$$

$$\text{Re } u = e^{-1.1t} \left\{ \frac{\left((22 + 7\sqrt{6} \sin[\sqrt{6}(x+t+1)]) \cos(x-16t) \right)}{7 + 2\sqrt{6} \sin[\sqrt{6}(x+t+1)]} - \frac{\sqrt{6} \cos[\sqrt{6}(x+t+1)] \times \sin(x-16)}{7 + 2\sqrt{6} \sin[\sqrt{6}(x+t+1)]} \right\} \quad (98)$$

$$\text{Im } u = e^{-1.1t} \left\{ \frac{\left((22 + 7\sqrt{6} \sin[\sqrt{6}(x+t+1)]) \sin(x-16t) \right)}{7 + 2\sqrt{6} \sin[\sqrt{6}(x+t+1)]} + \frac{\sqrt{6} \cos[\sqrt{6}(x+t+1)] \times \cos(x-16)}{7 + 2\sqrt{6} \sin[\sqrt{6}(x+t+1)]} \right\} \quad (99)$$

Fig.11 The cubic-quartic soliton of the Re. part of quadratic-low Eq.(98) in **2D** and **3D** at: $w_l = -8$
 $A_{-1}=0, A_0=4, A_1=1, a_1=0.6i, a_2=0.6i, b_1=-0.4i, e_1=d_1=k_1=q_1=\zeta_0=1, c_1=-6, \delta_1=-16+1.1i, t=0.1$ for **2D**

Fig.12 The cubic-quartic soliton of the Im. part of quadratic-low Eq.(99) in **2D** and **3D** at: $w_l = -8$
 $A_{-1}=0, A_0=4, A_1=1, a_1=0.6i, a_2=0.6i, b_1=-0.4i, e_1=d_1=k_1=q_1=\zeta_0=1, c_1=-6, \delta_1=-16+1.1i, t=0.1$ for **2D**

5. The Cubic-Quartic NLSE for the Parabolic-Low nonlinearity using the ESEM
The cubic-quartic NLSE in polarization-preserving fibers with parabolic law can be proposed in the form,

$$iQ_t + iaQ_{xxx} + bQ_{xxxx} = (c|Q^2| + d|Q^4|)Q \quad (100)$$

The cubic-quartic NLSE in birefringent fiber for parabolic-law is,

$$iu_t + ia_1 u_{xxx} + b_1 u_{xxxx} + c_1 |u|^2 u + d_1 |v|^2 u + (e_1 |u^4| + g_1 |u^2| |v^2| + h_1 |v^4|) u = 0 \quad (101)$$

$$iv_t + ia_1 v_{xxx} + b_1 v_{xxxx} + c_1 |v|^2 v + d_1 |u|^2 v + (e_1 |v^4| + g_1 |v^2| |u^2| + h_1 |u^4|) v = 0 \quad (102)$$

Via inserting the relations (13-26) into the equations (101) and (102) will transform them to the following real and imaginary equations respectively

$$\text{Re} \Rightarrow b_j k_j^4 \psi_j''' - (3a_j q_j k_j^2 + 6b_j k_j^2 q_j^2) \psi_j'' + (b_j q_j^4 + a_j q_j^3 - \delta_j) \psi_j' + (c_j + d_j) \psi_j^3 + (e + g + h) \psi_j^5 = 0 \quad (103)$$

$$\text{Im} \Rightarrow k_j^3 (a_j + 4b_j q_j) \psi_j''' + (w_j - 3a_j k_j q_j^2 - 4b_j k_j q_j^3) \psi_j' = 0 \quad (104)$$

Equation (104) implies $a_j = -4b_j q_j$, $w_j = 3k_j a_j q_j^2 + 4k_j b_j q_j^3$, hence under this constrain Eq.

(103) will be converted to,

$$b_j k_j^4 \psi_j''' + 6b_j q_j^2 k_j^2 \psi_j'' - (3k_j b_j q_j^4 + \delta_j) \psi_j' + (c_j + d_j) \psi_j^3 + (e + g + h) \psi_j^5 = 0 \quad (105)$$

Equation (105) represents either the first part Eq. (101) or the second part Eq. (102) which is the same, for similarity we will implement the solution for the first part $j = 1$ which is,

$$b_1 k_1^4 \psi_1''' + 6b_1 q_1^2 k_1^2 \psi_1'' - (3b_1 q_1^4 + \delta_1) \psi_1' + (c_1 + d_1) \psi_1^3 + (e + g + h) \psi_1^5 = 0 \quad (106)$$

Via balancing ψ_1''', ψ_1^5 at Eq. (106) leads to $5N = N + 4$ which implies that $N = 1$, hence the solution is:

$$\psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi \quad (107)$$

Where $\varphi' = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3$

Case 1: The first family, in which $a_1 = a_3 = 0 \Rightarrow \varphi' = a_0 + a_2 \varphi^2$, consequently

$$\psi_1' = -\frac{a_0 A_{-1}}{\varphi^2} + A_1 a_0 + A_1 a_2 \varphi^2 - a_2 A_{-1} \quad (108)$$

$$\psi_1'' = \frac{2a_0^2 A_{-1}}{\varphi^3} + \frac{2a_0 a_2 A_{-1}}{\varphi} + 2A_1 a_0 a_2 \varphi + 2A_1 a_2^2 \varphi^3 \quad (109)$$

$$\psi_1''' = \frac{24a_0^4 A_{-1}}{\varphi^5} + \frac{40a_0^3 a_2 A_{-1}}{\varphi^3} + \frac{16a_0^2 a_2^2 A_{-1}}{\varphi} + 16A_1 a_0^2 a_2^2 \varphi + 40A_1 a_0 a_2^3 \varphi^3 + 24A_1 a_2^4 \varphi^5 \quad (110)$$

Via inserting $\psi_1''', \psi_1'', \psi_1, \psi_1^3$ and ψ_1^5 into equation (106), collecting and equating the coefficients of various powers of φ^i to zero leads to system of equations, by solving this system many solutions will be achieved from which only 8-results are acceptable while the remaining are refused. These acceptable results are,

$$A_0 = 0, a_2 = \frac{a_0 A_{-1}}{A_{-1}}, \delta_1 = \frac{b_1 k_1 (16a_0^4 A_{-1}^2 k_1^3 - 24a_0^2 A_{-1} A_{-1} q_1^2 - 3A_{-1}^2 q_1^2)}{A_{-1}^2}, \quad (111)$$

$$c_1 = \frac{-A_{-1}^3 d_1 + 80a_0^4 A_{-1} b_1 k_1^4 - 12a_0^2 A_{-1} b_1 q_1^2 k_1^2}{A_{-1}^3}, g = \frac{-(e+h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4}$$

$$A_0 = 0, a_2 = \frac{-a_0 A_{-1}}{A_{-1}}, \delta_1 = \frac{-b_1 (-256a_0^4 A_{-1}^2 k_1^4 + 48a_0^2 A_{-1} A_{-1} k_1^2 q_1^2 + 3A_{-1}^2 q_1^4)}{A_{-1}^2}, \quad (112)$$

$$c_1 = \frac{-A_{-1}^3 d_1 + 160a_0^4 A_{-1} b_1 k_1^4 - 12a_0^2 A_{-1} b_1 k_1^2 q_1^2}{A_{-1}^3}, g = \frac{-(e+h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4}$$

$$A_0=0, a_2=\frac{ia_0A_1}{A_{-1}}, \delta_1=\frac{532b_1a_0^4A_1^2k_1^4}{3A_{-1}^2}, q_1=\frac{-(-1)^{\frac{3}{4}}k_1a_0\sqrt{\frac{10A_1}{3}}}{\sqrt{A_{-1}}} \quad (113)$$

$$c_1=\frac{-A_{-1}^3d_1+120a_0^4A_1b_1k_1^4}{A_{-1}^3}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

$$A_0=0, a_2=\frac{ia_0A_1}{A_{-1}}, \delta_1=\frac{532b_1a_0^4A_1^2k_1^4}{3A_{-1}^2}, q_1=\frac{(-1)^{\frac{3}{4}}k_1a_0\sqrt{\frac{10A_1}{3}}}{\sqrt{A_{-1}}} \quad (114)$$

$$c_1=\frac{-A_{-1}^3d_1+120a_0^4A_1b_1k_1^4}{A_{-1}^3}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

$$A_0=0, a_2=\frac{-ia_0A_1}{A_{-1}}, \delta_1=\frac{532b_1a_0^4A_1^2k_1^4}{3A_{-1}^2}, q_1=\frac{-(-1)^{\frac{1}{4}}k_1a_0\sqrt{\frac{10A_1}{3}}}{\sqrt{A_{-1}}} \quad (115)$$

$$c_1=\frac{-A_{-1}^3d_1+120a_0^4A_1b_1k_1^4}{A_{-1}^3}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

$$A_0=0, a_2=\frac{-ia_0A_1}{A_{-1}}, \delta_1=\frac{532b_1a_0^4A_1^2k_1^4}{3A_{-1}^2}, q_1=\frac{(-1)^{\frac{1}{4}}k_1a_0\sqrt{\frac{10A_1}{3}}}{\sqrt{A_{-1}}} \quad (116)$$

$$c_1=\frac{-A_{-1}^3d_1+120a_0^4A_1b_1k_1^4}{A_{-1}^3}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

$$A_{-1}=0, A_0=0, \delta_1=-b_1(-16a_0^2a_2^2k_1^4-12a_0a_2k_1^2q_1^2+3q_1^4), \quad (117)$$

$$c_1=\frac{-A_{-1}^2d_1-40a_0b_1a_2^3k_1^4-12a_2^2b_1k_1^2q_1^2}{A_{-1}^2}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

$$A_1=0, A_0=0, \delta_1=-b_1(-16a_0^2a_2^2k_1^4-12a_0a_2k_1^2q_1^2+3q_1^4), \quad (118)$$

$$c_1=\frac{-A_{-1}^2d_1-40a_2b_1a_0^3k_1^4-12a_0^2b_1k_1^2q_1^2}{A_{-1}^2}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

$$A_1=0, A_0=0, a_2=0, \delta_1=-3b_1q_1^4, c_1=\frac{-A_{-1}^2d_1-12b_1a_0^2k_1^2q_1^2}{A_{-1}^2}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4} \quad (119)$$

Now we will extract the different solutions corresponding to these 9-acceptable results, for simplicity we will choose only one result say the first one.

For the 1-st result which is,

$$A_0=0, a_2=\frac{a_0A_1}{A_{-1}}, \delta_1=\frac{b_1k_1(16a_0^4A_1^2k_1^3-24a_0^2A_1A_{-1}q_1^2-3A_{-1}^2q_1^2)}{A_{-1}^2},$$

$$c_1=\frac{-A_{-1}^3d_1+80a_0^4A_1b_1k_1^4-12a_0^2A_{-1}b_1q_1^2k_1^2}{A_{-1}^3}, g=\frac{-(e+h)A_{-1}^4-24b_1a_0^4k_1^4}{A_{-1}^4}$$

This result will generate 4- sub results which are,

$$A_0=0, A_{-1}=A_1=1, a_0=a_2=k_1=q_1=b_1=c_1=d_1=\zeta_0=1, g+h+e=-24, \delta_1=-9$$

$$A_0=0, A_{-1}=A_1=1, a_0=a_2=-1, k_1=q_1=b_1=c_1=d_1=\zeta_0=1, g+h+e=-24, \delta_1=-9$$

$$A_0=0, A_{-1}=A_1=1, a_0=a_2=0.4, k_1=b_1=q_1=c_1=d_1=\zeta_0=1, g+h+e=-0.8, \delta_1=-2.3$$

$$A_0=0, A_{-1}=A_1=1, a_0=a_2=-0.4, k_1=b_1=q_1=c_1=d_1=\zeta_0=1, g+h+e=-0.8, \delta_1=-2.3 \quad (120)$$

For simplicity we will take only one of these results say the first, and extracting the corresponding solution according to the point of view of the suggested method which is:

$$\varphi(\zeta) = \frac{\sqrt{a_0 a_2}}{a_2} \tan(\sqrt{a_0 a_2}(\zeta + \zeta_0)), a_0 a_2 \neq 0$$

$$\varphi(\zeta) = \tan[x - 8t + 1] \quad (121)$$

$$\psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi$$

$$\psi_1(\zeta) = \cot[x - 8t + 1] + \tan[x - 8t + 1] \quad (122)$$

$$u(x, t) = \psi_1(\zeta) e^{i\eta_1(x, t)}, \zeta = k_1 x + w_1 t, \eta_1 = q_1 x + \delta_1 t$$

$$u(x, t) = (\cot[x - 8t + 1] + \tan[x - 8t + 1]) e^{i[x - 9t]} \quad (123)$$

$$\text{Re } u = (\cot[x - 8t + 1] + \tan[x - 8t + 1]) \times \cos(x - 9t) \quad (124)$$

$$\text{Im } u = (\cot[x - 8t + 1] + \tan[x - 8t + 1]) \times \sin(x - 9t) \quad (125)$$

Fig.13 The cubic-quartic soliton of the Re. part of parabolic low Eq.(124) in 2D and 3D with values:

$$A_0=0, A_{-1}=A_1=1, a_0=a_2=k_1=q_1=b_1=c_1=d_1=\zeta_0=1, g+h+e=-24, \delta_1=-9, w_1=-8, t=0.1 \text{ for } 2D$$

Fig.14 The cubic-quartic soliton of the Im. part of parabolic low Eq.(125) in 2D and 3D with values:

$$A_0=0, A_{-1}=A_1=1, a_0=a_2=k_1=q_1=b_1=c_1=d_1=\zeta_0=1, g+h+e=-24, \delta_1=-9, w_1=-8, t=0.1 \text{ for } 2D$$

Via the same manner, we can plot the other solutions.

Case 2: The second family, in which $a_0 = a_3 = 0 \Rightarrow \varphi' = a_1 \varphi + a_2 \varphi^2$, consequently

$$\psi_1' = A_1 a_2 \varphi^2 + a_1 A_1 \varphi - \frac{A_{-1} a_1}{\varphi} - A_{-1} a_2 \quad (126)$$

$$\psi_1'' = 2A_1 a_2^2 \varphi^3 + 3A_1 a_1 a_2 \varphi^2 + A_1 a_1^2 \varphi + A_{-1} a_2 a_1 + \frac{a_1^2 A_{-1}}{\varphi} \quad (127)$$

$$\begin{aligned} \psi_1''' = & \frac{a_1^4 A_{-1}}{\varphi} + A_{-1} a_2 a_1^3 + 14 A_1 a_2 a_1^3 \varphi^2 + 50 A_1 a_1^2 a_2^2 \varphi^3 \\ & + A_1 a_1 (1 + 60 a_2^3) \varphi^4 + A_1 (a_1 + 24 a_2^4) \varphi^5 \end{aligned} \quad (128)$$

Via inserting $\psi_1''', \psi_1'', \psi_1, \psi_1^3$ and ψ_1^5 into equation (106), collecting and equating the coefficients of various powers of φ^i to zero leads to system of equations in which the coefficients of $\varphi^{-5}, \varphi^{-4}$ imply that $e + g + h = 0$ and the coefficients of $\varphi^{-3}, \varphi^{-2}$ imply $c_1 = -d_1$ and by using these new constrain at the other equations of this system it will be reduced to the following system,

$$\begin{aligned} a_1^4 A_{-1} b_1 k_1^4 + 6 a_1^2 A_{-1} b_1 k_1^2 q_1^2 - 3 A_{-1} b_1 k_1 q_1^4 - A_1 \delta_1 &= 0 \\ a_1 A_1 b_1 k_1^4 + 24 a_2^4 A_1 b_1 q_1^4 &= 0 \\ a_1 A_1 b_1 k_1^4 + 60 a_1 a_2^3 A_1 b_1 q_1^4 &= 0 \\ 50 a_1^2 a_2^2 A_1 b_1 k_1^4 + 12 a_2^2 A_1 b_1 k_1^2 q_1^2 &= 0 \\ 14 a_1^3 a_2 A_1 b_1 k_1^4 + 18 a_1 a_2 A_1 b_1 k_1^2 q_1^2 &= 0 \\ 6 a_1^2 A_1 b_1 k_1^2 q_1^2 - 3 A_1 b_1 q_1^4 - A_1 \delta_1 &= 0 \\ a_1^3 a_2 A_{-1} b_1 k_1^4 + 6 a_1 a_2 A_{-1} b_1 k_1^2 q_1^2 - 3 A_0 b_1 q_1^4 - A_0 \delta_1 &= 0 \end{aligned} \quad (129)$$

The solution of the reduced system (129) gives the following three results,

$$\begin{aligned} (1) A_1 &= 0, \delta_1 = \frac{-b_1 k_1^4 q_1^4}{12}, q_1 = \frac{-i a_1 k_1}{\sqrt{6}} \\ (2) A_1 &= 0, \delta_1 = \frac{-b_1 k_1^4 q_1^4}{12}, q_1 = \frac{i a_1 k_1}{\sqrt{6}} \\ (3) A_1 &= 0, A_0 = \frac{a_2 A_{-1}}{a_1}, \delta_1 = b_1 k_1^4 a_1^4 + 6 a_1^2 b_1 k_1^2 q_1^2 - 3 b_1 q_1^4 \end{aligned} \quad (130)$$

We will extract the solutions in the framework of the suggested method for the first and the third result.

For the first result which is, $A_1 = 0, \delta_1 = \frac{-b_1 k_1^4 q_1^4}{12}, q_1 = \frac{-i a_1 k_1}{\sqrt{6}}$

This result can be simplified to be,

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = \sqrt{6} i, a_2 = 0, k_1 = q_1 = d_1 = \zeta_0 = 1, \delta_1 = -0.1, c_1 = -d_1, e + g + h = 0$$

From the point of view of the suggested method the solution is,

$$\begin{aligned} \varphi(\zeta) &= \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0 \\ \varphi(\zeta) &= 2.45 i \text{Exp}[2.45 i(x - 8t + 1)] \end{aligned} \quad (131)$$

$$\begin{aligned} \psi_1(\zeta) &= \frac{-i \text{Exp}[-2.45 i(x - 8t + 1)]}{2.45} + 1 \\ \psi_1(\zeta) &= (1 - 0.4 \sin[2.45x - 19.6t + 2.45]) - 0.4 i \cos[2.45x - 19.6t + 2.45] \end{aligned} \quad (132)$$

$$u(x, t) = \psi_1(\zeta) e^{i m_1(x, t)}, \zeta = k_1 x + w_1 t, \eta_1 = q_1 x + \delta_1 t$$

$$u(x,t) = (1 - 0.4 \sin[2.45x - 19.6t + 2.45]) - 0.4i \cos[2.45x - 19.6t + 2.45] \times e^{i(x-0.1t)} \quad (133)$$

$$\begin{aligned} \text{Re } u &= (1 - 0.4 \sin[2.45x - 19.6t + 2.45]) \cos(x - 0.1t) \\ &\quad + (0.4 \cos[2.45x - 19.6t + 2.45]) \sin(x - 0.1t) \end{aligned} \quad (134)$$

$$\begin{aligned} \text{Im } u &= (1 - 0.4 \sin[2.45x - 19.6t + 2.45]) \sin(x - 0.1t) \\ &\quad - (0.4 \cos[2.45x - 19.6t + 2.45]) \cos(x - 0.1t) \end{aligned} \quad (135)$$

Fig.15 The cubic-quartic soliton of the Re. part of parabolic low Eq.(134) in **2D** and **3D** with values:

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = \sqrt{6}i, a_2 = 0, k_1 = q_1 = d_1 = \zeta_0 = 1, \delta_1 = -0.1, c_1 = -d_1, w_1 = -8, e + g + h = 0, t = 0.1 \text{ for } 2D$$

Fig.16 The cubic-quartic soliton of the Im. part of parabolic low Eq.(135) in **2D** and **3D** with values:

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = \sqrt{6}i, a_2 = 0, k_1 = q_1 = d_1 = \zeta_0 = 1, \delta_1 = -0.1, c_1 = -d_1, w_1 = -8, e + g + h = 0, t = 0.1 \text{ for } 2D$$

For the third result which is, $A_1 = 0, A_0 = \frac{a_2 A_{-1}}{a_1}, \delta_1 = b_1 k_1^4 a_1^4 + 6a_1^2 b_1 k_1^2 q_1^2 - 3b_1 q_1^4$

This result split into four sub results which are,

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = \sqrt{0.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0$$

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = -\sqrt{0.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0$$

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = i\sqrt{6.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0$$

$$A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = -i\sqrt{6.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0$$

We will plot only one say the first, from the point of view of the suggested method the solution is,

$$\begin{aligned} \varphi(\zeta) &= \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 \neq 0 \\ \varphi(\zeta) &= \frac{0.8 \text{Exp}[0.8x - 6.4t + 0.8]}{1 - 0.8 \text{Exp}[0.8x - 6.4t + 0.8]} \end{aligned} \quad (136)$$

$$\psi_1(\zeta) = 1.3 \text{Exp}[-0.8x + 6.4t - 0.8] - 1 \quad (137)$$

$$u(x,t) = (1.3 \text{Exp}[-0.8x + 6.4t - 0.8] - 1) \times e^{i(x+t)} \quad (138)$$

$$\text{Re } u = (1.3 \text{Exp}[-0.8x + 6.4t - 0.8] - 1) \times \cos(x+t) \quad (139)$$

$$\text{Im } u = (1.3 \text{Exp}[-0.8x + 6.4t - 0.8] - 1) \times \sin(x+t) \quad (140)$$

Fig.17 The cubic-quartic soliton of the Re. part of parabolic low Eq.(139) in **2D** and **3D** with values: $A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = 0.8, k_1 = q_1 = d_1 = \zeta_0 = \delta_1 = 1, c_1 = -d_1, w_1 = -8, e + g + h = 0, t = 0.1$ for **2D**

Fig.18 The cubic-quartic soliton of the Im. part of parabolic low Eq.(140) in **2D** and **3D** with values: $A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = 0.8, k_1 = q_1 = d_1 = \zeta_0 = \delta_1 = 1, c_1 = -d_1, w_1 = -8, e + g + h = 0, t = 0.1$ for **2D**

By the same manner we can plot the other remained cases.

6. The Cubic-Quartic NLSE for the Non-Local Low nonlinearity using the ESEM

The cubic-quartic NLSE in polarization-preserving fibers with non-local low be proposed in the form,

$$iQ_t + iaQ_{xxx} + bQ_{xxxx} + \left(c|Q|^2\right)_{xx} Q = 0 \quad (141)$$

The cubic-quartic NLSE in birefringent fiber non-local low is,

$$iu_t + ia_1u_{xxx} + b_1u_{xxxx} + \left(c_1(|u|^2)_{xx} + d_1(|v|^2)_{xx}\right)u = 0 \quad (142)$$

$$iv_t + ia_2v_{xxx} + b_1v_{xxxx} + \left(c_2(|v|^2)_{xx} + d_2(|u|^2)_{xx}\right)v = 0 \quad (143)$$

Via inserting the relations (13-26) into equation (142) and (143) will transform them to the following real and imaginary equations respectively:

$$\text{Re} \Rightarrow b_j k_j^4 \psi_j''' - (3a_j q_j k_j^2 + 6b_j k_j^2 q_j^2) \psi_j'' + (b_j q_j^4 + a_j q_j^3 - \delta_j) \psi_j + (2c_j + 2d_j) \psi_j \psi_j'^2 + (2c_j + 2d_j) \psi_j' \psi_j'^2 = 0 \quad (144)$$

$$\text{Im} \Rightarrow k_j^3 (a_j + 4b_j q_j) \psi_j''' + (w_j - 3a_j k_j q_j^2 - 4b_j k_j q_j^3) \psi_j' = 0 \quad (145)$$

Equation (145) implies $a_j = -4b_j q_j$, $w_j = 3k_j a_j q_j^2 + 4k_j b_j q_j^3$, hence under this constrain Eq. (144) will be converted to,

$$b_j k_j^4 \psi_j''' + 6b_j k_j^2 q_j^2 \psi_j'' - (3b_j q_j^4 + \delta_j) \psi_j + (2c_j + 2d_j) \psi_j \psi_j'^2 + (2c_j + 2d_j) \psi_j' \psi_j'^2 = 0 \quad (146)$$

Equation (146) represents either the first part Eq. (142) or the second part Eq (143) which is the same, for similarity we will implement the solution for the first part $j = 1$ which is,

$$b_1 k_1^4 \psi_1''' + 6b_1 k_1^2 q_1^2 \psi_1'' - (3b_1 q_1^4 + \delta_1) \psi_1 + (2c_1 + 2d_1) \psi_1 \psi_1'^2 + (2c_1 + 2d_1) \psi_1' \psi_1'^2 = 0 \quad (147)$$

Via balancing $\psi_j''', \psi_j'' \psi_j^2$ at Eq. (147) leads to $4N = 2N + N + 2$ which implies that $N = 2$, hence the solution is,

$$\psi_1(\zeta) = \frac{A_{-2}}{\varphi^2} + \frac{A_{-1}}{\varphi} + A_0 + A_1\varphi + A_2\varphi^2 \quad (148)$$

Where $\varphi' = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3$

Case 1: The first family, in which $a_1 = a_3 = 0 \Rightarrow \varphi' = a_0 + a_2\varphi^2$, consequently

$$\psi_1' = \frac{-2a_0A_{-2}}{\varphi^3} - \frac{a_0A_{-1}}{\varphi^2} - \frac{2a_2A_{-2}}{\varphi} + (A_1a_0 - a_2A_{-1}) + 2A_2a_0\varphi + (A_1a_2 + 2a_2A_2)\varphi^2. \quad (149)$$

$$\psi_1'' = \frac{6a_0^2A_{-2}}{\varphi^4} + \frac{2a_0^2A_{-1}}{\varphi^3} + \frac{8a_0a_2A_{-2}}{\varphi^2} + \frac{2a_0a_2A_{-1}}{\varphi} + 2(A_2a_0^2 + A_{-2}a_2^2) \quad (150)$$

$$2a_0a_2A_1\varphi + 8a_0a_2A_2\varphi^2 + 2a_2^2A_1\varphi^3 + 6a_2^2A_2\varphi^4.$$

$$\begin{aligned} \psi_1''' = & \frac{120a_0^4A_{-2}}{\varphi^6} + \frac{24a_0^4A_{-1}}{\varphi^5} + \frac{240a_0^3a_2A_{-2}}{\varphi^4} + \frac{40a_0^3a_2A_{-1}}{\varphi^3} \\ & + \frac{136a_0^2a_2^2A_{-2}}{\varphi^2} + \frac{16a_0^2a_2^2A_{-1}}{\varphi} + 16a_0a_2^3A_{-2} + 16a_0^3a_2A_2 + 16a_0^2a_2^2A_1\varphi \\ & + 136a_0^2a_2^2A_2\varphi^2 + 40a_0a_2^3A_1\varphi^3 + 240a_0a_2^3A_2\varphi^4 + 24a_2^4A_1\varphi^5 + 120a_2^4A_2\varphi^6 \end{aligned} \quad (151)$$

Via inserting $\psi_1''', \psi_1^2\psi_1'', \psi_1'', \psi_1\psi_1'^2$ and ψ_1^3 into equation (147), collecting and equating the coefficients of various powers of φ^i to zero leads to system of equations, by solving this system rejected results will be achieved because in all these results either $a_0 = 0$ or $a_2 = 0$ or $A_{-2} = A_{-1} = A_1 = A_2 = 0$, hence all these results will be refused.

Case 2: The second family, in which $a_0 = a_3 = 0 \Rightarrow \varphi' = a_1\varphi + a_2\varphi^2$, consequently

$$\psi_1' = A_1a_2\varphi^2 + a_1A_1\varphi - \frac{A_{-1}a_1}{\varphi} - A_{-1}a_2 \quad (152)$$

$$\psi_1'' = 2A_1a_2^2\varphi^3 + 3A_1a_1a_2\varphi^2 + A_1a_1^2\varphi + A_{-1}a_2a_1 + \frac{a_1^2A_{-1}}{\varphi} \quad (153)$$

$$\begin{aligned} \psi_1''' = & \frac{a_1^4A_{-1}}{\varphi} + A_{-1}a_2a_1^3 + 14A_1a_2a_1^3\varphi^2 + 50A_1a_1^2a_2^2\varphi^3 \\ & + A_1a_1(1 + 60a_2^3)\varphi^4 + A_1(a_1 + 24a_2^4)\varphi^5 \end{aligned} \quad (154)$$

Via inserting $\psi_1''', \psi_1^2\psi_1'', \psi_1'', \psi_1\psi_1'^2$ and ψ_1^3 into equation (147), collecting and equating the coefficients of various powers of φ^i zero, will lead to a system of equations, by solving this system only six acceptable solutions are achieved and the remain are refused because either $a_1 = 0$ for which there are no solution according to the suggested method or $A_{-2} = A_{-1} = A_1 = A_2 = 0$ or $b_1 = 0$.

The acceptable results are,

$$\begin{aligned}
(1) A_{-1} &= A_1 = A_2 = 0, a_2^2 = \frac{-5A_0(3b_1q_1^4 + \delta_1)}{108A_{-2}b_1k_1^2q_1^2}, a_1 = -1.1i \frac{q_1}{k_1}, c_1 = -d_1 \\
(2) A_{-1} &= A_1 = A_2 = 0, a_2^2 = \frac{-5A_0(3b_1q_1^4 + \delta_1)}{108A_{-2}b_1k_1^2q_1^2}, a_1 = 1.1i \frac{q_1}{k_1}, c_1 = -d_1 \\
(3) A_{-1} &= A_1 = A_2 = a_2 = 0, a_1 = -1.1i \frac{q_1}{k_1}, b_1 = \frac{-\delta_1}{3q_1^4}, c_1 = -d_1 \\
(4) A_{-1} &= A_1 = A_2 = a_2 = 0, a_1 = 1.1i \frac{q_1}{k_1}, b_1 = \frac{-\delta_1}{3q_1^4}, c_1 = -d_1 \\
(5) A_{-2} &= A_2 = a_2 = 0, A_{-1} = 0.3A_1, a_1 = -1.1i \frac{q_1}{k_1}, c_1 = -d_1 \\
(6) A_{-2} &= A_2 = a_2 = 0, A_{-1} = 0.3A_1, a_1 = 1.1i \frac{q_1}{k_1}, c_1 = -d_1
\end{aligned} \tag{155}$$

For simplicity and repetition we will establish the solutions corresponding to the first and the six results and plot them.

For the first result which is,

$$A_{-2} = A_2 = a_2 = 0, A_{-1} = 0.3A_1, a_1 = 1.1, c_1 = -d_1$$

This result can be simplified to be,

$$A_{-1} = A_1 = A_2 = 0, |a_1| = 1.1, |a_2| = 0.44, A_0 = A_{-2} = q_1 = k_1 = \delta_1 = 1, c_1 = -d_1 \tag{156}$$

The solution in the framework of the suggested method is,

$$\psi_1(\zeta) = \frac{A_{-2}}{\varphi^2} + \frac{A_{-1}}{\varphi} + A_0 + A_1\varphi + A_2\varphi^2$$

Where,

$$\varphi(\zeta) = \frac{1.1 \text{Exp}[1.1(\zeta + \zeta_0)]}{1 - 0.44 \text{Exp}[1.1(\zeta + \zeta_0)]} \tag{157}$$

$$\psi_1(x, t) = 0.8 \text{Exp}[-2.2(\zeta + \zeta_0)] - 0.7 \text{Exp}[-1.1(\zeta + \zeta_0)] + 1.2 \tag{158}$$

$$\psi_1(x, t) = (1.2 + 0.8 \text{Exp}[-2.2(\zeta + \zeta_0)] - 0.7 \text{Exp}[-1.1(\zeta + \zeta_0)]) \times e^{j(x+t)} \tag{159}$$

$$\text{Re}\psi_1(x, t) = (1.2 + 0.8 \text{Exp}[-2.2(\zeta + \zeta_0)] - 0.7 \text{Exp}[-1.1(\zeta + \zeta_0)]) \times \cos(x+t) \tag{160}$$

$$\text{Im}\psi_1(x, t) = (1.2 + 0.8 \text{Exp}[-2.2(\zeta + \zeta_0)] - 0.7 \text{Exp}[-1.1(\zeta + \zeta_0)]) \times \sin(x+t) \tag{161}$$

Fig.19 The cubic-quartic soliton of the Re. part of non-local low Eq.(160) in **2D** and **3D** with values:

$$A_{-1} = A_1 = A_2 = 0, |a_1| = 1.1, |a_2| = 0.44, A_0 = A_{-2} = q_1 = k_1 = \delta_1 = 1, c_1 = -d_1$$

Fig.20 The cubic-quartic soliton of the Im. part of non-local low Eq.(161) in **2D** and **3D** with values:

$$A_{-1}=A_1=A_2=0, |a_1|=1.1, |a_2|=0.44, A_0=A_2=q_1=k_1=\delta_1=1, c_1=-d_1$$

For the six result which is,

$$A_{-1}=A_1=A_2=0, a_2^2 = \frac{-5A_0(3b_1q_1^4 + \delta_1)}{108A_{-2}b_1k_1^2q_1^2}, a_1 = -1.1i \frac{q_1}{k_1}, c_1 = -d_1$$

This result can be simplified to be,

$$A_{-2}=A_0=A_2=a_2=0, A_{-1}=0.3, |a_1|=1.1, A_1=q_1=k_1=\delta_1=1, c_1=-d_1 \quad (162)$$

The solution in the framework of the suggested method is,

$$\varphi(\zeta) = 1.1 \text{Exp}[1.1(\zeta + \zeta_0)] \quad (163)$$

$$\psi_1(\zeta) = 0.3 \text{Exp}[-1.1(\zeta + \zeta_0)] + 1.1 \text{Exp}[1.1(\zeta + \zeta_0)] \quad (164)$$

$$\psi_1(x,t) = (0.3 \text{Exp}[-1.1(\zeta + \zeta_0)] + 1.1 \text{Exp}[1.1(\zeta + \zeta_0)]) \times e^{i(x+t)} \quad (165)$$

$$\text{Re}\psi_1(x,t) = (0.3 \text{Exp}[-1.1(\zeta + \zeta_0)] + 1.1 \text{Exp}[1.1(\zeta + \zeta_0)]) \times \cos(x+t) \quad (166)$$

$$\text{Im}\psi_1(x,t) = (0.3 \text{Exp}[-1.1(\zeta + \zeta_0)] + 1.1 \text{Exp}[1.1(\zeta + \zeta_0)]) \times \sin(x+t) \quad (167)$$

Fig.21 The cubic-quartic soliton of the Re. part of non-local low Eq.(160) in **2D** and **3D** with values:

$$A_{-2}=A_0=A_2=a_2=0, A_{-1}=0.3, |a_1|=1.1, A_1=q_1=k_1=\delta_1=1, c_1=-d_1$$

Fig.22 The cubic-quartic soliton of the Im. part of non-local low Eq.(161) in **2D** and **3D** with values:

$$A_{-2}=A_0=A_2=a_2=0, A_{-1}=0.3, |a_1|=1.1, A_1=q_1=k_1=\delta_1=1, c_1=-d_1$$

By the same manner we can plot the other results.

9-Conclusion

In this paper, multiple accurate types of solitons to the cubic-quartic NLSE in polarization-preserving fibers with four various forms are implemented via the ESEM. Firstly, through overview Fig. (1-10) it is clear ability of the suggested method to give good description of the new types of solitons to the cubic-quartic NLSE for the Kerr-Low NLSE in polarization-preserving fibers. Also, via overview Fig.(11-12) we are able to document exact types of solitons to the cubic-quartic NLSE for the Quadratic-Low in polarization-preserving fibers coupled equations by the same method. In related subject, the ESEM has been used effectively to establish new multiple accurate types of solitons to the cubic-quartic NLSE for the Parabolic-Low NLSE in polarization-preserving fibers coupled equations which appear clearly in Fig. (13-18). Finally, new types of solitons to the cubic-quartic NLSE for the Non-local Low in polarization-preserving fibers were achieved in the framework of the same method Fig (19-22). The obtained multiple new solitons which weren't achieved before by any other authors for the four forms mentioned before denote to the novelty of these results, especially compared with that achieved previously by [25-29] who used various manners to study these forms that emerged from this model. Consequently, new distinct and impressive visions of the solitons to the quartic-cubic NLSE for birefringent fibers of these four different forms have been constructed and the achieved solitons will add improved extended studies for all modern telecommunication engineering and all related phenomena.

Ethics Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

Availability of Data and Material: Not available.

Code availability : Not available.

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