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Growing scale-free simplices

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ABSTRACT

The past two decades have seen significant successes in our understanding of networked systems, from the mapping of real-world networks to the establishment of generative models recovering their observed macroscopic patterns. These advances, however, are restricted to pairwise interactions and provide limited insight into higher-order structures. Such multi-component interactions can only be grasped through simplicial complexes, which have recently found applications in social, technological and biological contexts. Here we introduce, study, and characterize a model to grow simplicial complexes of order two, i.e. nodes, links and triangles. Specifically, through a combination of preferential and/or non preferential attachment mechanisms, the model constructs networks with a scale-free degree distribution and an either bounded or scale-free generalized degree distribution. Allowing to analytically control the scaling exponents we arrive at a highly general scheme by which one is able to construct ensembles of synthetic complexes displaying desired statistical properties.

Introduction

All beauty, richness and harmony in the emergent dynamics of a complex system largely depend on the specific way in which its elementary components interact. The last twenty years have seen the birth and development of the multidisciplinary field of *Network Science*, wherein a variety of systems in physics, biology, social sciences and engineering has been modelled by networks of coupled units, in the attempt to unveil the mechanisms underneath the observed systems' functionality¹⁻⁶.

But the fundamental limit of such a representation is that networks capture only pairwise interactions, whereas the function of many real-world systems not only involves dyadic connections, but rather is the outcome of collective actions at the level of groups of nodes. For instance, in ecological systems, three or more species compete for food or territory⁷. Similar multi-component interactions appear in functional⁸⁻¹² and structural¹³ brain networks, protein interaction networks¹⁴, semantic networks¹⁵, multi-Authors scientific collaborations¹⁶, offline and online social networks^{17,18}, trigenic interactions in gene regulatory networks^{19,20}, and spreading of social behavior due to multiple, simultaneous, social interactions²¹.

Simplicial complexes (SCs), being structures formed by *simplices* of different dimensions (nodes, links, triangles, tetrahedra, etc.), can effectively map the relationships between any number of components. Originally introduced over two decades ago²², SCs are becoming increasingly relevant thanks to the enhanced resolution of current data sets and the recent advances in data analysis techniques^{23,24}. As real data is being accumulated, we encounter a theoretical challenge: how to synthesize SCs that

Figure 1. The structures generated in the preferential and the non preferential cases. The model without preferential attachment (panels **a** and **b**) and that with preferential attachment (panels **c** and **d**). Panels **a** and **c**: log-log plot of the degree distribution $P(k)$ of the resulting networks for different values of m_{tri} (see color code in the legend). Data are obtained as an ensemble average over 100 different realizations of a network with size $N = 10,000$ nodes. Dashed lines in **a** (**c**) correspond to the analytical predictions given by Eqs. (1) [(3)]. Panels **b** and **d**: schematic visualization of generated networks with $N = 200$ and $m_{tri} = 1$. The size of the nodes is proportional to 50 times the square root of the corresponding entry in the eigenvector centrality³⁶, the width of each link is proportional to $\sqrt{k_{ij}}$, and the color of the links encodes k_l as reported in the bars at the right of both panels.

faithfully reproduce the observed structural features. While significant progresses were already made in extending to SCs static graph models (such as random graphs^{25,26}, or the configuration model²⁷, or activity driven models²⁸), the same attention has not yet been paid to the study of growth models for SCs, despite the fact that in many circumstances the network is the result of a growing process, such as in the case of scientists collaborating with and citing each other. A step forward has been made in the field of network geometry²⁹ and its relation with quantum networks^{30,31} or complex materials³², where random graph models have been developed by aggregating SCs as fundamental building blocks.

In this paper we introduce and fully characterize several methods able to grow SCs of order two, i.e. structures made of nodes, links and triangles, making use of preferential and non preferential rules, as well as combinations of both. The resulting SCs are characterized by two distributions, the classic degree distribution $P(k)$, capturing the fraction of nodes with degree k , and the generalized degree distribution $P(k_l)$, where k_l characterizes the number of triangles supported by each link $l = (i, j)$. We show that our generative model always yields a power-law scaling in $P(k)$, recovering the ubiquitously observed scale-free property^{33,34}, and, at the same time, it allows full control over $P(k_l)$, i.e. bounded or scale-free with any desired scaling exponent. Indeed, $P(k_l)$ has been shown to play a crucial role in the emergence of collective behavior, such as synchronization³⁵.

Results

Simplicial complexes growth models. The purpose is to grow a network of N nodes featuring a transitivity coefficient $T = 1$ ^{37,38}, thus implying that each link is part of a connected triplet of nodes (a triad) which forms a triangle. This is done by starting at $t = 0$ with an elementary network seed, consisting in an initial clique (an all-to-all connected network) of size $N_0 \ll N$. Then, at each successive times $t = 1, 2, 3, \dots, N - N_0$ a new node is added to the graph. The added node selects m_{tri} already existing links, and forms connections with the $2m_{tri}$ nodes located at the ends of such links, thus generating m_{tri} new triangles (when $m_{tri} > 1$, a further condition is enforced that the m_{tri} selected links are not pairwise adjacent, to avoid multiple links from the added node to single existing nodes in the graph).

Such a procedure can be conducted with or without adopting a preferential attachment rule. In the first case, the probability that the node added at time t selects the specific link (i, j) to form its connections is taken to be $P_{ij}(t) = \frac{1}{N_0(N_0-1)/2 + 2m_{tri}(t-1)}$, i.e. the m_{tri} links are randomly selected among all those which already exist in the graph at time $t - 1$ with equal probability (and therefore without applying a preferential rule). In the second case, instead, one has that $P_{ij}(t) = \frac{k_{ij}(t-1)}{\sum_{i,j} k_{ij}(t-1)}$, which implies that the larger is the number of triangles a given edge ij is part of at time $t - 1$ the larger the probability for that edge of being selected to form a new triangle with the added node.

Figures 1a and 1c report the degree distribution $P(k)$ of the networks (of size $N = 10,000$ nodes) generated by the two methods. One immediately sees that in both cases the graphs feature a clear power-law scaling, i.e. the scale-freeness which is indeed characterizing the vast majority of real world networks^{1-6,33,34}. In Figs. 1b and 1d we report, instead, a visualization of a typical synthesized network with $N = 200$ and $m_{tri} = 1$.

We also measure the generalized degree k_l . Given a graph of N nodes and its adjacency matrix A (the $N \times N$ matrix with entries $a_{ij} = 1$ if nodes i and j are connected by a link, and $a_{ij} = 0$ otherwise), the generalized degree k_{ij} of the link between node i and j is the (i, j) -entry of the matrix $A \oslash (A^2)$ (with the symbol $\oslash$ standing for the Hadamard product). Looking at Figure 1, one can immediately see that while the two generated networks are essentially heterogeneous in the node degree, the properties of k_l appear instead to be very different. In particular, the non preferential case leads to a much more restricted range of k_l values as compared to that generated by the preferential rule (see the two color bars at the right of the panels). Moreover, the structure obtained with no preferential attachment is homogeneous in terms of k_l (almost all the six colors, each one representing a given value of k_l , are visible). On the opposite, the preferential rule generates a SC with a high heterogeneity in k_l : almost all links are red (they have the lowest as possible value of k_l), and only one link (the blue one) displays a value of k_l equal to the maximum in the distribution.

These latter features are more quantitatively visible in Figure 2, which reports the distribution $P(k_l)$ of the generalized degree k_l for the two cases. Comparing Figs. 2a and 2b, one immediately realizes that while $P(k_l)$ is exponentially decaying when the growth is realized with no preferential attachment (Fig. 2a), its scaling is a clear power-law in the presence of preferential attachment. A relevant conclusion at this stage is that the two cases are imprinting a completely different topology in the triangular structures (reflected by completely different scaling properties in the distribution of the generalized degree k_l).

Non preferential model. We now furnish a full analytic treatment, and provide the rigorous expressions for the distributions $P(k)$ and $P(k_l)$. We start with the case of no preferential attachment, and we call $N(k, t)$ the number of nodes with degree k at time t . Its rate equation reads

$$\frac{dN(k, t)}{dt} = \frac{2m_{tri}}{\sum_k kN(k, t)} [-kN(k, t) + (k-1)N(k-1, t)] + \delta_{k, 2m_{tri}},$$

where $\frac{dN(k, t)}{dt} \equiv N(k, t+1) - N(k, t)$, and δ is the Kronecker delta-function.

For $N(t) \equiv \sum_k N(k, t) \simeq t$, one seeks a solution of the form $N(k, t) = tP(k)$, where $P(k)$ is assumed to be time independent. Since the total number of edges is approximately $2m_{tri}t$, one has $\sum_k kN(k, t) \simeq 4m_{tri}t$, and the rate equation for $P(k)$ becomes

$$P(k) = \frac{k-1}{k+2}P(k-1) + \frac{2}{k+2}\delta_{k, 2m_{tri}}.$$

Such a latter equation is the same as Eq. (6) of Ref.³⁹, and its solution (for $k \geq 2m_{tri}$) is

$$P(k) = \frac{4m_{tri}(2m_{tri}+1)}{k(k+1)(k+2)} \sim k^{-3}, \quad (1)$$

which perfectly fits the data (see the dotted, dashed-dotted, and dashed black lines in Fig. 1a).

Figure 2. The distribution of the generalized degree k_l . The distribution $P(k_l)$ vs. the generalized degree k_l obtained by growing a network of size $N = 10,000$ with the non preferential model **a** and with the preferential model **b**. The data refer to an ensemble average over 100 different realizations of the growth process. Notice that panel **a** is in log-linear scale, whereas panel **b** is in log-log scale. Legends in both panels report the color code for the number m_{tri} used in the growth process. The dashed line in panel **a** is used for exponential solution given by Eq. (2), while the dashed line in panel **b** is used for the power-law solution given by Eq. (4).

Then one can consider $N_e(k_l, t)$ as the number of edges participating in k_l triangles at time t . Its rate equation is

$$\frac{dN_e(k_l, t)}{dt} = -m_{tri} \frac{N_e(k_l, t)}{N_e(t)} + m_{tri} \frac{N_e(k_l - 1, t)}{N_e(t)} + 2m_{tri} \delta_{k_l, 1} \pm \dots,$$

where $N_e(t) \simeq 2m_{tri}t$ and the unwritten terms account for the formation of triangles from two or more linked edges. For the distribution of the generalized degree k_l , one has that $P(k_l) = N_e(k_l, t)/N_e(t)$, and the recursive equation is

$$P(k_l) = \frac{1}{3}P(k_l - 1) + \frac{2}{3}\delta_{k_l, 1},$$

admitting the following solution:

$$P(k_l) = \frac{2}{3^{k_l}}, \quad k_l \geq 1. \quad (2)$$

The exponential function (2) is reported as a dashed line in Fig. 2a, and one can see that the fit with numerical simulations is rather good, especially for the case $m_{tri} = 1$.

Preferential model. The analytic treatment of the preferential attachment case is far more complicated, as it implies the demonstration of a couple of Theorems and the extensive use of a known Lemma. We limit here ourselves to furnish the main results without reporting all the (sometimes cumbersome) formal mathematical steps, whereas the interested reader can find the full details within the Methods section. Calling again $N(k, t)$ the number of vertices with degree k at time t , its recurrence relation (see details in Methods) can be written as

$$N(k, t+1) = N(k, t) \left(1 - \frac{k}{3t}\right) + N(k-2, t) \frac{k-2}{3t} + O\left(\frac{1}{t^2}\right)$$

for $k > 2m_{tri}$, and

$$N(2m_{tri}, t+1) = N(2m_{tri}, t) \left(1 - \frac{2m_{tri}}{3t}\right) + 1 + O\left(\frac{1}{t^2}\right)$$

for $k = 2m_{tri}$.

In order to obtain an expression for $P(k)$, one supposes that $\frac{N(k, t+1)}{t+1} = \frac{N(k, t)}{t}$ for large t . Therefore, one gets $P(2m_{tri}) = \frac{3}{3+2m_{tri}}$ and $P(k) = P(k-2) \frac{k/2+1}{k/2+1.5}$, which ultimately gives $P(k) = \frac{3}{3+2m_{tri}} \prod_{l=m_{tri}+1}^{k/2} \frac{l-1}{l+1.5}$, or alternatively

$$P(k) = \frac{3}{3+2m_{tri}} \frac{\Gamma(k/2)\Gamma(m_{tri}+2.5)}{\Gamma(m_{tri})\Gamma(k/2+2.5)} \sim k^{-2.5}, \quad (3)$$

Figure 3. The mixed model. The distributions $P(k)$ [panels a-c] and $P(k_l)$ [panels d-f] obtained by growing networks of size $N = 50,000$ with the mixed model, at three different values of B (reported in each panel's top-right corner). The data (blue lines) refer to an average over 100 different realizations of the growth process. Dotted lines report, for comparison, the scaling exponents predicted by Eq. (5).

where Γ is here the gamma function. The power law scaling predicted by Eq. (3) fits remarkably well the numerical data (see the dotted, dashed-dotted, and dashed black lines in Figure 1c).

As for $P(k_l)$, one calls again $N_e(k_l, t)$ the number of edges participating in k_l triangles at time t . The recurrence relation for $N_e(k_l, t)$ (see details in Methods) is

$$N_e(k_l, t+1) = N_e(k_l, t) \left(1 - \frac{k_l}{3t}\right) + N_e(k_l-1, t) \frac{(k_l-1)}{3t} + O\left(\frac{1}{t^2}\right)$$

for $k_l > 1$, and

$$N_e(1, t+1) = N_e(1, t) \left(1 - \frac{m_{tri}}{3m_{tri}t}\right) + 2m_{tri} + O\left(\frac{1}{t^2}\right)$$

for $k_l = 1$. Imposing that $\frac{N_e(k_l, t+1)}{t+1} = \frac{N_e(k_l, t)}{t}$ for large t , one gets an equation for $P(k_l)$ which reads as $P(k_l) = P(k_l-1) \frac{k_l-1}{k_l+3}$, with $P(1) = \frac{3}{4}$. The solution is

$$P(k_l) = \frac{3}{4} \prod_{l=2}^{k_l} \frac{l-1}{l+3} = \frac{3}{4} \frac{4!}{k_l(k_l+1)(k_l+2)(k_l+3)} \sim k_l^{-4}. \quad (4)$$

The power-law function (4) is reported as a dashed line in Fig. 2b, and one can observe that the fit is, once again, extremely good.

Mixed model. Finally, our study can be extended and generalized to a mixed model, through which it is possible to effectively imprint *any desired* power-law scaling in the triangular structure of the network. To do so, we consider the case in which the probability that the node added at time t selects the specific link (i, j) to form its connections is

$$P_{ij}(t) = A \frac{2}{N_l(t-1)} + B \frac{3k_{ij}(t-1)}{2\sum_{ij} k_{ij}(t-1)},$$

for some constants A and B . Here, B is non-negative, $N_l(t) \sim 2m_{tri}t$ is the number of links at moment t and $\sum_{ij} k_{ij}(t-1) = 3m_{tri}t$ is the sum of the generalized degrees of all edges. Notice that $A = 0$ and $B = \frac{2}{3}$ ($A = \frac{1}{2}$ and $B = 0$) recovers the preferential (non preferential) case discussed above.

From the constrain that the sum of all probabilities must be equal to 1, it follows that A and B must obey $2A + \frac{3}{2}B = 1$, so that $A = 1/2 - 3/4B$. Furthermore, $P_{ij}(t)$ must be non-negative for all ij (also those for which $k_{ij} = 1$), and this gives the following bounds for A and B : $-1 \leq A \leq \frac{1}{2}$ and $0 \leq B \leq 2$. Under these conditions, one can analytically demonstrate (see Methods for details) that, for strictly positive B values the resulting networks display scale-free distributions $P(k) \sim k^{-\gamma}$ and $P(k_l) \sim k_l^{-\gamma_l}$ with exponents given by

$$\gamma = 1 + \frac{1}{A+B} = 1 + \frac{4}{2+B}, \quad \gamma_l = 1 + \frac{2}{B}. \quad (5)$$

For $B = 0$, one has instead (see Methods for full details) $P(k_l) = \frac{2}{3k_l}$. Eq. (5) implies that γ values are between 2 ($B = 2$) and 3 ($B = 0$), whereas γ_l is equal to 2 for $B = 2$, and tends to infinity as B tends to 0. On its turn, this means that choosing B between 1 and 2, γ and γ_l can be pre-selected *ad libitum* between 2 and 3, i.e. the imprinted structures of links and triangles feature well defined mean values of the degrees, but unbounded fluctuations as the system grows in size.

In Figure 3 we report $P(k)$ and $P(k_l)$ for three distinct values of B . It is seen that the fit between analytic predictions and numerically generated data is always remarkably good. Moreover, the Figure demonstrates that our method constitutes a highly general scheme by means of which one can construct, in a fully flexible way, ensembles of synthetic complexes displaying any desired statistical properties [from the condition of Figure 3a and 3d featuring a super scale-freeness - where even the mean degrees diverge in the thermodynamic limit-, to any milder condition which characterizes in fact many networks from the real world].

Discussion

In summary, complex networks encode the basic architecture of social, biological and technological networks, touching upon the most crucial challenges of modern science – from the spread of epidemics in social networks^{40,41} to the resilience of our eco-systems and critical infrastructure⁴².

In the context of pair-wise interactions, the most natural measure of centrality is a node's individual degree, capturing its potential dynamic impact on the system⁴³. The discovery that most real world networks exhibit extreme levels of degree heterogeneity was disruptive – indicating that networks are highly centralized, with a potentially disproportionate role played by a small fraction of their components^{33,34}.

As we deepen our investigation into the interaction patterns of complex systems it becomes increasingly clear, however, that higher order structures, beyond pair-wise interactions, underlie much of the observed richness of real-world networks. Hence, we sought the fundamental rules that prescribe centrality, and govern its distribution, in a simplicial complex environment. A natural measure is the number of complexes, here triangles, that a link participates in. Indeed, a simplicial complex represents a potentially functional unit, such as a collaboration of a social team¹⁶, or a trio of interacting biochemical agents^{19,20}. A component that is part of many such complexes is, therefore, likely central in the functionality of the system.

Several growing network models have been introduced and studied, which help exposing the roots of simplicial complex heterogeneity, shedding light on the emergence of centrality beyond the degree distribution. As we seek to understand the behavior of complex systems, their resilience and dynamic functionality, we hope that our comprehension into the microscopic processes of their formation can provide meaningful macroscopic insights.

Methods

In this final Section, we furnish all the details of the analytical results regarding our models. The starting point is that, as the synthesized networks have transitivity coefficient $T = 1$ (i.e. no links exist which do not form part of at least a triangle), the degree of each vertex is the number of triangles containing that vertex multiplied by 2, and one has that $N_v(k_v, t) = N(2k, t)$, where $N_v(k_v, t)$ ($N(2k, t)$) is the number of nodes participating in k_v triangles (having degree $2k$) at time t . As a consequence one has that $P(k_v) = P(k)$, and is therefore entitled to concentrate on either one of such distributions, depending on which one finds the simpler analytical treatment.

The non preferential attachment case. Let $N_v(k, k_v, t)$ be the number of nodes with k neighbours participating in k_v triangles

at time t . The rate equation is

$$\begin{aligned} \frac{dN_v(k, k_v, t)}{dt} = & -\frac{2m_{tri}}{\sum_k kN(k, t)} kN_v(k, k_v, t) \\ & + \frac{2m_{tri}}{\sum_k kN(k, t)} (k-1)N_v(k-1, k_v-1, t) \\ & + \delta_{k, 2m_{tri}} \delta_{k_v, m_{tri}} \\ & \pm \dots, \end{aligned} \quad (6)$$

where the unwritten terms account for the formation of triangles from two or more linked edges. In the sequel, we assume that the chosen edges are not linked (none of the nodes of a selected edge is linked to any of the nodes of another selected edge, which is always the case for $m_{tri} = 1$). Hence, the number of edges and triangles related to a given node increases one by one. Only the new nodes entering the system have the number of edges ($2m_{tri}$) double than the number of triangles in which they are participating (m_{tri}). This way, for k big enough, one has

$$kN_v(k, k_v, t) \simeq k_v N_v(k, k_v, t). \quad (7)$$

After summing over all values of k , one obtains an approximate equation for $N_v(k_v, t)$, the number of nodes participating in k_v triangles:

$$\begin{aligned} \frac{dN_v(k_v, t)}{dt} \simeq & -\frac{2m_{tri}}{\sum_k kN(k, t)} k_v N_v(k_v, t) \\ & + \frac{2m_{tri}}{\sum_k kN(k, t)} (k_v-1)N_v(k_v-1, t) \\ & + \delta_{k_v, m_{tri}}. \end{aligned} \quad (8)$$

One then can proceed as in the case of the degree distribution $P(k)$ (see the main text), and seek a solution of the form $N_v(k_v, t) = tP(k_v)$:

$$P(k_v) = \frac{k_v-1}{k_v+2} P(k_v-1) + \frac{2}{2+k_v} \delta_{k_v, m_{tri}}. \quad (9)$$

The solution for $k_v \geq m_{tri}$ is

$$P(k_v) = \frac{2m_{tri}(m_{tri}+1)}{k_v(k_v+1)(k_v+2)} \sim k_v^{-3}.$$

Therefore, one has also that $P(k) \sim k^{-3}$ which coincides with Eq. (1) in the main text.

The preferential attachment case. In order to obtain $P(k)$ and $P(k_t)$, one needs here to make use of the following Lemma from Ref.⁴⁴:

Lemma 1 Suppose that a sequence $\{a_t\}$ satisfies a recurrence relation

$$a_{t+1} = \left(1 - \frac{b_t}{t+t_1}\right) a_t + c_t,$$

where t_0 and t_1 are arbitrary, positive, fixed, values. Furthermore, suppose that $\lim_{t \rightarrow \infty} b_t = b > 0$ and $\lim_{t \rightarrow \infty} c_t = c > 0$.

Then $\lim_{t \rightarrow \infty} \frac{a_t}{t}$ exists and one has

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \frac{c}{1+b}.$$

For instance (and indicating by $\mathbb{E}(\cdot)$ the expectation value), $P(k)$ for $k = 2m_{tri}$ can be obtained by setting $b_t = b = \frac{2m_{tri}}{3}$ and $c_t = c = 1$. Application of Lemma 1 ensures that the limit $\lim_{t \rightarrow \infty} \frac{\mathbb{E}(N(2m_{tri}, t))}{t}$ exists and is equal to $\frac{3}{3+2m_{tri}}$, leading to $P(1) = \frac{3}{3+2m_{tri}}$.

On the other hand, $P(k)$ for $k > 2m_{tri}$ is obtained assuming that $P(k-2)$ exists, and applying Lemma 1 again with $b_t = b = \frac{k}{3}$ and $c_t = \mathbb{E}(N(k-2, t)) \frac{k-2}{3t}$ (i.e. taking $c = P(k-2) \frac{k-2}{3}$). Such a choice, indeed, entitles one to write a recurrence relation, and to obtain an explicit formula for $P(k)$:

$$P(k) = \frac{3}{3+2m_{tri}} \frac{\Gamma(k/2)\Gamma(m_{tri}+2.5)}{\Gamma(m_{tri})\Gamma(k/2+2.5)} \sim k^{-2.5}, \quad (10)$$

which coincides with Eq. (3) of the main text. Here, Γ is the gamma function.

Furthermore, one can demonstrate that $P(k)$ is sharp, by the use of the following Theorem:

Theorem 1 For any fixed $\varepsilon > 0$ and $\delta > 0$ and for any large enough t , the difference between the number of vertices with degree k at time t and $P(k)t$ is smaller than εt with probability larger than $1 - \delta$.

The theorem can be proved by considering the martingale $X_l = \mathbb{E}(N(k, t) | \mathcal{F}_l)$ (where $\mathcal{F}_l$ is the σ -algebra generated by the probability space at time l). It is rather easy to show that $|X_{l+1} - X_l|$ is bounded by 4, and therefore the theorem follows from the Azuma-Hoeffding inequality^{45,46}.

When trying to obtain $P(k_l)$, one can encounter a problem in the case in which the triangles added at step t have one or more common edges, so that the number of different edges added at time t might not be equal to $2m_{tri}$.

Denoting by $e(t, k_l)$ the number of added edges of degree k_l at time t , the more accurate recurrence relation for $k_l > 1$ is the following:

$$\begin{aligned} \mathbb{E}(N_e(k_l, t+1)) = & \mathbb{E}(N_e(k_l, t)) \left(1 - \frac{k_l}{3t}\right) + \\ & \mathbb{E}(N_e(k_l - 1, t)) \frac{(k_l - 1)}{3t} + O\left(\frac{1}{t^2}\right) + \mathbb{E}(e(t, k_l)), \end{aligned}$$

and for $k_l = 1$ one has

$$\mathbb{E}(N_e(1, t+1)) = \mathbb{E}(N_e(1, t)) \left(1 - \frac{1}{3t}\right) + O\left(\frac{1}{t^2}\right) + \mathbb{E}(e(t, 1)).$$

Using Theorem 1 (and some non trivial math, of which we omit the technical details), one can prove that $\lim_{t \rightarrow \infty} e(t, 1) = 2m_{tri}$ and $\lim_{t \rightarrow \infty} e(t, k_l) = 0$ for $k_l > 1$, so that one has $\lim_{t \rightarrow \infty} \mathbb{E}(e(t, 1)) = 2m_{tri}$ and $\lim_{t \rightarrow \infty} \mathbb{E}(e(t, k_l)) = 0$ for $k_l > 1$.

Furthermore, one can prove another theorem:

Theorem 2 The number of different edges divided by t converges always to $2m_{tri}$.

It follows that $P(k_l) = \lim_{t \rightarrow \infty} \frac{\mathbb{E}(N_e(k_l, t))}{2m_{tri}t}$.

Finally, one can apply Lemma 1 with $b_t = b = \frac{1}{3}$ and $c = \lim 1 + O\left(\frac{1}{t^2}\right) = 1$ to get that $P(1) = \lim_{t \rightarrow \infty} \frac{\mathbb{E}(N_e(1, t))}{2m_{tri}t}$ exists and is equal to $\frac{3}{4}$. For $k_l > 1$, one assumes that $P(k_l - 1) \lim_{t \rightarrow \infty} \frac{\mathbb{E}(N_e(k_l - 1, t))}{2m_{tri}t}$ exists, and applies again Lemma 1 with $b_t = b = \frac{k_l}{3}$ and $c = \lim c_t = P(k_l - 1) \frac{(k_l - 1)}{3t}$. Hence, $P(k_l) = \lim_{t \rightarrow \infty} \frac{\mathbb{E}(N_e(k_l, t))}{2m_{tri}t}$ exists and is equal to $P(k_l - 1) \frac{k_l - 1}{k_l + 3}$. From such a recurrence relation, one finally obtain the explicit formula for $P(k_l)$:

$$P(k_l) = \frac{18}{k_l(k_l + 1)(k_l + 2)(k_l + 3)} \sim k_l^{-4}, \quad (11)$$

which is identical to Eq. (4) of the main text.

Finally, since the degree of the vertex is the number of triangles containing this vertex multiplied by 2, one has $N_v(k_v, t) = N(2k, t)$ and as a consequence one obtains

$$P(k_v) = \frac{3}{3 + 2m_{tri}} \frac{\Gamma(k_v) \Gamma(m_{tri} + 2.5)}{\Gamma(m_{tri}) \Gamma(k_v + 2.5)} \sim k_v^{-2.5}. \quad (12)$$

The mixed case. The recurrence relation for $N_v(k_v, t)$ (the number of vertices participating in k_v triangles at time t) is

$$\begin{aligned} \mathbb{E}(N_v(k_v, t+1)) = & \mathbb{E}(N_v(k_v, t)) \left(1 - A \frac{m_{tri}}{t} - (A+B) \frac{k_v}{t}\right) + \\ & \mathbb{E}N_v(k_v - 1, t) \left(A \frac{m_{tri}}{t} + (A+B) \frac{(k_v - 1)}{t}\right) + O\left(\frac{1}{t^2}\right). \end{aligned}$$

For $t > 0$ and $k_v = m_{tri}$ one has

$$\mathbb{E}(N_{t+1}^v(m_{tri})) = \mathbb{E}(N_v(m_{tri}, t)) \left(1 - A \frac{m_{tri}}{t} - (A+B) \frac{m_{tri}}{t}\right) + 1.$$

Furthermore, the use of Lemma 1 gives

$$P(m_{tri}) = \frac{1}{1 + 2Am_{tri} + Bm_{tri}}$$

and

$$P(k_v) = P(k_v - 1) \frac{Am_{tri} + (A + B)(k_v - 1)}{1 + Am_{tri} + (A + B)k_v}.$$

Therefore, one has

$$P(k_v) = \frac{1}{1 + 2Am_{tri} + Bm_{tri}} \prod_{l=m_{tri}+1}^{k_v} \frac{Am_{tri} + (A + B)(l - 1)}{1 + Am_{tri} + (A + B)l}. \quad (13)$$

or

$$\begin{aligned} P(k_v) &= \frac{1}{1 + 2Am_{tri} + Bm_{tri}} \cdot \prod_{l=m_{tri}+1}^{k_v} \frac{\frac{Am_{tri}}{A+B} + l - 1}{\frac{1+Am_{tri}}{A+B} + l} = \\ &= \frac{1}{1 + 2Am_{tri} + Bm_{tri}} \cdot \frac{\Gamma(k_v + \frac{Am_{tri}}{A+B})\Gamma(m_{tri} + \frac{1+Am_{tri}}{A+B} + 1)}{\Gamma(m_{tri} + \frac{Am_{tri}}{A+B})\Gamma(k_v + \frac{1+Am_{tri}}{A+B} + 1)} \sim k_v^{-\gamma_v}, \end{aligned}$$

where $\gamma_v = 1 + \frac{1}{A+B}$.

On the other hand, one can consider $N_l(k_l, t)$, the number of links with degree k_l at moment t . Its recurrence equation is

$$\begin{aligned} \mathbb{E}(N_l(k_l, t + 1)) &= \mathbb{E}N_l(k_l, t) \left(1 - A\frac{1}{t} - B\frac{k_l}{2t}\right) + \\ &\quad \mathbb{E}N_l(k_l - 1, t) \left(A\frac{1}{t} + B\frac{k_l - 1}{2t}\right) + O\left(\frac{1}{t^2}\right). \end{aligned}$$

For $k_l = 1$ one has

$$\mathbb{E}(N_l(1, t + 1)) = \mathbb{E}(N_l(1, t)) \left(1 - A\frac{1}{t} - B\frac{1}{2t}\right) + 2m_{tri} + O\left(\frac{1}{t^2}\right).$$

One can apply again Lemma 1, and obtain

$$P(1) = \frac{2m_{tri}}{1 + A + \frac{1}{2}B}, \quad P(k_l) = P(k_l - 1) \frac{2A + B(k_l - 1)}{2 + 2A + Bk_l}.$$

Therefore, one gets

$$P(k_l) = \frac{2m_{tri}}{1 + A + \frac{1}{2}B} \prod_{l=2}^{k_l} \frac{2A + B(l - 1)}{2 + 2A + Bl}. \quad (14)$$

Let us consider separately the cases $B = 0$ (the fully non preferential case) and $B > 0$.

For $B = 0$, one has

$$P(k_l) = \frac{2m_{tri}}{1 + A} \prod_{l=2}^{k_l} \frac{2A}{2 + 2A}.$$

Since $A = \frac{1}{2}$, one obtains

$$P(k_l) = \frac{4m_{tri}}{3} \prod_{l=2}^{k_l} \frac{1}{3} = \frac{4m_{tri}}{3} \frac{1}{3^{k_l-1}},$$

which fully coincide with the exponential scaling derived in the main text for the non preferential case.

On the other hand, when instead $B > 0$, one has

$$\begin{aligned} P(k_l) &= \frac{2m_{tri}}{1 + A + \frac{1}{2}B} \prod_{l=2}^{k_l} \frac{\frac{2A}{B} + l - 1}{\frac{2}{B} + \frac{2A}{B} + l} = \\ &= \frac{2m_{tri}}{1 + A + \frac{1}{2}B} \frac{\Gamma(\frac{2+2A}{B} + 2)\Gamma(\frac{2A}{B} + k_l)}{\Gamma(\frac{2A}{B} + 1)\Gamma(\frac{2+2A}{B} + k_l + 1)} \sim k_l^{-\gamma_l}, \end{aligned}$$

where $\gamma_l = 1 + \frac{2}{B}$.

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Author contributions statement

K.K., I.S.N. and N.K. equally contributed to the manuscript. S.B. and B.B. equally contributed to the manuscript. S.B., B.B. and I.S.N. conceived the research study. K.K. and N.K. carried out all the analytical treatments. I.S.N. wrote the codes and conducted the simulations and the analysis of the data. A.M.R. proved Theorem 2. All Authors discussed the results, drawn the conclusions, and jointly wrote the Manuscript.

Additional information

Supplementary Information accompanies this paper with the codes used in this study.

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Figures

Figure 1

The structures generated in the preferential and the non preferential cases. The model without preferential attachment (panels a and b) and that with preferential attachment (panels c and d). Panels a and c: log-log plot of the degree distribution $P(k)$ of the resulting networks for different values of m_{tri} (see color code in the legend). Data are obtained as an ensemble average over 100 different realizations of a network with size $N = 10,000$ nodes. Dashed lines in a (c) correspond to the analytical predictions given by Eqs. (1) [(3)]. Panels b and d: schematic visualization of generated networks with $N = 200$ and $m_{tri} = 1$.

The size of the nodes is proportional to 50 times the square root of the corresponding entry in the eigenvector centrality³⁶, the width of each link is proportional to p_{kl} , and the color of the links encodes k_l as reported in the bars at the right of both panels.

Figure 2

The distribution of the generalized degree k_l . The distribution $P(k_l)$ vs. the generalized degree k_l obtained by growing a network of size $N = 10,000$ with the non preferential model a and with the preferential model b. The data refer to an ensemble average over 100 different realizations of the growth process. Notice that panel a is in log-linear scale, whereas panel b is in log-log scale. Legends in both panels report the color code for the number m_{tri} used in the growth process. The dashed line in panel a is used for exponential solution given by Eq. (2), while the dashed line in panel b is used for the power-law solution given by Eq. (4).

Figure 3

The mixed model. The distributions $P(k)$ [panels a-c] and $P(k_l)$ [panels d-f] obtained by growing networks of size $N = 50,000$ with the mixed model, at three different values of B (reported in each panel's top-right corner). The data (blue lines) refer to an average over 100 different realizations of the growth process. Dotted lines report, for comparison, the scaling exponents predicted by Eq. (5).