Riemann-Hilbert Approach of the Complex Sharma-Tasso-Olver Equation and its N -soliton solutions *

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Abstract: In this paper, we use Riemann-Hilbert method to study the N -soliton solutions of the complex Sharma-Tasso-Olver(cSTO) equation. And then, based on analyzing the spectral problem of the Lax pair, the matrix Riemann-Hilbert problem for this integrable equation can be constructed, the N -soliton solutions about this system are given explicitly under the relationship of scattering matrix. At last, under the condition that some specific parameter values are given, the three-dimensional diagrams of the 2-soliton solution and the trajectory of the soliton solution will be simulated.

Keywords: the complex Sharma-Tasso-Olver equation; the Riemann-Hilbert problem; the scattering relationship; N -soliton solutions.

MSC codes: 35Q51; 35Q15; 37K10

1 Introduction

As we all know, the Sharma-Tasso-Olver(STO) equation

$$u_y + \frac{1}{2}(u_{xxx} - 3(u_x u)_x - 3u^2 u_x) = 0, \quad (1.1)$$

was first proposed to an example of odd members of Burgers hierarchy, which extends the "linearization" achieved via the Cole-Hopf ansatz to equations consisting of highest derivatives odd space derivatives. Moreover, due to the application of this equation in mathematical physics, many physicists and mathematicians showed great interest in it [1]. In this paper, we consider the complex Sharma-Tasso-Olver(cSTO) equation that was derived by Fan [2]

$$u_y + \frac{1}{2}(u_{xxx} - 3i(u_x u)_x - 3u^2 u_x) = 0, \quad (1.2)$$

and given the initial value $u(x, 0) = u_0(x) \in \mathbb{S}(\mathbb{R})$, where $\mathbb{S}(\mathbb{R}) = \{f(x) | \int_{-\infty}^{\infty} [1 + |x|^p f(x)] dx < \infty, p > 1\}$ is the Schwartz space. What's more, with the fast development of the integrable

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systems, it plays an important role in mathematical physics in recent years. In fact, it is significant to find the solutions and symmetries of the cSTO equation.

Therefore, constructing the N -soliton solutions of the cSTO equation via Riemann-Hilbert approach is very meaningful. Such problem goes back to 1997, when Fokas proposed the matrix Riemann-Hilbert problem(RHp) [3], which can be used not only to solve the initial-boundary value problems [4–9] and the long-time asymptotic behavior of the solutions [10], but also to obtain the solutions of the integral equations by the theory of inverse scattering [11–14]. It is because the Riemann-Hilbert approach is so widely used that it has developed into an important tool for solving some problems in applied mathematics and other areas.

Moreover, in addition to the Riemann-Hilbert method, there are bilinear method, Darboux transformation, etc., which are frequently used to solve soliton solutions of nonlinear partial differential equations, such as the multi-component AKNS integrable hierarchies [15], the N -coupled Hirota equations [16], the coupled nonlinear Schrödinger equations [17], the nonlocal Fokas-Lenells equation [18], the generalized Sasa-Satsuma equation [19], the coupled derivative Schrödinger equations [20], the (4+1)-dimensional Fokas equation [21], the (3+1)-dimensional extended Jimbo-Miwa equation [22] and so on.

The content of this paper is divided into the following parts. In the **Section 2**, firstly, based on the Lax pair of the cSTO equation, we have the Riemann-Hilbert problem for the cSTO equation, meanwhile, the variable transformation and the spectrum analysis can be applied in this process. In the **Section 3**, through Riemann-Hilbert method and the scattering relationship, the N -soliton solutions of the cSTO equation can be obtained in some detail, then under the results obtained above, we can give some three-dimensional figures for the case of some specific parameters. A brief conclusion for this paper and some discussion about this equation can be given in the **last Section**.

2 Fundamentals of the cSTO equation

Firstly, let's introduce some of the basic of RHp:

Definition 1. *Let the contour Γ be the union of a finite number of smooth and oriented curves on the Riemann sphere $\mathbb{C}$, such that $\mathbb{C} \setminus \Gamma$ has only a finite number of connected components. Let $J(\lambda)$ be a 2×2 matrix defined on the contour Γ , the RHP (Γ, J) is the problem that devotes to find a 2×2 matrix-valued function $M(\lambda)$ that satisfies:*

1. $M(\lambda)$ is analytic for all $k \in \mathbb{C} \setminus \Gamma$, and extends continuously to the contour Γ ;
2. $M_+(\lambda) = M_-(\lambda)J(\lambda)$, $\lambda \in \Gamma$;
3. $M(\lambda) \rightarrow I$, as $\lambda \rightarrow \infty$.

2.1 Lax pair for the cSTO equation

The cSTO equation admits the Lax pair

$$\begin{cases} \varphi_x = M\varphi = (-i\lambda^2\sigma_3 - \frac{1}{2}iu\sigma_3 + \lambda R)\varphi, \\ \varphi_y = N\varphi = (-2i\lambda^6\sigma_3 + U + V\sigma_3)\varphi, \end{cases} \quad (2.1)$$

where $R = \begin{pmatrix} 0 & u \\ 2 & 0 \end{pmatrix}$, define $R = R^H$, the superscript H represents conjugate transpose, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, σ_3 is the third Pauli's matrix, $U = 2\lambda^5 R + \lambda^3(iR_x + uR) + \lambda(-\frac{1}{2}R_{xx} + \frac{3}{8}i(R^3)_x - \frac{1}{2}iRR_x\sigma_- + \frac{1}{2}u^2R)$, $V = -2i\lambda^4 u + \lambda^2(u_x - iu^2) + \frac{1}{4}iu_{xx} + \frac{3}{4}uu_x - \frac{1}{4}iu^3$, and $\lambda \in \mathbb{C}$ is the spectral parameter. Then we can check that the compatibility condition of the Lax pair in (2.1) about the cSTO equation generates the zero-curvature equation $M_y - N_x + [M, N] = 0$, and it can yield the cSTO equation (1.2), the square bracket is the usual matrix commutator defined as $[M, N] = MN - NM$ in the equation we mentioned above. In what follows, we're going to extend the column vector φ to a 2×2 matrix and introduce a matrix $J_{2 \times 2}$ for convenience, so the formula can be shown

$$J = \varphi e^{i\lambda^2 x \sigma_3 + 2i\lambda^6 y \sigma_3}, x \rightarrow \pm\infty, \quad (2.2)$$

according to the Equ.(2.2), the Lax pair (2.1) can be rewritten as follows:

$$\begin{cases} J_x + i\lambda^2[\sigma_3, J] = M' J, \\ J_y + 2i\lambda^6[\sigma_3, J] = N' J, \end{cases} \quad (2.3)$$

where $M' = -\frac{1}{2}iu\sigma_3 + \lambda R$, $N' = U + V\sigma_3$, $[\sigma_3, J] = \sigma_3 J - J\sigma_3$. Indeed, it can be written in full derivative form:

$$d(e^{i(\lambda^2 x + 2\lambda^6 y)\hat{\sigma}_3} J) = W, \quad (2.4)$$

where

$$W = e^{i(\lambda^2 x + 2\lambda^6 y)\hat{\sigma}_3} (M' dx + N' dy). \quad (2.5)$$

Here it should be noted that $\hat{\sigma}_3$ represents the matrix commutator of σ_3 , i.e., $\hat{\sigma}_3 \kappa = [\sigma_3, \kappa]$, and $e^{\hat{\sigma}_3}(\kappa) = e^{\sigma_3} \kappa e^{-\sigma_3}$ (κ is a 2×2 matrix).

2.2 Associated Riemann-Hilbert problems for the cSTO equation

Since the initial-value $u(x, 0)$ belongs to the Schwartz space, we can then get the asymptotic condition on the basis of the matrix spectral problems of the Lax pair (2.1), and have the following equation:

$$\varphi \sim e^{-i\lambda^2 x \sigma_3 - 2i\lambda^6 y \sigma_3}, \quad (2.6)$$

where $x, y \rightarrow \pm\infty$. According to the equivalent variable transformation and $\text{tr}(M') = \text{tr}(N') = 0$, we can obtain that

$$\det J = 1, \quad (2.7)$$

founded on the generalized Liouville's formula. Due to the importance of the spectrum problem of the cSTO equation, the analysis and symmetry of the spectral function should be put on the first step, and then, we can introduce the following two related Volterra integral equations:

$$J_L(x, \lambda) = I + \int_{-\infty}^x e^{i\lambda^2(x-\tau)\hat{\sigma}_3} (M'(\tau) J_L(\tau, \lambda)) d\tau, \quad (2.8)$$

$$J_R(x, \lambda) = I - \int_x^{\infty} e^{i\lambda^2(x-\tau)\hat{\sigma}_3} (M'(\tau) J_R(\tau, \lambda)) d\tau, \quad (2.9)$$

where I is an identity matrix. Define

$$J_L = (J_L^1, J_L^2) = \begin{pmatrix} J_L^{11} & J_L^{12} \\ J_L^{21} & J_L^{22} \end{pmatrix}, J_R = (J_R^1, J_R^2) = \begin{pmatrix} J_R^{11} & J_R^{12} \\ J_R^{21} & J_R^{22} \end{pmatrix}, \quad (2.10)$$

where the first column of J_L and J_R are J_L^1 and J_R^1 , J_L^2 , J_R^2 are the second column of J_L and J_R , respectively. According to the Volterra integral equation (2.8), the formula $e^{i\lambda^2(x-\tau)\sigma_3} M' e^{-i\lambda^2(x-\tau)\sigma_3}$ can be calculated, and then the following formula is obtained

$$e^{i\lambda^2(x-\tau)\sigma_3} M' e^{-i\lambda^2(x-\tau)\sigma_3} = \begin{pmatrix} -\frac{1}{2}iu & \lambda u e^{2i\lambda^2(x-\tau)} \\ 2\lambda e^{-2i\lambda^2(x-\tau)} & \frac{1}{2}iu \end{pmatrix}. \quad (2.11)$$

Given the above calculation results and the equation (2.10) defines the formula, we can obtain the facts that J_L^1 is analytic in $D_1 = \{\lambda^2 | \text{Im } \lambda^2 < 0\}$, J_L^2 is analytic in $D_2 = \{\lambda^2 | \text{Im } \lambda^2 > 0\}$, J_R^1 is analytic in D_2 , J_R^2 is analytic in D_1 , respectively. Actually, due to the Abel's formula and $\text{tr}(M') = 0$, it can derive that the determinants of J_L and J_R are independent for all x ,

$$\begin{cases} J_L \rightarrow I, x \rightarrow -\infty, \\ J_R \rightarrow I, x \rightarrow +\infty. \end{cases} \quad (2.12)$$

From two matrix solutions of (2.12), we can express them as

$$\varphi_L = J_L E, \varphi_R = J_R E, E = e^{i\lambda^2 x \sigma_3}, \quad (2.13)$$

because of their linear correlation, the formula about them can be put as follows

$$J_L E = J_R E R(\lambda), \quad (2.14)$$

where $R(\lambda) = (r_{ij})_{2 \times 2}$ is the scattering matrix. In fact, it is evident that $\det R(\lambda) = 1$ as $\det J_L = 1$ and $\det J_R = 1$. In what follows, assuming that

$$J_L^{-1} = \begin{pmatrix} (J_L^1)^{-1} \\ (J_L^2)^{-1} \end{pmatrix}, J_R^{-1} = \begin{pmatrix} (J_R^1)^{-1} \\ (J_R^2)^{-1} \end{pmatrix}, \quad (2.15)$$

analytical analysis of each column vector of J_L and J_R according to the above. It is accessible that $(J_L^1)^{-1}$ is analytic in D_2 , $(J_L^2)^{-1}$ is analytic in D_1 , $(J_R^2)^{-1}$ is analytic in D_2 , $(J_R^1)^{-1}$ is analytic in D_1 . In accordance with the equation of (2.14), we can obtain that the analyticity of $R(\lambda)$ in the following

$$ER(\lambda)E^{-1} = J_R^{-1} J_L = \begin{pmatrix} (J_R^1)^{-1} J_L^1 & (J_R^1)^{-1} J_L^2 \\ (J_R^2)^{-1} J_L^1 & (J_R^2)^{-1} J_L^2 \end{pmatrix}, \quad (2.16)$$

indeed, r_{11} is analytic in D_1 , r_{22} is analytic in D_2 . Using Riemann-Hilbert method is very extensive and important for solving a class of equations, so we will construct the RHP for the cSTO equation in this section. First of all, we can introduce two matrix functions that consisted of J_L , J_R , $(J_L)^{-1}$, $(J_R)^{-1}$ and letting

$$\begin{aligned} P_R &= (J_L^1, J_R^2) \\ &= J_L L_1 + J_R L_2, \\ P_L &= ((J_L^1)^{-1}, (J_R^2)^{-1}) \\ &= L_1 (J_L)^{-1} + L_2 (J_R)^{-1}, \end{aligned}$$

where $L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Furthermore, it implies that P_R is analytic in D_1 , P_L is analytic in D_2 and $\begin{cases} P_R \rightarrow I \\ P_L \rightarrow I \end{cases}$, where $\lambda \rightarrow \infty$. In the context of the previous preparation knowledge of the cSTO equation, the related Riemann-Hilbert problem can be given as below:

1. P_R is analytic in D_1 , P_L is analytic in D_2 , respectively;
2. $P_L P_R = G(x, y; \lambda)$;
3. $P \rightarrow I$, when $\lambda \rightarrow \infty$,

where $G(x, y; \lambda) = e^{(i\lambda^2 x + 2i\lambda^6 y)\sigma_3} \begin{pmatrix} 1 & h_{12} \\ r_{21} & 1 \end{pmatrix} e^{-(i\lambda^2 x + 2i\lambda^6 y)\sigma_3}$ is the jump matrix and $R^{-1} = H = (h_{ij})_{1 \leq i, j \leq 2}$. On account of $\det R(\lambda) = 1$, we can get that $h_{11}r_{11} + h_{12}r_{21} = 1$.

3 N-soliton solutions for the cSTO equation

In this section, through the Riemann-Hilbert problem, the cSTO equation can be given specific soliton solutions. Now let's assume that the Riemann-Hilbert is irregular, and then, we can obtain that $\det P_R$ and $\det P_L$ make sense in their own analytic areas except for a few zeros. On the basis of the definitions of P_R, P_L and the scattering relationship (2.14), we can get the following form

$$\begin{cases} \det P_L = \det(L_1 J_L^{-1} + L_2 J_R^{-1}) = h_{11}(\lambda), \\ \det P_R = \det(J_L L_1 + J_R L_2) = r_{11}(\lambda), \end{cases} \quad (3.1)$$

it is obvious that the zeros of $\det P_L$ and $\det P_R$ are the same as $h_{11}(\lambda)$ and $r_{11}(\lambda)$, respectively. Furthermore, since the relation

$$J_{R,L}^H(\lambda^*) = J_{R,L}^{-1}(\lambda), \quad (3.2)$$

the following facts can be obtained that

$$P_R^H(\lambda^*) = P_L^{-1}(\lambda), R^H(\lambda^*) = R^{-1}(\lambda), \lambda \in D_2. \quad (3.3)$$

Indeed, according to the equation of (3.3), the formula that we can obtain

$$h_{11}^H(\lambda^*) = r_{11}(\lambda), \quad (3.4)$$

and each zero $\pm\lambda_k^*$ of h_{11} corresponds to each zero $\pm\lambda_k$ of r_{11} . Suppose that $\det(P_R)$ exists N simple zeros $\tilde{\lambda}_1 \cdots \tilde{\lambda}_N \in D_1$ and $\det(P_L)$ exists N simple zeros $\lambda_1^* \cdots \lambda_N^* \in D_2$, then, it is shown that

$$\lambda_k^* = \tilde{\lambda}_k, \quad 1 \leq k \leq N. \quad (3.5)$$

Now, assume that η_k is nonzero column vector and η_k^* is nonzero row vector. What follows presents the linear equations about η_k and η_k^* .

$$\begin{cases} P_R(\lambda_k)\eta_k(\lambda_k) = 0, \\ \eta_k^*(\lambda_k^*)P_L(\lambda_k^*) = 0. \end{cases} \quad (3.6)$$

From the formulas above, we can get the relation of the eigenvectors

$$\eta_k^* = \eta_k^H, \quad 1 \leq k \leq N, \quad (3.7)$$

and

$$\begin{cases} P_R(\lambda_k)(\frac{\partial \eta_k}{\partial x} - i\lambda_k^2 \sigma_3 \eta_k) = 0, \\ P_R(\lambda_k)(\frac{\partial \eta_k}{\partial y} - 2i\lambda_k^6 \sigma_3 \eta_k) = 0. \end{cases} \quad (3.8)$$

Comparing with the Lax pair (2.3), (3.6) and (3.7), we can get the expression of η_k and η_k^*

$$\begin{cases} \eta_k = e^{i\lambda_k^2 x \sigma_3 + 2i\lambda_k^6 y \sigma_3} \eta_{k,0}, \quad 1 \leq k \leq N, \\ \eta_k^* = \eta_{k,0}^H e^{-i(\lambda_k^*)^2 x \sigma_3 - 2i(\lambda_k^*)^6 y \sigma_3}, \quad 1 \leq k \leq N, \end{cases} \quad (3.9)$$

where $\eta_{k,0} = (\alpha_k, \beta_k)^T$ ($1 \leq k \leq N$) is two-dimension constant column vector. When the scattering data satisfies the condition $r_{21} = 0$, the soliton solutions can be presented explicitly. Hence, the solutions of the Riemann-Hilbert problem can be given in the following

$$\begin{cases} P_R(\lambda) = I - \sum_{k,j=1}^N \frac{\eta_k \eta_j^* (D^{-1})_{kj}}{\lambda - \lambda_j^*}, \\ P_L(\lambda) = I + \sum_{k,j=1}^N \frac{\eta_k \eta_j^* (D^{-1})_{kj}}{\lambda - \lambda_k}, \end{cases} \quad (3.10)$$

where D is a $N \times N$ matrix that can be defined by

$$d_{kj} = \frac{\eta_k^* \eta_j}{\lambda_j - \lambda_k^*}, \quad 1 \leq k, j \leq N. \quad (3.11)$$

In fact, we can express the asymptotic expansion of $P_R(\lambda)$ as

$$P_R = I + \frac{P_R^1}{\lambda} + \frac{P_R^2}{\lambda^2} + O(\frac{1}{\lambda^3}), \quad \lambda \in D_1 \rightarrow \infty. \quad (3.12)$$

Moreover, the equation of (3.9) can be substituted into the first equation of (2.3), and then, the formula we can get as below

$$u = 2i(P_R^1)_{12}, \quad (3.13)$$

where $(P_R^1)_{12}$ is the $(1,2)$ -element of P_R^1 . Meanwhile, through the equation of (3.10), we can deduce the expression of P_R^1 as follows

$$P_R^1 = - \sum_{k=1}^N \sum_{j=1}^N \eta_k \eta_j^* (D^{-1})_{kj}. \quad (3.14)$$

And so we can obtain the N -soliton solutions of the cSTO equation (1.1)

$$u(x, y) = -2i \sum_{k=1}^N \sum_{j=1}^N \eta_k \eta_j^* (D^{-1})_{kj}, \quad (3.15)$$

then, under the condition of the formula (3.15) established above, when $N = 2$, the 2-soliton solution of the cSTO equation (1.1) can be given by

$$u(x, y) = -\frac{2i}{d_{11}d_{22} - d_{12}d_{21}}(\alpha_1\beta_2^*d_{12}e^{\vartheta_2^* - \vartheta_1} - \alpha_1\beta_1^*d_{22}e^{\vartheta_1^* - \vartheta_1} + \alpha_2\beta_1^*d_{21}e^{\vartheta_1^* - \vartheta_2} - \alpha_2\beta_2^*d_{11}e^{\vartheta_2^* - \vartheta_2}),$$

where $d_{11} = \frac{1}{\lambda_1 - \lambda_1^*}(|\alpha_1|^2 e^{-\vartheta_1^* - \vartheta_1} + |\beta_1|^2 e^{\vartheta_1^* + \vartheta_1})$, $d_{12} = \frac{1}{\lambda_2 - \lambda_1^*}(\alpha_1^* \alpha_2 e^{-\vartheta_1^* - \vartheta_2} + \beta_1^* \beta_2 e^{\vartheta_1^* + \vartheta_2})$, $d_{21} = \frac{1}{\lambda_1 - \lambda_2^*}(\alpha_2^* \alpha_1 e^{-\vartheta_2^* - \vartheta_1} + \beta_2^* \beta_1 e^{\vartheta_2^* + \vartheta_1})$, $d_{22} = \frac{1}{\lambda_2 - \lambda_2^*}(|\alpha_2|^2 e^{-\vartheta_2^* - \vartheta_2} + |\beta_2|^2 e^{\vartheta_2^* + \vartheta_2})$ and $\vartheta_1 = i\lambda_1^2 x + 2i\lambda_1^6 y$, $\vartheta_2 = i\lambda_2^2 x + 2i\lambda_2^6 y$. What's more, under the condition that some specific parameter values are given, some three-dimensional plots of the 2-soliton solutions and the trajectory of the soliton solution can be shown in Figure 1 and Figure 2.

Figure 1: Evolution plot of the 2-soliton solutions to the cSTO equation with parameters by $\alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = -i, \alpha = 1, \lambda_1 = -\frac{1}{2} - \frac{1}{5}i, \lambda_2 = -\frac{3}{5} - \frac{2}{5}i$.

Figure 2: The soliton of $|u|$ along x axis with different y .

4 Conclusion and Discussion

In this paper, according to the Riemann-Hilbert method, the N -soliton of the cSTO equation can be constructed. And the next step is to try to generalize the cSTO equation in multiple components, such as the multi-component generalization of the Kundu-NLS equation [23]. In fact, the cSTO equation is a kind of significant equation in mathematical physics, so it is very meaningful to do something similar to the STO equation. Moreover, we can use Riemann-Hilbert method to solve nonlocal integrable equations and integrable hierarchies. Therefore, it is important to continue to study it. Meanwhile, the mathematical structure and physical properties of the cSTO equation are also need to study.

5 Conflict of Interest

The authors declare that they have no conflict of interest.

6 Data availability statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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