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Yang Liu (✉ 2016liuyang@tongji.edu.cn)

Liaocheng University

Liying Liu

Liaocheng University

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Lookback option pricing problem of uncertain mean-reverting currency model

Yang Liu^{a*}, Liying Liu^b

a. School of Business, Liaocheng University, Liaocheng 252059, China*

b. School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China

2016liuyang@tongji.edu.cn, lyliu001@163.com

Abstract

A lookback option is a maturity option that pays off based on the maximum or minimum stock price over the life of the option. This paper investigates the problem of pricing a lookback option based on the uncertain mean-reverting currency model and designs the algorithms to calculate the formulations. Furthermore, discussions about parameters and results are drawn in the paper.

Keywords: uncertainty theory, uncertain differential equation, lookback option pricing, currency model

1 Introduction

The option pricing problem which was proposed by Bachelier [1] in 1900 was constantly popular. But it has not been solved scientifically until stochastic process differential equation was founded in 1951 by Ito [2]. Following that, Black and Scholes [3] investigated the famous Black-Scholes option pricing formula in 1973, which assumed the stock price followed a geometric Wiener process. Considering the jump phenomena of stock price, Merton [4] reconstructed the model. After that, amount of stochastic differential equations had been used to describe the stock price and various option pricing formulas were derived.

In the application of Black-Scholes option pricing formula, investors pointed out the strict restriction of the prerequisite to use the formula. In addition, two paradoxes were founded by Liu [5] show that the stock price in real financial market is impossible to follow stochastic differential equations.

To deal with the conflicts between real mark and probability theory, uncertainty theory was built by Liu [6] in 2007, which was based on normality, duality, subadditivity and product axioms. Then it was applied to amount of fields, such as uncertain programming [7], uncertain logic [8], uncertain reliability analysis and uncertain risk analysis [9]. The basic concept uncertain variable in uncertainty theory was defined by Liu [6] to model the quantity associated with human uncertainty. Meanwhile, uncertainty distribution was given to describe uncertain variables, excepted value was provided to rank uncertain variables.

Uncertain process, a sequence of uncertain variables driven by time, was explored by Liu [10] in 2008 to model the evolution of uncertain phenomena. As a special and important type of uncertain process, Liu [7] designed Liu process within the framework of uncertainty theory as a counterpart of Wiener process. Following that, Liu [11] began the pioneer work of uncertain finance based on the assumption of the stock price's fluctuation follows a geometric Liu process. After that, its European option, American option [12] and Asian option [13] [14] pricing formulas were derived respectively.

Lookback option is a path-dependent option that offers the payoff depends on the maximum or minimum

value of the underlying asset price. The pricing problem of lookback option has been widely discussed since it was first valued by Goldman et al. [15]. Conze and Viswannathan [16] used the joint distribution of the spot price and its historical maximum (or minimum) to obtain the closed-form valuation formulas. Quanto lookback option price formulas were derived by Dai et al. [17]. With framework of uncertainty theory, Zhang et al. [18] firstly investigated the lookback option problem. Based on that, Gao et al. [19] developed the lookback call and put option pricing formulas following the uncertain exponential Ornstein-Uhlenbeck model.

By viewing the foreign exchange rate as uncertain process and describing an uncertain differential equation as Liu process, an uncertain currency model was proposed by Liu et al. [20] in 2015. Based on the foundation, Shen and Yao [21] presented a mean-reverting currency model by considering the exchange rate in the long term in the real global market. Sheng [22] investigated the Asian currency option pricing problem following the mean-reverting stock model. Then Wang and Ning [23] built a uncertain currency model with floating interest rates by considering the uncertain fluctuations in the financial market from time to time.

The existing scholarship in this field, however, tended to overlook the combination of lookback option and uncertain mean-reverting currency model. Following the previous work on option pricing problem within framework of uncertainty theory lead, this paper will combine lookback option and uncertain mean-reverting currency model to derive the lookback option pricing formulas for the uncertain mean-reverting currency model. The rest of the paper is organized as follows. In next section, this paper will review some preliminary knowledge in uncertainty theory. In Section 3, the lookback call option will be introduced and its pricing formula will be derived. An algorithm will be designed to calculate the price numerically, a numerical experiment will be given and the sensibility of single variable will be analysed finally. In Section 4, the lookback put option will be discussed in the similar way. In Section 5, the conclusion will give some conclusions.

2 Preliminaries

In this section, some preliminary concepts from uncertainty theory as needed are reviewed for further understanding the paper.

2.1 Uncertain variable

Definition 1. (Liu [6]) Let Γ be a nonempty set, and $\mathcal{L}$ be a σ -algebra over Γ . A set function $\mathcal{M} : \mathcal{L} \rightarrow [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ .

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ .

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \dots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Note that the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Besides, the product uncertain measure on the product σ -algebra was defined by Liu [11] as follows.

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots, n$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \dots, n$, respectively.

Definition 2. (Liu [6]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B , the set $\{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event in $\mathcal{L}$.

Definition 3. (Liu [6]) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x .

Liu and Lio[24] proposed that A real-valued function $\Phi(x)$ on $\mathfrak{R}$ is an uncertainty distribution if and only if it is a monotone increasing function satisfying

$$0 \leq \Phi(x) \leq 1, \quad \Phi(x) \not\equiv 0, \quad \Phi(x) \not\equiv 1, \quad \Phi(x_0) = 1$$

if $\Phi(x) = 1$ for any $x > x_0$.

Definition 4. (Liu [25]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ .

Example 1. An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x - a)/(b - a), & \text{if } a \leq x < b \\ 1, & \text{if } x \geq b \end{cases}$$

denoted by $\mathcal{L}(a, b)$, where a and b are real numbers with $a < b$. The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

Example 2. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp \left(\frac{\pi(e - x)}{\sqrt{3}\sigma} \right) \right)^{-1}, \quad x \in \mathfrak{R}$$

denoted by $\mathcal{N}(e, \sigma)$, where e and σ are real numbers with $\sigma > 0$. The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

Definition 5. (Liu [6]) Let ξ be an uncertain variable. Then the expected value $E[\xi]$ is

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\} dx$$

provided that at least one of the two integrals is finite.

Theorem 2.1. (Liu [6]) Let ξ be an uncertain variable with uncertainty distribution Φ . Then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx.$$

2.2 Uncertain differential equaiton

An uncertain process is a sequence of uncertain variables indexed by time. A formal definition of uncertain process is stated as follows.

Definition 6. (Liu [10]) Let T be a totally ordered set (e.g. time) and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t .

Definition 7. (Liu [26]) Uncertain processes $X_{1t}, X_{2t}, \dots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \dots, t_k$, the uncertain vectors

$$\xi_i = (X_{it_1}, X_{it_2}, \dots, X_{it_k}), \quad i = 1, 2, \dots, n$$

are independent, i.e., for any Borel sets $B_1, B_2, \dots, B_n$ of k -dimensional real vectors, we have

$$\mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^n \mathcal{M} \{ \xi_i \in B_i \}.$$

An uncertain process X_t is said to have independent increments if

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \dots, t_k$ are any times with $t_0 < t_1 < \dots < t_k$. An uncertain process X_t is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Definition 8. (Liu [11]) An uncertain process C_t is said to be a Liu process if

- (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
- (ii) C_t has stationary and independent increments,
- (iii) every increment $C_{t+s} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp \left(-\frac{\pi x}{\sqrt{3}t} \right) \right)^{-1}, \quad x \in \mathbb{R}.$$

Definition 9. (Liu [11]) Let X_t be an uncertain process and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \dots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t with respect to C_t is

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Theorem 2.2. (Yao-Chen [27]) Let Z_t be an independent sample-continuous increment process. Then Z_t has a regular uncertainty distribution $\Phi_t(x)$. Let $J(x)$ be a real value strictly increasing function. It shows that the time integral

$$Z_T = \int_0^T J(Z_t) dt$$

has an inverse uncertainty distribution

$$\Psi_T^{-1}(\alpha) = \int_0^T J(\Psi_t^{-1}(\alpha)) dt.$$

Definition 10. (Liu [10]) Let C_t be a Liu process, assume f and g are continuous functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies the equation identically in t .

Definition 11. (Yao-Chen[27]) The α -path ($0 < \alpha < 1$) of an uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

with an initial value X_0 is a deterministic function X_t^α with respect to t that solves the corresponding ordinary differential equation

$$dX_t^\alpha = f(t, X_t^\alpha)dt + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt.$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, 0 < \alpha < 1$$

Theorem 2.3. (Yao-Chen [27]) Let X_t and X_t^α be the solution and α -path of the uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

respectively. Then,

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

This formula is well known as Yao-Chen formula.

Theorem 2.4. (Yao-Chen [27]) Let X_t and X_t^α be the solution and α -path of the uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Theorem 2.5. (Yao[28]) Let X_t and X_t^α be the solution and α -path of the uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

respectively. Then, for any time $s > 0$ and strictly increasing function $J(x)$, the supremum

$$\sup_{0 \leq t \leq s} J(X_t)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} J(X_t^\alpha);$$

and the infimum

$$\inf_{0 \leq t \leq s} J(X_t)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \inf_{0 \leq t \leq s} J(X_t^\alpha).$$

2.3 Uncertain mean-reverting currency model

Assuming that the stock price follows a geometric Liu process, Liu [11] built the first uncertain stock model, which was applied widely

$$\begin{cases} dX_t = rX_t dt \\ dY_t = eY_t dt + rY_t dC_t. \end{cases} \quad (1)$$

The bond price X_t and the stock price Y_t are determined by this model, where r is the risk-free interest rate, e is the stock drift, σ is the stock diffusion and C_t is a canonical Liu process. Then the model was derived in the international financial market by Liu et al. [20]

$$\begin{cases} dX_t = \mu X_t dt \\ dY_t = \nu Y_t dt \\ dZ_t = eZ_t dt + \sigma Z_t dt. \end{cases} \quad (2)$$

In this model, X_t represents the domestic currency with a domestic interest rate μ , Y_t is the foreign currency with a foreign interest rate ν , Z_t represents the exchange rate (the domestic currency price of one unit of foreign currency) at time t . It means that one unit of foreign currency equals to Z_t unit of domestic currency.

It is found that the change of foreign exchange rate stays around an average level in the long term in real global financial market. Therefore mean-reverting model was proposed by Shen and Yao [21] in 2016, in which $\frac{m}{a}$ means the average level

$$\begin{cases} dX_t = \mu X_t dt \\ dY_t = \nu Y_t dt \\ dZ_t = e(m - aZ_t)dt + \sigma Z_t dt. \end{cases} \quad (3)$$

3 Lookback call option pricing formula with the fixed strike

3.1 The pricing formula of lookback call option

A lookback call option provides the holder the right to sell a certain asset at the highest price during a certain period. Assume that a buyer can buy one unit of foreign lookback option during $t \in [0, T]$ with a fixed striking price K , which T is the expiration time. Then the payoff of the buyer is

$$(\sup_{0 \leq t \leq T} Z_t - K)^+,$$

where Z_t is the currency at time t . Considering the time value of money resulted from the bond, the present value of the payoff is

$$\exp(-\mu T) (\sup_{0 \leq t \leq T} Z_t - K)^+.$$

Let f_c present the price of the call option, then the expected profit of the buyer is

$$-f_c + \exp(-\mu T) (\sup_{0 \leq t \leq T} Z_t - K)^+$$

and the expected profit of the bank is

$$f_c - Z_0 \exp(-\nu T) (1 - \frac{K}{\sup_{0 \leq t \leq T} Z_t})^+.$$

According to the fair price principle, the buyer and the bank should gain the same expected profit

$$E[-f_c + \exp(-\mu T)(\sup_{0 \leq t \leq T} Z_t - K)^+] = E[f_c - Z_0 \exp(-\nu T)(1 - \frac{K}{\sup_{0 \leq t \leq T} Z_t})^+].$$

This the pricing formula can be presented as follows:

$$f_c = \frac{1}{2} \exp(-\mu T) E[(\sup_{0 \leq t \leq T} Z_t - K)^+] + \frac{1}{2} Z_0 \exp(-\nu T) E[(1 - \frac{K}{\sup_{0 \leq t \leq T} Z_t})^+].$$

Theorem 3.1. Assume the lookback call option for the stock model (3) has a strike price K and an expiration time T . Then the price of the lookback call option is

$$f_c = \frac{1}{2} \exp(-\mu T) \int_0^1 \sup_{0 \leq t \leq T} (\Psi_t^{-1}(\alpha) - K)^+ d\alpha + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \sup_{0 \leq t \leq T} (1 - \frac{K}{\Psi_t^{-1}(\alpha)})^+ d\alpha,$$

where

$$\Psi_t^{-1}(\alpha) = Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at)).$$

Proof: By solving the ordinary differential equation

$$Z_t^\alpha = (m - \sigma Z_t^\alpha + \sigma \Psi_t^{-1}(\alpha)) dt,$$

we get the α -path of Z_t :

$$\begin{aligned} Z_t^\alpha &= Z_0 \exp(-at) + \frac{m + \sigma \Psi_t^{-1}(\alpha)}{a} (1 - \exp(-at)) \\ &= Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at)). \end{aligned}$$

By Theorem 2.4, the foreign exchange rate Z_t has inverse distribution function

$$\Psi_t^{-1}(\alpha) = Z_t^\alpha = Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a}.$$

And since

$$(\sup_{0 \leq t \leq T} Z_t - K)^+ = (\sup_{0 \leq t \leq T} (Z_t - K))^+ = \sup_{0 \leq t \leq T} (Z_t - K)^+,$$

by Theorem 2.5, the inverse distribution of

$$(\sup_{0 \leq t \leq T} Z_t - K)^+$$

is

$$\sup_{0 \leq t \leq T} (\Psi_t^{-1}(\alpha) - K)^+.$$

The inverse distribution of

$$(1 - \frac{K}{\sup_{0 \leq t \leq T} Z_t})^+$$

is

$$\sup_{0 \leq t \leq T} (1 - \frac{K}{\Psi_t^{-1}(\alpha)})^+.$$

Therefore,

$$\begin{aligned}
f_c &= \frac{1}{2} \exp(-\mu T) \int_0^1 \sup_{0 \leq t \leq T} (\Psi_t^{-1}(\alpha) - K)^+ d\alpha + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \sup_{0 \leq t \leq T} (1 - \frac{K}{\Psi_t^{-1}(\alpha)})^+ d\alpha \\
&= \frac{1}{2} \exp(-\mu T) \int_0^1 \sup_{0 \leq t \leq T} (Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at)) - K)^+ d\alpha \\
&\quad + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \sup_{0 \leq t \leq T} (1 - \frac{K}{Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at))})^+ d\alpha.
\end{aligned}$$

The lookback call option pricing formula is verified.

3.2 Numerical experiment

An algorithm is designed to calculate the lookback call option price f_c .

Step 0 Set the initial parameters as $K=4$, $Z_0 = 5$, $\mu = 0.05$, $\nu = 0.04$, $\sigma = 0.1$, $T=1$, $m=6$, $a=1$.

Step 1 Set $\alpha_i = i/N$ and $t_j = jT/M$, $i = 1, 2, \dots, N-1$, $j = 1, 2, \dots, M$, where N and M are two large numbers.

Step 2 Set $i = 0$, $j = 0$.

Step 3 Set $i \leftarrow i + 1$.

Step 4 Set $j \leftarrow j + 1$.

Step 5 Set $P_{t_0}^{\alpha_i} = 0$.

Step 6 Calculate the price at time t_j

$$P_{t_j}^{\alpha_i} = \Psi_{t_j}^{-1}(\alpha) = Z_0 \exp(-at_j) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha_i}{1-\alpha_i}}{a} (1 - \exp(-at_j)).$$

If $P_{t_j}^{\alpha_i} > P_{t_{j-1}}^{\alpha_i}$ and $j < M$, then return to Step 4. If $P_{t_j}^{\alpha_i} \geq P_{t_{j-1}}^{\alpha_i}$ and $j = M$, then jump to Step 8.

Step 7 Set $P_{t_j}^{\alpha_i} \leftarrow P_{t_{j-1}}^{\alpha_i}$. If $j < M$, then return to Step 4.

Step 8 Calculate $P_{t_j}^{\alpha_i} - K$, $1 - \frac{K}{P_{t_j}^{\alpha_i}}$.

Step 9 Set $P^{\alpha_i} \leftarrow \max(0, P_{t_j}^{\alpha_i} - K)$, $Q^{\alpha_i} \leftarrow \max(0, 1 - \frac{K}{P_{t_j}^{\alpha_i}})$. If $i < N-1$, then return to Step 2.

Step 10 Calculate the lookback call option price

$$f_c \leftarrow \frac{1}{2} \exp(-\mu T) \frac{1}{N-1} \sum_{i=1}^{N-1} P^{\alpha_i} + \frac{1}{2} Z_0 \exp(-\nu T) \frac{1}{N-1} \sum_{i=1}^{N-1} Q^{\alpha_i}.$$

Example 3.1. Suppose that the initial exchange rate $Z_0 = 5$, the domestic interest rate $\mu = 0.05$, the foreign interest rate $\nu = 0.04$, the log-diffusion $\sigma = 0.1$, the strike price $K=4$, the expiration time $T=1$, $m=6$, $a=1$. Then the price of lookback call currency option is

$$f_c \approx 1.4719.$$

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3.3 Sensibility analysis

Theorem 3.2. (Monotonicity of lookback call option pricing model) Suppose the lookback call option price follows the stock model (3), then f_c has the following properties:

1. f_c is an increasing function of Z_0 ;
2. f_c is a decreasing function of K ;
3. f_c is a decreasing function of μ ;
4. f_c is a decreasing function of ν .

Proof: 1. Since

$$\begin{aligned} \frac{df_c}{dZ_0} &= \frac{1}{2} \exp(-\mu T) \int_0^1 \sup_{0 \leq t \leq T} \exp(-at) d\alpha + \frac{1}{2} \exp(-\nu T) \int_0^1 \sup_{0 \leq t \leq T} \left(1 - \frac{K}{\Psi_t^{-1}(\alpha)}\right)^+ d\alpha \\ &+ \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \sup_{0 \leq t \leq T} \left(\frac{K \exp(-at)}{\Psi_t^{-1}(\alpha)^2}\right) d\alpha > 0. \end{aligned}$$

It is obvious that f_c is an increasing function of Z_0 .

Under the assumption that other parameters except Z_0 are constants, change Z_0 from 0 to 6 with step 0.001, we can draw the figure of lookback call option price f_c (Fig 1).

Figure 1: Price with respect to Z_0

Table 1: f_c with the change of Z_0

Z_0	4.5	5	5.5	6	6.5
f_c	1.263	1.472	1.689	1.912	2.142

2. Since

$$\frac{df_c}{dK} = -\frac{1}{2} \exp(-\mu T) - \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \sup_{0 \leq t \leq T} \left(\frac{1}{\Psi_t^{-1}(\alpha)}\right) d\alpha < 0,$$

f_c is a decreasing function of K .

Under the assumption that other parameters except K are constants, change K from 3 to 6 with step 0.001, we can draw the figure of lookback call option price f_c (Fig 2).

Table 2: f_c with the change of K

K	3	3.5	4	4.5	5
f_c	2.374	1.923	1.472	1.021	0.570

3. Since

$$\frac{df_c}{d\mu} = -\frac{T}{2} \exp(-\mu t) \int_0^1 \sup_{0 \leq t \leq T} (\Psi_t^{-1}(\alpha) - K)^+ d\alpha < 0,$$

f_c is a decreasing function of μ .

Figure 2: Price with respect to K

Figure 3: Price with respect to μ

Table 3: f_c with the change of μ					
μ	0.02	0.05	0.08	0.11	0.14
f_c	1.496	1.472	1.449	1.427	1.405

Under the assumption that other parameters except μ are constants, change μ from 0 to 1 with step 0.001, we can draw the figure of lookback call option price f_c (Fig 3).

4. Since

$$\frac{df_c}{d\nu} = -\frac{T}{2} \exp(-\nu t) \int_0^1 \sup_{0 \leq t \leq T} (1 - \frac{K}{\Psi_t^{-1}(\alpha)})^+ d\alpha < 0,$$

f_c is a decreasing function of ν .

Under the assumption that other parameters except ν are constants, change ν from 0 to 1 with step 0.001, we can draw the figure of lookback call option price f_c (Fig 4). $\square$

Figure 4: Price with respect to ν

Table 4: f_c with the change of ν					
ν	0.01	0.04	0.07	0.10	0.13
f_c	1.493	1.472	1.451	1.431	1.412

4 Lookback put option pricing formula with the fixed strike

4.1 The pricing formula of lookback put option

Assume that a foreign lookback put option has a fixed striking price K and an expiration time T . Then the payoff of a buyer is

$$(K - \inf_{0 \leq t \leq T} Z_t)^+.$$

Considering the time value of money resulted from the bond, the present value of the payoff is

$$\exp(-\mu T)(K - \inf_{0 \leq t \leq T} Z_t)^+.$$

Let f_p present the price of the lookback put option, then the expected profit of the buyer is

$$-f_p + \exp(-\mu T)(K - \inf_{0 \leq t \leq T} Z_t)^+$$

and the expected profit of the bank is

$$f_p - Z_0 \exp(-\nu T) \left(\frac{K}{\inf_{0 \leq t \leq T} Z_t} - 1 \right)^+.$$

According to the fair price principle, the buyer and the bank should gain the same expected profit

$$E[-f_p + \exp(-\mu T)(K - \inf_{0 \leq t \leq T} Z_t)^+] = E[f_p - Z_0 \exp(-\nu T)(\frac{K}{\inf_{0 \leq t \leq T} Z_t} - 1)^+].$$

This the pricing formula can be presented as follows:

$$f_p = \frac{1}{2} \exp(-\mu T) E[(K - \inf_{0 \leq t \leq T} Z_t)^+] + \frac{1}{2} Z_0 \exp(-\nu T) E[(\frac{K}{\inf_{0 \leq t \leq T} Z_t} - 1)^+].$$

Theorem 4.1. Assume the lookback put option for the stock model (3) has a strike price K and an expiration time T . Then the price of the lookback put option is

$$f_p = \frac{1}{2} \exp(-\mu T) \int_0^1 \inf_{0 \leq t \leq T} (K - \Psi_t^{-1}(\alpha))^+ d\alpha + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \inf_{0 \leq t \leq T} (\frac{K}{\Psi_t^{-1}(\alpha)} - 1)^+ d\alpha,$$

where

$$\Psi_t^{-1}(\alpha) = Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at)).$$

Proof: We have known that the foreign exchange rate Z_t has inverse distribution function

$$\Psi_t^{-1}(\alpha) = Z_t^\alpha = Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a}.$$

Since

$$(K - \inf_{0 \leq t \leq T} Z_t)^+ = (\inf_{0 \leq t \leq T} (K - Z_t))^+ = \inf_{0 \leq t \leq T} (K - Z_t)^+,$$

the inverse distribution of

$$(K - \inf_{0 \leq t \leq T} Z_t)^+$$

is

$$\inf_{0 \leq t \leq T} (K - \Psi_t^{-1}(\alpha))^+.$$

The inverse distribution of

$$(\frac{K}{\inf_{0 \leq t \leq T} Z_t} - 1)^+$$

is

$$\inf_{0 \leq t \leq T} (\frac{K}{\Psi_t^{-1}(\alpha)} - 1)^+.$$

Therefore,

$$\begin{aligned} f_p &= \frac{1}{2} \exp(-\mu T) \int_0^1 \inf_{0 \leq t \leq T} (K - \Psi_t^{-1}(\alpha))^+ d\alpha + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \inf_{0 \leq t \leq T} (\frac{K}{\Psi_t^{-1}(\alpha)} - 1)^+ d\alpha \\ &= \frac{1}{2} \exp(-\mu T) \int_0^1 \inf_{0 \leq t \leq T} (K - Z_0 \exp(-at) - \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at)))^+ d\alpha \\ &\quad + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \inf_{0 \leq t \leq T} (\frac{K}{Z_0 \exp(-at) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}}{a} (1 - \exp(-at))} - 1)^+ d\alpha. \end{aligned}$$

The lookback put option pricing formula is verified.

4.2 Numerical experiment

An algorithm is designed to calculate the lookback put option price f_p .

Step 0 Set the initial parameters as $K=5$, $Z_0 = 4$, $\mu = 0.05$, $\nu = 0.04$, $\sigma = 0.1$, $T=1$, $m=6$, $a=1$.

Step 1 Set $\alpha_i = i/N$ and $t_j = jT/M$, $i = 1, 2, \dots, N-1$, $j = 1, 2, \dots, M$, where N and M are two large numbers.

Step 2 Set $i = 0$, $j = 0$.

Step 3 Set $i \leftarrow i + 1$.

Step 4 Set $j \leftarrow j + 1$.

Step 5 Set $P_{t_0}^{\alpha_i} = K$.

Step 6 Calculate the price at time t_j

$$P_{t_j}^{\alpha_i} = \Psi_{t_j}^{-1}(\alpha) = Z_0 \exp(-at_j) + \frac{m + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha_i}{1-\alpha_i}}{a} (1 - \exp(-at_j)).$$

If $P_{t_j}^{\alpha_i} \leq P_{t_{j-1}}^{\alpha_i}$ and $j < M$, then return to Step 4. If $P_{t_j}^{\alpha_i} \leq P_{t_{j-1}}^{\alpha_i}$ and $j = M$, then jump to Step 8.

Step 7 Set $P_{t_j}^{\alpha_i} \leftarrow P_{t_{j-1}}^{\alpha_i}$. If $j < M$, then return to Step 4.

Step 8 Calculate $K - P_{t_j}^{\alpha_i}$, $\frac{K}{P_{t_j}^{\alpha_i}} - 1$.

Step 9 Set $P^{\alpha_i} \leftarrow \max(0, K - P_{t_j}^{\alpha_i})$, $Q^{\alpha_i} \leftarrow \max(0, \frac{K}{P_{t_j}^{\alpha_i}} - 1)$. If $i < N-1$, then return to Step 2.

Step 10 Calculate the lookback put option price

$$f_p \leftarrow \frac{1}{2} \exp(-\mu T) \frac{1}{N-1} \sum_{i=1}^{N-1} P^{\alpha_i} + \frac{1}{2} Z_0 \exp(-\nu T) \frac{1}{N-1} \sum_{i=1}^{N-1} Q^{\alpha_i}.$$

Example 4.1. Suppose that the initial exchange rate $Z_0 = 4$, the domestic interest rate $\mu = 0.05$, the foreign interest rate $\nu = 0.04$, the log-diffusion $\sigma = 0.1$, the strike price $K=5$, the expiration time $T=1$, $m=6$, $a=1$. Then the price of lookback put option is

$$f_p \approx 0.9595.$$

4.3 Sensibility analysis

Theorem 4.2. (Monotonicity of lookback put option pricing model) Suppose the lookback put currency option price follows the stock model (3), then f_p has the following properties:

1. f_p is an increasing function of K ;
2. f_p is a decreasing function of Z_0 ;
3. f_p is a decreasing function of μ ;
4. f_p is a decreasing function of ν .

Proof: 1. Since

$$\frac{df_p}{dK} = \frac{1}{2} \exp(-\mu T) + \frac{1}{2} Z_0 \exp(-\nu T) \int_0^1 \inf_{0 \leq t \leq T} \left(\frac{1}{\Psi_t^{-1}(\alpha)} \right) d\alpha > 0,$$

f_p is an increasing function of K .

Under the assumption that other parameters except K are constants, change K from 3 to 6 with step 0.001, we can draw the figure of lookback put option price f_p (Fig 5)

2. Under the assumption that other parameters except Z_0 are constants, change Z_0 from 3 to 6 with step 0.001, we can draw the figure of lookback put option price f_p (Fig 6).

Figure 5: Price with respect to K

Table 5: f_p with the change of K

K	4	4.5	5	5.5	6
f_p	0.005	0.482	0.960	1.437	1.915

Figure 6: Price with respect to Z_0

Table 6: f_p with the change of Z_0

Z_0	3	3.5	4	4.5	5
f_p	1.914	1.437	0.960	0.482	0.005

3. Since

$$\frac{df_p}{d\mu} = -\frac{T}{2} \exp(-\mu t) \int_0^1 \inf_{0 \leq t \leq T} (K - \Psi_t^{-1}(\alpha))^+ d\alpha < 0,$$

so f_p is a decreasing function of μ .

Under the assumption that other parameters except μ are constants, change μ from 0 to 1 with step 0.001, we can draw the figure of lookback put option price f_p (Fig 7).

Figure 7: Price with respect to μ

Table 7: f_p with the change of μ

μ	0.02	0.05	0.08	0.11	0.14
f_p	0.974	0.960	0.945	0.932	0.918

4. Since

$$\frac{df_p}{d\nu} = -\frac{T}{2} \exp(-\nu t) \int_0^1 \inf_{0 \leq t \leq T} \left(\frac{K}{\Psi_t^{-1}(\alpha)} - 1 \right)^+ d\alpha < 0,$$

f_p is a decreasing function of ν .

Under the assumption that other parameters except ν are constants, change ν from 0 to 1 with step 0.001, we can draw the figure of lookback put option price f_p (Fig 8). $\square$

Table 8: f_p with the change of ν

ν	0.01	0.04	0.07	0.10	0.13
f_p	0.974	0.960	0.945	0.932	0.918

5 Conclusion

This paper mainly presented lookback options for the uncertain mean-reverting currency model. It considered lookback call currency option and lookback put option with the fixed strike, meanwhile derived their option pricing formulas. Moreover, the paper designed numerical methods to calculate the prices. Further research may consider lookback options of various uncertain stock models.

Figure 8: Price with respect to ν

Compliance with ethical standards

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