Multipolar Fuzzy KU-ideals in KU-algebras

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Multipolar fuzzy KU-ideals in KU-algebras

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Abstract

The notion of an m-polar fuzzy KU-ideal is introduced and its properties are investigated. The relationship between m-polar fuzzy KU-subalgebra, m-polar fuzzy KU-ideal, are discussed.

Keywords: KU-algebras, m-Polar fuzzy set, m-Polar fuzzy KU-subalgebra, m-Polar fuzzy KU-ideal

Mathematics Subject Classification: 06F35 · 03G25 · 08A72

1. Introduction

In order to deal with possibilistic uncertainty which is connected with perceptions, imprecision of states, and preferences, fuzzy set is useful tool and it is introduced by Zadeh [11]. Since then, fuzzy set theory has become an active research area in various fields, including graph theory, statistics, life and medical sciences, engineering, social sciences, decision making, computer network, robotics and automata theory, artificial intelligence, and pattern recognition, etc. In [8] and [9] constructed a new algebraic structure which is called KU-algebras. Mostafa et al [7] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals.

Recently, the notion of m-polar fuzzy set theory was applied to graph theory (Akram and Sarwar, 2018[3]), Al-Masarwah and Ahmad in 2019 [4] discussed the notion of m-polar fuzzy sets with an application to BCK/BCI-algebras. We introduce the notions of m-polar fuzzy subalgebras and m-polar fuzzy (closed, commutative) ideals, and then we investigate several properties.

The purpose of this manuscript is to apply the notion of m-polar fuzzy set to fuzzy KU-ideal in KU-algebras. We introduce the notions of an m-polar fuzzy KU-ideal, and investigate their properties. We examine the relationship between m-polar fuzzy KU-subalgebra and m-polar fuzzy KU-ideal. We show the relationship of an m-polar fuzzy KU-ideal and ideal of BCK/BCI-algebras.

2. Preliminaries

We first recall some elementary aspects which are used to present the paper. Throughout this paper, X always denotes a KU-algebra without any specifications.

Definition 2.1. [8] Let X be a nonempty set with a binary operation $*$ and a constant 0 , then

$(X, *, 0)$ is called a KU -algebra, if for all $u, v, w \in X$ the following axioms are holds:

$$(ku\ 1) (u * v) * [(v * w) * (u * w)] = 0 ,$$

$$(ku\ 2) u * 0 = 0,$$

$$(ku\ 3) 0 * u = u ,$$

$$(ku\ 4) u * v = 0 \text{ and } v * u = 0 \text{ implies } u = v,$$

$$(ku\ 5) u * u = 0$$

On a KU-algebra $(X, *, 0)$ we can define a binary relation $\leq$ on X by putting:

$u \leq v \Leftrightarrow v * u = 0$. Then $(X, \leq)$ is a partially ordered set and 0 is its smallest element. Thus $(X, *, 0)$ satisfies the following conditions: for all $u, v, w \in X$.

$$(1): (v * w) * (u * w) \leq (u * v)$$

$$(2): 0 \leq u$$

$$(3): u \leq v, v \leq u \text{ implies } u = v ,$$

$$(4): v * u \leq u.$$

A subset S of a KU-algebra X is called KU-subalgebra of X , if $u, v \in S$, implies $u * v \in S$.

A non- empty subset I of a KU-algebra X is said to be an KU-ideal of X if it satisfies:

$$(K1) 0 \in I,$$

$$(K2) u * (v * w) \in I \text{ and } v \in I \text{ imply } u * w \in I \text{ for all } u, v \text{ and } w \in X.$$

Theorem 2.2. [7] In a KU-algebra $(X, *, 0)$, the following axioms are satisfied: For all $u, v, w \in X$,

$$(1) u \leq v \text{ imply } v * w \leq u * w ,$$

$$(2) u * (v * w) = v * (u * w), \text{ for all } u, v, w \in X ,$$

$$(3) ((v * u) * u) \leq v$$

Definition 2.3: [7] Let μ be a fuzzy set on a KU-algebra X , then μ is called a fuzzy KU-subalgebra of X if $\mu(u * v) \geq \min\{\mu(u), \mu(v)\}$, for all $u, v \in X$.

Definition 2.4: [7] Let X be a KU-algebra. A fuzzy set μ in X is called a fuzzy KU-ideal of X if it satisfies:

$$(FK1) \mu(0) \geq \mu(u),$$

$$(FK2) \mu(u * w) \geq \min\{\mu(u * (v * w)), \mu(v)\}, \text{ for all } u, v \text{ and } w \in X.$$

Lemma 2.5: [7] If A fuzzy KU-subalgebra of X , then $\mu(0) \geq \mu(u)$, for all $u \in X$.

Proposition 2.6: [7] If A fuzzy KU-ideal of X and $u \leq v$, then $\mu(u) \geq \mu(v)$, for all $u, v \in X$.

Theorem 2.7: [7] A fuzzy KU-ideal of X is a fuzzy KU-subalgebra of X .

By a m -polar fuzzy set a set X (see [5]), we mean a function $\hat{O} : X \rightarrow [0, 1]^m$. The membership value of every element $u \in X$ is denoted by

$$\hat{O}(u) := ((\pi_1 \circ \hat{O})(u), (\pi_2 \circ \hat{O})(u), \dots, (\pi_m \circ \hat{O})(u)),$$

Where $\pi_i : [0, 1]^m \rightarrow [0, 1]$ is the i -th projection for all $i = 1, 2, \dots, m$.

Given an m -polar fuzzy set on a set X , we consider the set

$$U(\hat{O}; \hat{r}) := \{u \in X \mid \hat{O}(u) \geq \hat{r}\}$$

that is,

$$U(\hat{O}; \hat{r}) := \{u \in X \mid (\pi_i \circ \hat{O})(u) \geq r_i, i = 1, 2, \dots, m\},$$

which is called a m -polar $\hat{r}$ -level cut set of $\hat{O}$.

By an m -polar fuzzy point a set X , we mean an m -polar fuzzy set $\hat{O}$ on X of the form

$$\hat{O}(v) = \begin{cases} \hat{t} = (t_1, t_2, \dots, t_m) \in (0, 1]^m & \text{if } u = v, \\ \hat{0} = (0, 0, \dots, 0) & \text{if } u \neq v \end{cases}$$

and it is denoted by $u_{\hat{t}}$. We say that u is the support of $u_{\hat{t}}$, and $\hat{t}$ is the value of $u_{\hat{t}}$.

We say that an m -polar fuzzy point $u_{\hat{t}}$ is contained in an m -polar fuzzy set $\hat{O}$, denoted by $u_{\hat{t}} \in \hat{O}$, if $\hat{O}(u) \geq \hat{t}$, that is, $(\pi_i \circ \hat{O})(u) \geq t_i$ for all $i = 1, 2, \dots, m$.

Definition 2.8. [4] An m -polar fuzzy set $\hat{O}$ of BCK/BCI-algebra X is called an m -polar fuzzy subalgebra if the following assertion is valid:

$$\hat{O}(u * v) \geq \min\{\hat{O}(u), \hat{O}(v)\}$$

that is,

$$(\pi_i \circ \hat{O})(u * v) \geq \min\{(\pi_i \circ \hat{O})(u), (\pi_i \circ \hat{O})(v)\}$$

For all $u, v \in X, i = 1, 2, \dots, m$.

Definition 2.9. [4] An m -polar fuzzy set $\hat{O}$ of BCK/BCI-algebra X is called an m -polar fuzzy ideal if the following assertion is valid:

$$\hat{O}(0) \geq \hat{O}(u) \geq \min\{\hat{O}(u * v), \hat{O}(v)\}$$

that is,

$$(\pi_i \circ \widehat{O})(0) \geq (\pi_i \circ \widehat{O})(u) \geq \min\{(\pi_i \circ \widehat{O})(u * v), (\pi_i \circ \widehat{O})(v)\}$$

For all $u, v \in X, i = 1, 2, \dots, m$.

3. m-Polar fuzzy KU-subalgebras and KU-ideals

In this section, we introduce the notions of m-polar fuzzy KU-subalgebras, m-polar fuzzy KU-ideals in KU-algebras and investigate some of their related properties.

Definition 3.1. An m-polar fuzzy set $\widehat{O}$ of X is called an m-polar fuzzy KU-subalgebra if the following assertion is valid for all $u, v \in X$.

$$\widehat{O}(u * v) \geq \min\{\widehat{O}(u), \widehat{O}(v)\}$$

that is,

$$(\pi_i \circ \widehat{O})(u * v) \geq \min\{(\pi_i \circ \widehat{O})(u), (\pi_i \circ \widehat{O})(v)\}$$

For all $u, v \in X, i = 1, 2, \dots, m$.

Example 3.2: Let $X = \{0, 1, 2, 3, 4\}$ be KU-algebra with a binary operation $*$ defined by the following table

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	3
2	0	0	0	1	4
3	0	0	0	0	3
4	0	0	0	0	0

Define a 3-polar fuzzy set $\widehat{O} = X \rightarrow [0, 1]^3$ by:

$$\widehat{O}(u) = \begin{cases} (0.3, 0.4, 0.6) & \text{if } u = 0 \\ (0.2, 0.3, 0.2) & \text{if } u = 1 \\ (0.1, 0.2, 0.3) & \text{if } u = 2 \\ (0.2, 0.3, 0.4) & \text{if } u = 3 \\ (0.2, 0.3, 0.5) & \text{if } u = 4 \end{cases}$$

It is routine to verify that $\widehat{O}$ is a 3-polar fuzzy KU-subalgebra of X .

Theorem 3.3: Let $\widehat{O}$ be an m-polar fuzzy set of X . Then $\widehat{O}$ is an m-polar fuzzy KU-subalgebra of X if and only if $U(\widehat{O}; \hat{r}) \neq \emptyset$ is a KU-subalgebra of X for all $\hat{r} = (r_1, r_2, \dots, r_m) \in [0, 1]^m$.

Proof. Assume that $\widehat{O}$ is an m-polar fuzzy subalgebra of X and let $\hat{r} \in [0, 1]^m$ be such that $U(\widehat{O}; \hat{r}) \neq \emptyset$. Let $u, v \in U(\widehat{O}; \hat{r})$. Then $\widehat{O}(u) \geq \hat{r}$ and $\widehat{O}(v) \geq \hat{r}$. It follows from definition 3.1 that $\widehat{O}(u * v) \geq \min\{\widehat{O}(u), \widehat{O}(v)\} \geq \hat{r}$, so that $u * v \in U(\widehat{O}; \hat{r})$. Hence $U(\widehat{O}; \hat{r})$ is a subalgebra of X .

Conversely, assume that $U(\hat{O}; \hat{r})$ is a subalgebra of X . Suppose that there exist $u, v \in X$ such that $\hat{O}(u * v) < \min\{\hat{O}(u), \hat{O}(v)\}$. Then there exists $\hat{r} = (r_1, r_2, \dots, r_m) \in [0, 1]^m$ such that $\hat{O}(u * v) < \hat{r} \leq \min\{\hat{O}(u), \hat{O}(v)\}$. It follows that $u, v \in U(\hat{O}; \hat{r})$, but $u * v \notin U(\hat{O}; \hat{r})$. This is a contradiction, and so $\hat{O}(u * v) \geq \min\{\hat{O}(u), \hat{O}(v)\}, \forall u, v \in X$. Therefore $\hat{O}$ is an m-polar fuzzy KU-subalgebra of X . ■

Lemma 3.4. Every m-polar fuzzy subalgebra $\hat{O}$ of X satisfies the following inequality:

$$(\forall u \in X)(\hat{O}(0) \geq \hat{O}(u))$$

that is,

$$(\pi_i \circ \hat{O})(0) \geq (\pi_i \circ \hat{O})(u)$$

For all $u, v \in X, i = 1, 2, \dots, m$.

Proof. Note that $u * u = 0$ for all $u \in X$. Using definition 3.1, we have

$$\hat{O}(0) = \hat{O}(u * u) \geq \min\{\hat{O}(u), \hat{O}(u)\} = \hat{O}(u).$$

for all $u \in X$. ■

Proposition 3.5. If every m-polar fuzzy subalgebra $\hat{O}$ of X satisfies the following inequality:

$$(\forall u, v \in X)(\hat{O}(u * v) \geq \hat{O}(v)) \quad (1)$$

Then $\hat{O}(u) = \hat{O}(0)$.

that is,

$$(\pi_i \circ \hat{O})(u * v) \geq (\pi_i \circ \hat{O})(v)$$

Then $(\pi_i \circ \hat{O})(u) = (\pi_i \circ \hat{O})(0)$

For all $u, v \in X, i = 1, 2, \dots, m$.

Proof. Let $u \in X$. Using (ku 2) and (1), we have $\hat{O}(u) = \hat{O}(u * 0) \geq \hat{O}(0)$. It follows from Lemma 3.4 that $\hat{O}(u) = \hat{O}(0)$. ■

Definition 3.6. An m-polar fuzzy set $\hat{O}$ of X is called an m-polar fuzzy KU-ideal if the following conditions are valid:

$$(\forall u \in X)(\hat{O}(0) \geq \hat{O}(u))$$

$$(\forall u, v \text{ and } w \in X)(\hat{O}(u * w) \geq \min\{\hat{O}(u * (v * w)), \hat{O}(v)\}). \quad (2)$$

that is,

$$(\forall u \in X)((\pi_i \circ \hat{O})(0) \geq (\pi_i \circ \hat{O})(u))$$

$$(\forall u, v \text{ and } w \in X)((\pi_i \circ \hat{O})(u * w) \geq \min\{(\pi_i \circ \hat{O})(u * (v * w)), (\pi_i \circ \hat{O})(v)\}).$$

For all $i = 1, 2, \dots, m$.

Proposition 3.7. If $\hat{O}$ is an m-polar fuzzy KU-ideal of X and $u \leq v$, then

$$(\hat{O}(u) \geq \hat{O}(v)) (\forall u, v \in X)$$

that is,

$$((\pi_i \circ \hat{O})(u) \geq (\pi_i \circ \hat{O})(v)) (\forall u, v \in X, i = 1, 2, \dots, m)$$

Proof. If $u \leq v$, then $v * u = 0$ and $(ku \ 3) \ 0 * u = u$. Since $\hat{O}$ is an m-polar fuzzy KU-ideal of X , we get

$$\begin{aligned} \hat{O}(0 * u) &= \hat{O}(u) \geq \min\{\hat{O}(0 * (v * u)), \hat{O}(v)\} = \min\{\hat{O}(0 * 0), \hat{O}(v)\}. \\ &= \min\{\hat{O}(0), \hat{O}(v)\} = \hat{O}(v). \end{aligned}$$

For all $u, v \in X$. ■

Proposition 3.8. Let $\hat{O}$ be an m-polar fuzzy KU-ideal of X . If $u * v \leq w$, holds in X , then

$$(\hat{O}(v) \geq \min\{\hat{O}(u), \hat{O}(w)\}) (\forall u, v \text{ and } w \in X)$$

That is,

$$((\pi_i \circ \hat{O})(v) \geq \min\{(\pi_i \circ \hat{O})(u), (\pi_i \circ \hat{O})(w)\}) (\forall u, v \text{ and } w \in X, i = 1, 2, \dots, m)$$

Proof: Assume that the inequality $u * v \leq w$, holds in X . Then $w * (u * v) = 0$ and (2)

$$\begin{aligned} \hat{O}(u * v) &\geq \min\{\hat{O}(u * (w * v)), \hat{O}(w)\} = \min\{\hat{O}(w * (u * v)), \hat{O}(w)\} \\ &= \min\{\hat{O}(0), \hat{O}(w)\} = \hat{O}(w) \quad (3) \end{aligned}$$

Now, $\hat{O}(0 * v) = \hat{O}(v) = \min\{\hat{O}(0 * (u * v)), \hat{O}(u)\} = \min\{\hat{O}(u * v), \hat{O}(u)\} \geq \min\{\hat{O}(w), \hat{O}(u)\}$ (by using (3)), i.e. $\hat{O}(v) \geq \min\{\hat{O}(u), \hat{O}(w)\}$. This completes the proof. ■

Theorem 3.9. If $\hat{O}$ is an m-polar fuzzy KU-subalgebra of X satisfies the condition in Proposition 3.8, then $\hat{O}$ is an m-polar fuzzy KU-ideal of X .

Proof. Let $\hat{O}$ be an m-polar fuzzy KU-subalgebra of X satisfies the condition in Proposition 3.8, by lemma 3.4. We have $\hat{O}(0) \geq \hat{O}(u)$ for all $u \in X$. By theorem 2.2(3), we have $(u * (v * w)) * (u * w) \leq v$, for all $u, v, w \in X$. it follows from Proposition 3.8, that $\hat{O}(u * w) \geq \min\{\hat{O}(u * (v * w)), \hat{O}(v)\}$, for all $u, v, w \in X$. Therefore, $\hat{O}$ is an m-polar fuzzy KU-ideal of X . ■

Proposition 3.10. Every m-polar fuzzy KU-ideal of X is m-polar fuzzy ideal.

Proof. Clear.

Proposition 3.11. If $\hat{O}$ is an m-polar fuzzy KU-ideal of X, then

$$(\hat{O}(u * (u * v)) \geq \hat{O}(v)) (\forall u, v \in X)$$

That is,

$$((\pi_i \circ \hat{O})(u * (u * v)) \geq (\pi_i \circ \hat{O})(v))$$

For all $u, v \in X, i = 1, 2, \dots, m$.

Proof. Let $\hat{O}$ be an m-polar fuzzy KU-ideal of a KU-algebra X and let $u, v, w \in X$. Taking $w = u * v$ in (2) and using (ku 2), we get

$$\begin{aligned} \hat{O}(u * (u * v)) &\geq \min\{\hat{O}(u * (v * (u * v))), \hat{O}(v)\} \\ &= \min\{\hat{O}(u * (u * (v * v))), \hat{O}(v)\} \\ &= \min\{\hat{O}(u * (u * 0)), \hat{O}(v)\} \\ &= \min\{\hat{O}(0), \hat{O}(v)\} = \hat{O}(v). \blacksquare \end{aligned}$$

Proposition 3.12. If $\hat{O}$ is an m-polar fuzzy KU-ideal of X, then the set $B = \{u \in X : \hat{O}(u) = \hat{O}(0)\}$ is an m-polar KU-ideal.

Proof. Since $0 \in X$, then $\hat{O}(0) = \hat{O}(0)$ implies $0 \in B$, so $B \neq \emptyset$. Let $u * (v * w) \in B$ and $v \in B$ implies $\hat{O}(u * (v * w)) = \hat{O}(0)$ and $\hat{O}(v) = \hat{O}(0)$. Since $\hat{O}$ is an m-polar fuzzy KU-ideal of X, then

$$\hat{O}(u * w) \geq \min\{\hat{O}(u * (v * w)), \hat{O}(v)\} = \hat{O}(0).$$

But $\hat{O}(0) \geq \hat{O}(u * w)$. Then $\hat{O}(0) = \hat{O}(u * w)$, it follows that $u * w \in B$, for all $u, v, w \in X$. Hence, the set B is an m-polar KU-ideal. ■

Conclusion

An m-polar fuzzy model is a generalized form of a bipolar fuzzy model. The m-polar fuzzy models provide more precision, flexibility and compatibility to the system when more than one agreement is to be dealt with. In this article, we have discussed the KU-ideal of KU-algebras based on m-polar fuzzy sets. We have introduced the notions of m-polar fuzzy KU-subalgebras and m-polar fuzzy KU-ideals, and investigated several properties.

Compliance with ethical standards

Conflict of interest: The author declare that there is no conflict of interest regarding the publication of this paper.

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