Heron's cubic root iteration method with a new approach

S. Gadtia
Veer Surendra Sai University of Technology
S. K. Padhan (skpadhan_math@vssut.ac.in)
Veer Surendra Sai University of Technology https://orcid.org/0000-0002-3284-3016

Research Article

Keywords: Even order roots, Odd order roots, Higher order roots, Heron's method. Mathematics Subject Classi cations 2010: 01A30, 01A35, 11A07

Posted Date: November 3rd, 2021

DOI: https://doi.org/10.21203/rs.3.rs-792197/v1

Heron's cubic root iteration method with a new approach

S. Gadtia[†] and S.K. Padhan[*]

[†,*]*Department of Mathematics, Veer Surendra Sai University of Technology, Burla-768018, India*

[†] *Department of Mathematics, Agalpur Panchayat Samiti College, Roth, Balangir-767061, India*

Abstract

Heron's cubic root iteration formula conjectured by Wertheim is proved and extended for any odd order roots. Some possible proofs are suggested for the roots of even order. An alternative proof of Heron's general cubic root iterative method is explained. Further, Lagrange's interpolation formula for *nth* root of a number is studied and found that Al-Samawal's and Lagrange's method are equivalent. Again, counterexamples are discussed to justify the effectiveness of the present investigations.

Keywords: Even order roots; Odd order roots; Higher order roots; Heron's method.
Mathematics Subject Classifications 2010: 01A30; 01A35; 11A07

1 Introduction and preliminaries

In Metrika III. 20, Heron described a procedure to calculate an approximate cube root of 100 with the help of a cone. Deslauriers and Dubuc [12] reported that Heron used a general iteration formula to determine the cube root of a number N. That was $\sqrt[3]{N} = a + \frac{bd}{bd + aD}(b-a)$, where $a^3 < N < b^3$, $d = N - a^3$ and $D = b^3 - N$. Heath [19] expressed that Wertheim [37] made a conjecture about Heron's cube root iteration formula. Wertheim [37] conjecture that cube root of a number A was given by $\sqrt[3]{A} = a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2}$,

[*] All Correspondence to: E-mail: skpadhan_math@vssut.ac.in

where $a^3 < A < (a+1)^3$, $A - a^3 = d_1$ and $(a+1)^3 - A = d_2$. Later, Eneström [13] proved the Wertheim's conjecture by using some elementary considerations.

There is considerable interest in Heron's work. Some of the works are found in [8, 20, 36]. Heron developed an iteration formula to find the approximate cube root of a number. In this paper, a general iteration formula is established to find the approximate odd order roots of a number. The possible proofs for even order roots are also provided in a similar way as in [35]. Further, Lagrange's interpolation formula for *nth* root of a number is studied and we have found that Al-Samawal's method and Lagrange's method give the same result. Furthermore, many examples are discussed in support of the results proved here. Moreover, it is observed that our method give better result than Al-Samawal's method.

As root extraction always plays a major role in algebra, number theory, numerical analysis and many more branches of mathematics, several attempts have been made by different authors from ancient to modern times to find the square, cube and higher order roots using many techniques. Hence, a brief historical background on root extraction is presented in the next section.

2 Historical background

About 1800BC, the Babylonians used an iterative method to find the square root of a number. The iterative method used by the Mesopotamians to extract square root is used in digital computers till today. The method is as follows:

Let us find the square root of a positive integer N. Let a be the positive number whose square is close to N.

Then $N = a^2 + e$,where e is an error. Suppose

$$\begin{aligned} \sqrt{N} = a + b \quad &\Rightarrow \quad N = (a+b)^2 = a^2 + e \\ &\Rightarrow \quad 2ab + b^2 = e. \end{aligned}$$

As b^2 is very small as compared to $2ab$, so neglecting it, we get $2ab = e$

$$\begin{aligned} &\Rightarrow \quad b = \frac{e}{2a} \\ &\Rightarrow \quad \sqrt{N} = a + b \approx a + \frac{e}{2a}. \end{aligned}$$

Considering $a_1 = a + \frac{e}{2a}$ as the new approximation and repeat the process again and again to obtain better and better approximations which is very much closed to the actual root (Joseph [22], Page 144).

Around 800-500BC, the value of $\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = 1.4142156...$ is evaluated in the earliest mathematical writings of the Indians "The Sulbasutras" (Joseph [22], Page 334). In the 1st Century AD, Heron of Alexandria ([19], page 324) used the same procedure for square root extraction as described by Babylonians which is presently known as the famous Heron's square root formula i.e.

$$\sqrt{N} = a + b \approx a + \frac{e}{2a} = a + \frac{N - a^2}{2a} = \frac{a^2 + N}{2a} = \frac{1}{2}\left(a + \frac{N}{a}\right)$$

In the later half of 1st Century AD, the methods of square root (khai fang) and cube root (khai li fang) extraction appear in China in the book "Chiu-Chang Suan Shu" (Bag [3], page 82).

During 200-400 AD, Bakhshali Manuscript was written in India which is famous for computation of square roots of non-square numbers and this method is quadratically convergent [4, 22]. According to this

$$\sqrt{N} = \sqrt{a^2 + e} \approx a + \frac{e}{2a} = a + \frac{e}{2a} - \frac{(\frac{e}{2a})^2}{2(a + \frac{e}{2a})}.$$

It seems that it is an extended work on square root formula in "The Sulbasutras" (Joseph [22], page 364). In 250AD, Liu Hui of North-Central China [31] pictured the geometrical basis of the square root formula using different colors. Theon of Alexandria [3, 18, 34] in 390AD, illustrated the Ptolemy's method for extracting square roots using sexagesimal system of fractions and found the square root of 4500^0 to be $67^0 4' 55"$ in sexagesimal unit. This method is purely geometrical and depended on Euclidean concept.

In 499 AD, Aryabhata I [7] gave a rule to extract the square and the cube root of positive integers. Other Indian Scholars Mahavira (850AD), Sridhara (900AD), Aryabhata II (950AD), Bhaskara II (1150AD)and Kamalakaran (1616-1700AD) have also given the same procedure for square root extraction. Some scholars Bramhagupta (598AD), Mahavira, Aryabhata II, Bhaskara I etc. have also attempted the same cube root formula derived by Aryabhata I. Although Bramhagupta does not give any rule for square root, but he tried to express the cube root method of Aryabhata I (Bag [3], page 78, Mishra [27]) using different expressions.

Abraham Ibn Ezra (1090-1167) of Spain, who is famous for "Hebrew Mathematical Tradition" in his only mathematical work "Sefer Ha-mispar" (Section-7) described square root algorithm as $\sqrt{N} = \sqrt{a^2 \pm e} \approx a \pm \frac{e}{2a}$. Two examples quoted there are as follows:

$$\sqrt{7500} = \sqrt{8100 - 600} \approx 90 - \frac{600}{180} = 86$$

and

$$\sqrt{600000} = \sqrt{640000 - 40000} \approx 800 - \frac{40000}{1600} = 774.$$

This identity and approximations were known to Babylonians and Greeks long years back. Al-Samwal (1172), Nasir al-Din-al-Tulsi (1265) were respectively formulated the rule to find fifth and fourth order roots of a real number. Levi ben Gershan (1288-1344) of South France in his book "Maaseh Hoshev" (1321) (Section.e) explained the root extractions of perfect square and cube numbers. Jasmid al-kashi in 1427 extracted the $5th$ root of a decimal number and $6th$ root of a sexagecimal number respectively. Many researchers Tonstall (1522), Rcorde (1542), Buteo (1599) and De lagmy (1650) were looking at cube root extraction in a different way. Some of them prefer to use the table of cubes for cube root extraction (Smith [34]). In 1685, the so called Newton's method was published for the first time. After that Thomas Simpson described it using Calculus in 1740. Similarly Halley (1694) and Householder also developed root finding algorithms using Calculus.

In the $4th$ volume of Merifetname, Ibrahim Hakki of Erzurum, Turkey (Born in 1703) described a method of finding square root. According to him if $x < y$, then $\sqrt{y} \approx \sqrt{x} + \frac{y-x}{2\sqrt{x}+1}$ i.e.$\sqrt{7} \approx \sqrt{4} + \frac{7-4}{2\sqrt{4}+1} = 2 + \frac{3}{5}$. In 1819, English Mathematician George Horner published a numerical method for finding the roots of nth degree equations and an Italian, Paolo Ruffini (1765-1822) independently discovered the same, which is presently named as Horner-Ruffini method. But the same computational technique was using by Chinese more than 500 years ago.

Also in 20th Century, many researchers like [1, 2, 3, 4, 9, 10, 11, 17, 23, 25, 26, 30, 32, 33] have worked on surds, square and cube root extraction methods of Hindu Mathematics. Knudsen [24] has given a detailed analysis of square roots in the Sulbasutras. Parakh [29] discussed Aryabhata's square and cube root algorithms and their computational complexities. Mishra [27] presented a brief account of square root method. A special algorithm to approximate the square root of positive integer was given by Goo [16] in 2013. Izmirli [21] has proved some results about the digital roots. Taisbak [35] established a conjecture about Heron's method and gave possible proofs of the same using difference operators in

2014. Shanty [31] discussed about the implementation of instructed activities of square root method of Liu Hui in a geometrical manner. Also Padhan *et al.* [28] have proposed a general formula for cube and higher roots of real numbers and implemented in FPGA. Cho et al. [6] presented a refinement of Muller's algorithm for the computation of cube root of c, where c is a cubic residue $(mod\ p)$. Gadtia et al. [15] proposed two new iterative algorithms for square root formula found in Śulbasũtras and Bakhshãlĩ Manuscript. Recently, Faisal et al. [14] examined the solubility of $x^3 = a$ in general finite fields and gave some results on cube roots of cubic residue.

In this paper, Wertheim's conjecture on Heron's cubic root method is proved and extended to higher order roots. Also it is shown that Lagrange's and Al-samawal's method for nth root extraction are equivalent. Appropriate numerical examples are illustrated to justify the fresh findings.

3 Iteration methods for odd order roots

In this section, a general formula for all odd order roots of a number is studied. Counterexamples are also given in support of study.

3.1 Wertheim's conjecture for cube root

Theorem 3.1 If $a^3 < A < (a+1)^3$, then approximate cube root of A is defined as

$$\sqrt[3]{A} = a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2},$$

where $d_1 = A - a^3$ and $d_2 = (a+1)^3 - A$.

Proof: Now

$$\begin{aligned} d_1 &= A - a^3 \\ &= (a+x)^3 - a^3 \\ &= 3a^2x + 3ax^2 + x^3. \end{aligned} \tag{3.1}$$

Again

$$\begin{aligned} d_2 &= (a+1)^3 - A \\ &= (a+1)^3 - (a+1-y)^3 \\ &= 3(a+1)^2y - 3(a+1)y^2 + y^3. \end{aligned} \tag{3.2}$$

Using eqns. (3.1) and (3.2), we have

$$\begin{aligned}\frac{d_2}{d_1} &= \frac{3(a+1)^2y - 3(a+1)y^2 + y^3}{3a^2x + 3ax^2 + x^3} \\ &= \frac{3(a+1)y(a+1-y)}{3ax(a+x)} \\ &\quad \big(\text{neglecting the very small terms } y^3 \text{ and } x^3\big) \\ &= \frac{(a+1)y(a+1-y)}{ax(a+x)} \\ &= \frac{(a+1)y}{ax}. \ (\because a+x = a+1-y) \end{aligned} \qquad (3.3)$$

From eqn. (3.3), we get

$$\begin{aligned} & \frac{ad_2}{(a+1)d_1} = \frac{y}{x} \\ \Rightarrow\ & \frac{(a+1)d_1 + ad_2}{(a+1)d_1} = \frac{y+x}{x} \\ \Rightarrow\ & x = \frac{(a+1)d_1}{(a+1)d_1 + ad_2} \ \ (\because x+y=1) \\ \Rightarrow\ & \sqrt[3]{A} = a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2} \end{aligned} \qquad \square$$

Remark 3.1 Heron's cubic root iteration formula conjectured by Wertheim [37] can also be written as

$$\sqrt[3]{A} = \frac{(a+1)^2d_1 + a^2d_2}{(a+1)d_1 + ad_2}.$$

Example 3.1 Evaluation of cube root of 100.
It is obvious that $4^3 < 100 < 5^3$ that is $64 < 100 < 125$. According to Theorem 3.2, $a = 4,\ a+1 = 5,\ d_1 = 100 - 64 = 36,\ d_2 = 125 - 100 = 25.$
Therefore,

$$\begin{aligned}\sqrt[3]{100} &= 4 + \frac{5 \times 36}{5 \times 36 + 4 \times 25} \\ &= 4.6428571428571.\end{aligned}$$

It can be easily seen that $(4.6428571428571)^3 = 100.08199708455$ and the error is very minimum that is 0.08199708455.

3.2 $5th$ root extraction

Theorem 3.2 If $a^5 < A < (a+1)^5$, then approximate $5th$ root of A is defined as

$$\sqrt[5]{A} = a + \frac{(a+1)^2 d_1}{(a+1)^2 d_1 + a^2 d_2},$$

where $d_1 = A - a^5$ and $d_2 = (a+1)^5 - A$.

Proof: Suppose that x is the $5th$ root of A. Assume that $(x-a)^5 = \delta_1$ and $(a+1-x)^5 = \delta_2$. Now

$$\begin{aligned} & (x-a)^5 = \delta_1 \\ \Rightarrow\ & x^5 - 5x^4 a + 10x^3 a^2 - 10x^2 a^3 + 5xa^4 - a^5 = \delta_1 \\ \Rightarrow\ & 5xa(x^3 - 2x^2 a + 2xa^2 - a^3) = d_1 - \delta_1,\ \text{where } x^5 - a^5 = d_1. \end{aligned} \tag{3.4}$$

Again

$$\begin{aligned} & (a+1-x)^5 = \delta_2 \\ \Rightarrow\ & 5(a+1)x\{(a+1)^3 - 2(a+1)^2 x + 2(a+1)x^2 - x^3\} = d_2 - \delta_2, \\ & \text{where } (a+1)^5 - x^5 = d_2. \end{aligned} \tag{3.5}$$

Using eqns. (3.4) and (3.5), we have

$$\begin{aligned} \frac{d_2 - \delta_2}{d_1 - \delta_1} &= \frac{(a+1)\{(a+1-x)^3 + (a+1)^2 x - (a+1)x^2\}}{a\{(x-a)^3 + x^2 a - xa^2\}} \\ &= \frac{(a+1)\{(a+1)^2 x - (a+1)x^2\}}{a(x^2 a - xa^2)} \\ & \big(\text{neglecting the very small terms } (a+1-x)^3 \text{ and } (x-a)^3\big) \\ &= \frac{(a+1)^2(a+1-x)}{a^2(x-a)} \\ &= \frac{(a+1)^2}{a^2} \times \frac{1-(x-a)}{x-a} \\ &= \frac{(a+1)^2}{a^2}\big(\frac{1}{x-a} - 1\big). \end{aligned} \tag{3.6}$$

As the value of δ_1 and δ_2 are very small, from eqn. (3.6), we get

$$
\begin{aligned}
& \frac{d_2}{d_1} = \frac{(a+1)^2}{a^2}\Big(\frac{1}{x-a} - 1\Big) \\
\Rightarrow\ & \frac{1}{x-a} = \frac{a^2 d_2}{(a+1)^2 d_1} + 1 \\
\Rightarrow\ & x - a = \frac{(a+1)^2 d_1}{a^2 d_2 + (a+1)^2 d_1} \\
\Rightarrow\ & x = a + \frac{(a+1)^2 d_1}{a^2 d_2 + (a+1)^2 d_1} \\
\Rightarrow\ & \sqrt[5]{A} = a + \frac{(a+1)^2 d_1}{(a+1)^2 d_1 + a^2 d_2}. \qquad \square
\end{aligned}
$$

Example 3.2 Evaluation of $5th$ root of 100.
It is clear that $2^5 < 100 < 3^5$ that is $32 < 100 < 243$. According to Theorem 3.2, $a = 2,\ a + 1 = 3,\ d_1 = 100 - 32 = 68,\ d_2 = 243 - 100 = 143$.
Therefore,

$$
\begin{aligned}
\sqrt[5]{100} &= 2 + \frac{3^2 \times 68}{3^2 \times 68 + 2^2 \times 143} \\
&= 2.5168918918919.
\end{aligned}
$$

It can be easily verified that $(2.5168918918919)^5 = 101.00033364761$ and the error is very minimum that is 1.00033364761.

3.3 nth root extraction

Here the general formula for odd order roots ($n = 2m + 1,\ m \in \mathbb{Z}_+$) of a number is discussed.

Theorem 3.3 If $a^n < A < (a+1)^n$, $n = 2m + 1$, then approximate nth root of A is defined as

$$\sqrt[n]{A} = a + \frac{(a+1)^m d_1}{(a+1)^m d_1 + a^m d_2},$$

where $d_1 = A - a^n$ and $d_2 = (a+1)^n - A$.

Proof: Let x be the nth root of A. Suppose that $(x-a)^{2m+1}=\delta_1$ and $(a+1-x)^{2m+1}=\delta_2$. Now

$$\begin{aligned}
& (x-a)^{2m+1}=\delta_1 \\
\Rightarrow\ & x^{2m+1}-(2m+1)_{C_1}x^{2m}a+(2m+1)_{C_2}x^{2m-1}a^2-...+(2m+1)_{C_{2m}}xa^{2m}-a^{2m+1}=\delta_1 \\
\Rightarrow\ & (2m+1)_{C_1}x^{2m}a-(2m+1)_{C_2}x^{2m-1}a^2+...-(2m+1)_{C_{2m}}xa^{2m}=x^{2m+1}-a^{2m+1}-\delta_1 \\
\Rightarrow\ & (2m+1)_{C_1}xa(x^{2m-1}-mx^{2m-2}a+...-a^{2m-1})=d_1-\delta_1 \qquad (3.7)
\end{aligned}$$

where $x^{2m+1}-a^{2m+1}=d_1$.

Similarly, taking $(a+1)^{2m+1}-x^{2m+1}=d_2$, we have

$$(2m+1)_{C_1}(a+1)x\{(a+1)^{2m-1}-m(a+1)^{2m-2}x+...-x^{2m-1}\}=d_2-\delta_2. \qquad (3.8)$$

Dividing eqn. (3.8) by eqn. (3.7), we get

$$\begin{aligned}
\frac{d_2-\delta_2}{d_1-\delta_1} &= \frac{(2m+1)_{C_1}(a+1)x\{(a+1)^{2m-1}-m(a+1)^{2m-2}x+...-x^{2m-1}\}}{(2m+1)_{C_1}xa(x^{2m-1}-mx^{2m-2}a+...-a^{2m-1})} \\
&= \frac{(a+1)\{(a+1)^{2m-1}-m(a+1)^{2m-2}x+...-x^{2m-1}\}}{a(x^{2m-1}-mx^{2m-2}a+...-a^{2m-1})} \\
&\quad \Big(\text{neglecting the very small terms } (x-a)^3,\ (x-a)^5,...,(x-a)^{2m-1} \\
&\quad \text{and } (a+1-x)^3,\ (a+1-x)^5,...,(a+1-x)^{2m-1} \text{ and simplifyng }\Big) \\
&= \frac{(a+1)^m}{a^m}\Big(\frac{1}{x-a}-1\Big). \qquad (3.9)
\end{aligned}$$

As the value of δ_1 and δ_2 are very small, from eqn. (3.9), we get

$$\begin{aligned}
& \frac{d_2}{d_1}=\frac{(a+1)^m}{a^m}\Big(\frac{1}{x-a}-1\Big) \\
\Rightarrow\ & \frac{1}{x-a}=\frac{a^m d_2}{(a+1)^m d_1}+1 \\
\Rightarrow\ & x-a=\frac{(a+1)^m d_1}{a^m d_2+(a+1)^m d_1} \\
\Rightarrow\ & x=a+\frac{(a+1)^m d_1}{a^m d_2+(a+1)^m d_1} \\
\Rightarrow\ & \sqrt[n]{A}=a+\frac{(a+1)^m d_1}{(a+1)^m d_1+a^m d_2}. \qquad \square
\end{aligned}$$

Remark 3.2 When $m=1$, Theorem 3.3 reduces to Heron's cubic root iteration formula conjectured by Wertheim [37]. That is

$$\sqrt[3]{A}=a+\frac{(a+1)d_1}{(a+1)d_1+ad_2},$$

where $a^3<A<(a+1)^3$, $A-a^3=d_1$ and $(a+1)^3-A=d_2$.

4 Iteration methods for even order roots

In this section, it is verified through counterexamples that our proposed methods can also work well for even order roots. That is, if $a^n < A < (a+1)^n$, $n = 2m$, then

$$\sqrt[n]{A} = a + \frac{(a+1)^m d_1}{(a+1)^m d_1 + a^m d_2}, \tag{4.1}$$

where $d_1 = A - a^n$ and $d_2 = (a+1)^n - A$.

Example 4.1 (4*th* order root extraction). Evaluation of $4th$ root of 100.
It is clear that $3^4 < 100 < 4^4$ that is $81 < 100 < 256$. According to eqn. (4.1), $a = 3,\ a+1 = 4,\ d_1 = 100 - 81 = 19,\ d_2 = 256 - 100 = 156$.
Therefore,

$$\begin{aligned} \sqrt[4]{100} &= 3 + \frac{4^2 \times 19}{4^2 \times 19 + 3^2 \times 156} \\ &= 3.1779859484778. \end{aligned}$$

It can be easily checked that $(3.1779859484778)^4 = 102.00181287647$ and the error is 2.00181287647.

Example 4.2 (6*th* order root extraction). Evaluation of $6th$ root of 100.
It is seen that $2^6 < 100 < 3^6$ that is $64 < 100 < 729$. According to eqn. (4.1), $a = 2,\ a+1 = 3,\ d_1 = 100 - 64 = 36,\ d_2 = 729 - 100 = 629$.
Therefore,

$$\begin{aligned} \sqrt[6]{100} &= 2 + \frac{3^3 \times 36}{3^3 \times 36 + 4^3 \times 629} \\ &= 2.161892071952. \end{aligned}$$

It can be easily shown that $(2.161892071952)^6 = 102.09490123878$ and the error is 2.09490123878.

4.1 $4th$ order root extraction using difference operators

Some possible proofs are given in order to extract the $4th$ root of a number using difference operators. This approach is similar to Taisbak [35].
First part: Let $a-1,\ a,\ a+1$ be 3 successive positive integers. To find the ratio between the difference of their $4th$ powers.

Assertion: If $S = a^4 - (a-1)^4$ and $D = (a+1)^4 - a^4$, then $S : D \geq (a-1)^2 : (a+1)^2$.
Proof:

$$\begin{aligned} S &= a^4 - (a-1)^4 \\ &= a^4 - (a^4 - 4a^3 + 6a^2 - 4a + 1) \\ &= 4a^3 - 6a^2 + 4a - 1. \end{aligned}$$

$$\begin{aligned} D &= (a+1)^4 - a^4 \\ &= a^4 + 4a^3 + 6a^2 + 4a + 1 - a^4 \\ &= 4a^3 + 6a^2 + 4a + 1. \end{aligned}$$

Ignoring ± 1, we get

$$\begin{aligned} \frac{S}{D} &\geq \frac{4a^3 - 6a^2 + 4a}{4a^3 + 6a^2 + 4a} \\ &= \frac{2(a-1)^2 + (a-1) + 1}{2(a+1)^2 - (a+1) + 1} \\ &= \frac{2(a-1)^2 + (a-1)}{2(a+1)^2 - (a+1)} \ \big(\text{ignoring } +1\big) \\ &= \frac{(a-1)\{2(a-1)+1\}}{(a+1)\{2(a+1)-1\}} \\ &= \frac{(a-1)^2}{(a+1)^2}. \ \big(\text{ignoring } \pm 1\big) \end{aligned}$$

Therefore, $\dfrac{S}{D} \geq \dfrac{(a-1)^2}{(a+1)^2}$.
Second part: To determine the approximate $4th$ root of a non-$4th$ power integer A. Let it be a, whose neighbors are $a - f$ and $a + g$ ($f,\ g$ are numbers < 1).
Assertion: If $P = a^4 - (a-f)^4$ and $Q = (a+g)^4 - a^4$, then $P : Q \geq f(a-f)^2 : g(a+g)^2$.
Proof: $P = a^4 - (a-f)^4 = 4a^3f - 6a^2f^2 + 4af^3 - f^4$, $Q = (a+g)^4 - a^4 = 4a^3g + 6a^2g^2 + 4ag^3 + g^4$.

Ignoring $-f^4$ and $+g^4$, we get

$$\begin{aligned}
\frac{P}{Q} &\geq \frac{4a^3f - 6a^2f^2 + 4af^3}{4a^3g + 6a^2g^2 + 4ag^3} \\
&= \frac{f\{2(a-f)^2 + af\}}{g\{2(a+g)^2 - ag\}} \\
&= \frac{f\{2(a-f)^2 + (a-f)f + f^2\}}{g\{2(a+g)^2 - (a+g)g + g^2\}} \\
&= \frac{f\{2(a-f)^2 + (a-f)f\}}{g\{2(a+g)^2 - (a+g)g\}} \quad \big(\text{ignoring } +f^2 \text{ and } +g^2\big) \\
&= \frac{f(a-f)\{2(a-f) + f\}}{g(a+g)\{2(a+g) - g\}} \\
&= \frac{f(a-f) \times 2(a-f)}{g(a+g) \times 2(a+g)} \quad \big(\text{ignoring } +f \text{ and } -g\big) \\
&= \frac{f(a-f)^2}{g(a+g)^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{P}{Q} \geq \frac{f(a-f)^2}{g(a+g)^2} \\
\Rightarrow\ & (a+g)^2P : (a-f)^2Q \geq f : g.
\end{aligned}$$

Taking the case of $\sqrt[4]{100}$, it can be observed that $a + g = 4$, $a - f = 3$, $P = 19$, $Q = 156$ and the ratio $f : g = 304 : 1404$. So the ratio $f : (f + g) = 304 : 1708$. As $f + g = 1$, so f is a fraction less than 1 and hence $a = 3 + f$ is found.

5 Heron's Cubic Root Iteration Method

In this section, we shall prove Heron's general cubic root iteration method.

Theorem 5.1 If $a^3 < A < b^3$, then approximate cube root of A is defined as

$$\sqrt[3]{A} = a + \frac{bd_1}{bd_1 + ad_2}(b - a),$$

where $d_1 = A - a^3$ and $d_2 = b^3 - A$.

Proof: Now

$$\begin{aligned}
d_1 &= A - a^3 \\
&= (a + x)^3 - a^3 \\
&= 3a^2x + 3ax^2 + x^3. \qquad (5.1)
\end{aligned}$$

Again

$$\begin{aligned} d_2 &= b^3 - A \\ &= b^3 - (b-y)^3 \\ &= 3b^2y - 3by^2 + y^3. \end{aligned} \tag{5.2}$$

Using eqns. (5.1) and (5.2), we have

$$\begin{aligned} \frac{d_2}{d_1} &= \frac{3b^2y - 3by^2 + y^3}{3a^2x + 3ax^2 + x^3} \\ &= \frac{3by(b-y)}{3ax(a+x)} \\ &\quad \big(\text{neglecting the very small terms } y^3 \text{ and } x^3\big) \\ &= \frac{by(b-y)}{ax(a+x)} \\ &= \frac{by}{ax}. \ (\because a + x = b - y) \end{aligned} \tag{5.3}$$

From eqn. (5.3), we get

$$\begin{aligned} & \frac{ad_2}{bd_1} = \frac{y}{x} \\ \Rightarrow \ & \frac{bd_1 + ad_2}{bd_1} = \frac{y+x}{x} \\ \Rightarrow \ & x = \frac{bd_1}{bd_1 + ad_2}(b-a) \ \ (\because x + y = b - a) \\ \Rightarrow \ & \sqrt[3]{A} = a + \frac{bd_1}{bd_1 + ad_2}(b-a). \end{aligned}$$

□

Remark 5.1 Heron's general cubic root iteration formula can also be written as

$$\sqrt[3]{A} = \frac{b^2d_1 + a^2d_2}{bd_1 + ad_2}.$$

6 A study on existing methods

In this section, we study some known methods and obtain relations between them.

Al-samawal's method:[5] If a is the integer part of nth root of a number A, then approximate nth root of A is

$$\sqrt[n]{A} = a + \frac{A - a^n}{(a+1)^n - a^n}.$$

Lagrange's interpolation method: If $a^n < A < (a+1)^n$, then by Lagrange's interpolation, the approximate nth root of A is

$$\sqrt[n]{A} = \frac{A-(a+1)^n}{a^n-(a+1)^n} \times a + \frac{A-a^n}{(a+1)^n-a^n} \times (a+1).$$

Remark 6.1 Al-samawal's method and Lagrange's interpolation method for nth order root of a number are equivalent.

Proof: For $a^n < A < (a+1)^n$, Lagrange's interpolation method is

$$\begin{aligned}
\sqrt[n]{A} &= \frac{A-(a+1)^n}{a^n-(a+1)^n} \times a + \frac{A-a^n}{(a+1)^n-a^n} \times (a+1) \\
&= \frac{-aA + a(a+1)^n + (a+1)A - (a+1)a^n}{(a+1)^n - a^n} \\
&= \frac{-aA + a(a+1)^n + aA + A - (a+1)a^n}{(a+1)^n - a^n} \\
&= \frac{A + a(a+1)^n - aa^n - a^n}{(a+1)^n - a^n} \\
&= \frac{(A-a^n) + a\{(a+1)^n - a^n\}}{(a+1)^n - a^n} \\
&= a + \frac{A-a^n}{(a+1)^n - a^n}.
\end{aligned}$$

Hence, Al-samawal's method and Lagrange's interpolation method are equivalent. □

7 Comparison with our proposed methods

It is natural to wonder how our method compares with other known methods. We provide below a comparison and show that our method yields better results. In the table given below, we have compared Al-samawal's method [5] with our proposed method.

	Al-samawal's method		Our proposed method	
$\sqrt[n]{A}$	Approximate Value	Error	Approximate Value	Error
$\sqrt[3]{100}$	4.5901639344262	-3.28705926928	4.6428571428571	0.08199708455
$\sqrt[4]{100}$	3.1085714285714	-6.622250226676	3.1779859484778	2.00181287647
$\sqrt[5]{100}$	2.3222748815166	-32.4589009001	2.5168918918919	1.00033364761
$\sqrt[6]{100}$	2.0541353383459	-24.87675760439	2.16189207152	2.09490123878

From the above table it is cleared that our proposed method gives better approximation than previously reported methods [5].

8 Conclusion

In this paper we have extended Heron's method to obtain higher order roots. Some possible proofs for roots of even order are also mentioned. It is also shown that both Al-samawal's method and Lagrange's method are same. The method of our paper is compared with Al-samawal's method and we observe that it gives minimum error. It is also found that the method conjectured by Wertheim [37] is a particular case of the present work. Several counterexamples are discussed in lieu of the present investigation. The convergency of the investigated methods are yet to be studied.

9 Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This material has not been published in whole or in part elsewhere and do not communicate any other journal for publication. The author has not received funding from any sources for this work. There is no research involving human participants and or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

References

[1] A. Ahmad, On Vedic Square-Circle Conversion, Ganita Bharati, 4 (1980) 72-77.

[2] A. Ahmad, On Babylonian and Vedic Square-Root of 2, Ganita Bharati, 16 (1994) 1-4.

[3] A.K. Bag, Mathematics in Ancient and Medieval India, Chukhambha Orientalia, Varanasi, 1979.

[4] M.N. Channabasappa, On the square root formula in the Bakhshali manuscript, Indian Journal of History of Science, 11 (1976) 112-124.

[5] K. Chemla, Similarities between Chinese and Arabic mathematical writings. I. Root extraction, Arabic Sciences and Philosophy, 4 (1994) 207-266.

[6] G.H. Cho, S. Kwon and H.S. Lee, A refinement of Müller's cube root algorithm, Finite Fields and Their Applications, 67 (2020) Article 101708.

[7] W.E. Clark, The Aryabhatiya of Aryabhata, University of Chicago Press, Chicago, 1930.

[8] K.D. Crisman and M.H. Veatch, Reinventing Heron, The college Mathematics Journal, 45 (2014) 191-197.

[9] B. Datta, The Science of Śulba, Calcutta University, 1930.

[10] B. Datta and A.N. Singh, Surds in Hindu Mathematics(revised by KS Shukla), Indian Journal of History of Science, 28 (1993) 253-264.

[11] B. Datta and A.N. Singh, Approximate values of Surds in Hindu Mathematics (revised by KS Shukla), Indian Journal of History of Science, 28 (1993) 265-275.

[12] G. Deslauriers and S. Dubuc, Le calcul de la racine cubique selon Heron, Elemente der Mathematik, 51 (1996) 28-34.

[13] G. Eneström, Bibliotheca Mathematica, VIII (1907-8) 412-413.

[14] Faisal, Rojali and M.S.B. Mohamad, An Algorithms for Finding the Cube Roots in Finite Fields, Procedia Computer Science, 179 (2021) 838-844.

[15] S. Gadtia, S.K. Padhan and B. Bhoi, Square root formulae in the Śulbasũtras and Bakhshãlĩ Manuscript, Journal of Interdisciplinary Mathematics, 23 (2020) 1-10.

[16] H.M. Goo, A Special Algorithm to Approximate the Square Root of Positive Integer, International Journal of Mathematical and Computational Sciences, 7 (2013) 245-248.

[17] R.C. Gupta, Vedic Mathematics from the Śulbasũtras, Indian Journal of Mathematics Education, 9 (1989) 1-10.

[18] T. Heath, A history of Greek Mathematics, Vol.-I, 1921.

[19] T. Heath, A history of Greek Mathematics, Vol.-II, 1923.

[20] A. Hess, A highway from Heron to Brahmagupta, Forum Geometricorum, 12 (2012) 191-192.

[21] I.M. Izmirli, On Some Properties of Digital Roots, Advances in Pure Mathematics, 4 (2014) 295-301.

[22] G.G. Joseph, The crest of the peacock : Non-European roots of mathematics, [3rd ed.], 1991.

[23] S. Kak, Computational aspects of Aryabhata algorithm, Indian Jornal of History of Science, 21 (1986) 62-71

[24] T.L. Knudsen, Historical Notes: Square roots in the Sulbasutras, Indian Journal of History of Science, 40 (2005) 107-111.

[25] R.S. Lal, Extraction of Square-Root in Ancient Hindu Mathematics, Mathematics Education, 18 (1984) 130-135.

[26] R.S. Lal, Extraction of Cube-Root in Ancient Hindu Mathematics, Mathematics Education, 19 (1985) 153-160.

[27] V. Mishra, Computing $\sqrt{N}$: A Modern Generalization of Ancient Technique, Indian Journal of History of Science, 46 (2011) 49-61.

[28] S.K. Padhan, S. Gadtia and B. Bhoi, FPGA based implementation for extracting the roots of real number, Alexandria Engineering Journal, 55 (2016) 2849-2854.

[29] A. Parakh,$\bar{A}$ryabhata's root extraction methods, Indian Journal of History of Science, 42 (2007) 149-161.

[30] S.C. Sarabhai, The Cube-Root Method of Mahav$\bar{\imath}$r$\bar{a}$c$\bar{a}$rya, Mathematics Education, 20 (1986) 14-18.

[31] N.O. Shanty, The use of history of mathematics in learning the extraction of square roots, Proceedings of ADVED 2016, Istanbul, Turkey (2016) 42-45.

[32] S.L. Singh and R.Chand, An Extraction of Ancient Indian Cube-Root Method, Journal of Natural and Physical Sciences, 3 (1989) 78-89.

[33] A.N. Singh, On Indian Method of Root Extraction, Bulletin of Calcutta Mathematical Society, 18 (1927) 123-140.

[34] D.E. Smith, History of Mathematics, Volume.II, Dover, New York 1958.

[35] C.M. Taisbak,Cube roots of integers. A conjecture about Herons method in Metrika III. 20, Historia Mathematica, 41 (2014) 103-106

[36] C.M. Taisbak, An Archimedean proof of Heron's formula for the area of a triangle: heuristics reconstructed. From Alexandria, through Bagdad, Springer, Heidelberg, (2014) 189-198

[37] G. Wertheim, Zeitschr. f. Math. u. Physik, Hist. -Litt. Abt., XLIV (1899) 1-3.