Diffusion and memory effect in a stochastic processes and the correspondence to an information propagation in a social system

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Abstract A generalized Langevin equation is suggested to describe a diffusion particle system with memory. The equation can be transformed into the Fokker-Planck equation by using the Kramers-Moyal expansion. The solution of Fokker-Planck equation can describe not only the diffusion of particles but also that of opinion particles based on the similarities between the two. We find that the memory can restrain some non-equilibrium phenomena of velocity distribution in the system, without memory, induced by correlation between the noise and space[1]. However, the memory can enhance the effective collision among particles as shown by the variation of diffusion coefficients, and changes the diffusion mode between the dissipative and pumping region by comparing with that in the aforementioned system without memory. As the discussions in this physical system is paralleled to a social system, the random diffusion of social ideology, such as the information propagation, can be suppressed by the correlation between the noise and space.

Keywords Fokker-Planck equation · positive damping · negative damping · memory effects · opinion particle

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1 Introduction

Diffusion is one of universal phenomena in nature and therefore is one of the classical subjects of physics. It has been successfully described by continuous/discrete time random walk[2–5] and other models expressed by the stochastic differential equations[6–11]. As is well known, the diffusion in an open system is often influenced by stochastic fluctuation, internal and/or external force(noise). The internal and external fluctuations synchronously arise in an open-system, and influence upon each other. So the additive noise and multiplicative noise are often considered as concomitant.

It is noted that there are different representations of particle motion on two time scales, short-time and long-time, in a stochastic system. The particle trajectory in the short-time is non-differentiable because of frequent collisions of it with others, so the changing state can not be described by certain differential equations, while the particle motion in the long-time can be described analytically because the non-differentiability can be eliminated or neglected in a large time scale. Therefore, the particle trajectory is non-differentiable in microscopic scale but differentiable in macroscopic scale[12–14, 1]. In other words, the randomness of a system can be ascribed to some extent to the memory of the microscopic dynamics[15]. However, the differentiability of a particle motion is guaranteed by the long-time scale, which may be compatible with the statistical description of a system with memory due to the fact that the statistical properties are generally embodied in macroscopic scale.

The memory effects in relevant systems have drawn much attention of physicists. Stanislavsky proved that the non-differentiable nature of the microscopic dynamics can be transmitted to the macroscopic level[16]. Maes investigated extension of fluctuation-response relations for non-equilibrium diffusion in stochastic systems with memory, and applied the relation to modify the Sutherland-Einstein relation due to the strong memory[17]. The memory effects of anomalous diffusions are numerically simulated by Chattopadhyay[18]. The recursive relationship in a random walk model[19, 20] was transformed into the Fokker-Planck equation by Silva et al. to describe novel phenomena caused by memory[20]. Based on the elephant random walk model, the dependence of the diffusion on the initial condition and memory parameters is investigated[5, 21].

As is well known that the memory of individuals in social societies deeply affects the human behaviors such as opinion formation[22, 23], cultural dynamics[24], crowd behaviors[25], etc., which are characterized by the complex outcome of many physiological and psychological processes. It is also known that physicists have made many attempts to describe the human behaviors within the framework of statistical physics based on the similarity between the components of a physical system and a social system, the Brownian particles and the individuals[26]. As far as the specific methodology, the popular ones are master equation[1, 27], the Fokker-Planck equation transformed from the Langevin equation[26], as well as the extended versions of the equations expressing the dynamics on the complex networks[28, 29]. Some new conceptions such as social space, opinion space, information space, social force, etc., are proposed to characterize the state of social particle, for instance, opinion particle[1, 30, 31].

It seems almost inevitable that the memory of microscopic dynamics will give rise to some peculiar effects in both of physical and social systems. Aiming at revealing the memory effect in diffusion process of a stochastic system, this paper suggests a stochastic dynamic model with memory described by a generalized Langevin equation, and investigate the spatial-temporal characteristics induced by the stochastic fluctuation or memory via solving a generalized Fokker-Planck-Kolmogorov equation derived from the generalized Langevin equation. And then, the results will be mapped to a social system to understand some of public opinions associated with the memory of individuals.

2 Stochastic dynamical model with memory

The motion of a particle with memory is essentially a non-Markovian process, which can be described by a generalized Langevin equation as following

$$\begin{cases} \dot{x} = v & (a) \\ \dot{v} = -\int_0^t u(t-t')\gamma(v)v dt' - \nabla U(x) + G(v)\xi(t) & (b) \end{cases} \quad (1)$$

where x and v denote a displacement and velocity of a particle, respectively. $u(t)$ represents the memory function. For the classical ideal Markov process (without memory), the memory function is defined as $u(t-t') = \gamma_0 \delta(t-t')$, where $\delta(t-t')$ is the Delta function, and the Laplace transform of the memory function can be written as $\bar{u}(s) = \frac{1}{s}$. For a real system, the memory function is between the one without memory and the ideal one of which the value is unity. So, the memory function can be taken as a step function. And then, the Laplace transform of the memory function can be written as $\bar{u}(s) = \frac{1}{s^\nu}$, $0 < \nu < 1$ [18, 21, 32]. Similarly, the memory function here is selected as the fractional order of which the Laplace transform is

$$\bar{u}(s) = \begin{cases} \gamma_0 \frac{1}{s^\nu}, & s \geq 0 \\ 0, & s < 0 \end{cases} \quad (2)$$

where γ_0 represents the damping coefficient, ν the power-law exponent of which the value ranges in $(-1, 0)$.

In Eq.1, $\gamma(v)$ describes the non-linear damping function, which is selected as the form suggested by Rayleigh-Helmholtz, $\gamma(v) = v^2 - \alpha$. Where α is a positive constant which divides the velocity space into two regions, the positive damping region and the negative pumping one [33, 34]. $U(x)$ denotes a generalized potential, and is taken as $\frac{\omega^2}{2}x^2$ as well as linear gradient force $\nabla U(x) = \omega_0^2 x$. $G(v)\xi(t)$ stands for random quantity corresponding to the Langevin force induced by fluctuation. $G(v)$ is the strength of noise and $\xi(t)$ the Gaussian white noise that is characterized by zero mean value, $\langle \xi(t) \rangle = 0$. $G(v)\xi(t)$ is actually composed of the additive noise $\zeta(t)$ and multiplicative $g(v)\zeta(t)$ ($g(v)$ is the strength of multiplicative noise, which can be selected as the linear function of velocity v). Therefore, based on the statistical relevance theory, the strength is given by $G(v) = \sqrt{2D_0v^2 + 4\theta\sqrt{D_0Q}v + 2Q}$, where θ denotes the

correlation strength factor between different noises, and satisfies with $0 < \theta < 1$. The detailed calculation please refer to Ref.[35].

The integration term in the time domain is introduced into the model because of the memory effects in the diffusion process. Therefore, the generalized Langevin equation proves to be the integro-differential one. In fact, the integration term $R(t)$ represents the time correlation function between the damping term and the memory, that is,

$$R(t) = \int_0^t u(t-t') \gamma(v(t')) v(t') dt' \quad (3)$$

To simplify Eq.1, the Laplace transform is needed to be carried out on this function, and the inverse Laplace transform can be given by

$$\begin{aligned} R(t) &= \mathcal{L}^{-1}[\bar{R}(s)] \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \bar{u}(s) * \overline{\gamma(v(s))v(s)} e^{st} ds \\ &= \frac{\gamma_0}{\Gamma(v)} \int_0^t (t-t')^v \gamma(v(t')) v(t') dt' \\ &= \frac{\gamma_0}{\Gamma(v)} t^v * (\gamma(v(t))v(t)) \end{aligned} \quad (4)$$

where $\Gamma(v)$ denotes the Gamma function, $\mathcal{L}^{-1}$ the inverse Laplace transform and $\bar{R}(s)$ the Laplace transform that is needed to be carried out on the integration term, that is, the Laplace transform of function $R(t)$.

From Eq.4, the memory function in a general form can be written as

$$u(t, t') = \frac{\gamma_0}{\Gamma(v)} (t-t')^v \quad (5)$$

Then Eq.1 will be transformed into the corresponding Fokker-Planck equation

$$\frac{\partial P(x, v, t)}{\partial t} = \sum_i^{a,b} \sum_{n=1}^{\infty} (-\nabla)^n \left(D_i^{(n)}(x, v, t) P(x, v, t) \right) \quad (6)$$

where $P(x, v, t)$ denotes the probability of particle with velocity v at time t and position x , $D_i^{(n)}(x, v, t-t')$ the Kramers-Moyal expansion coefficients, the subscript $i = a, b$ denotes the expansions derived respectively from differential equations (a) and (b) in Eq.1, and the superscript n the order number. Defining the coefficients as the first time derivative of the central moment and divided by $n!$ [36], we have

$$D_i^{(n)}(x, v, t) = \frac{1}{n!} \frac{d}{dt} M_n(t) \Big|_{t=0}, i = a, b \quad (7)$$

where $M_n(t)$ is the n th moment of variables. To simplify each moment, let the right terms in Eq.1(b) be written as

$$\begin{cases} H(x, v, t) = \frac{\gamma_0}{\Gamma(v)} t^v * (\gamma(v)v) + \nabla U(x) \\ \Psi(v, t) = G(v) \xi(t) \end{cases} \quad (8)$$

A Taylor expansion of function $H(x, v, t)$ and $\Psi(v, t)$ at $x(t')$ and $v(t')$ can be respectively given by (It is needed to point out that there are equations $\frac{\partial H(x(t), v(t), t)}{\partial x(t)} \equiv \frac{\partial H(x(t'), v(t'), t')}{\partial x(t')}$, $\frac{\partial H(x(t), v(t), t)}{\partial v(t)} \equiv \frac{\partial H(x(t'), v(t'), t')}{\partial v(t')}$ and $\frac{\partial \Psi(v(t), t)}{\partial v(t)} \equiv \frac{\partial \Psi(v(t'), t')}{\partial v(t')}$)

$$\begin{cases} H(x(t), v(t), t) = H(x(t'), v(t'), t') + \frac{\partial H(x(t'), v(t'), t')}{\partial x(t')} (x(t) - x(t')) \\ \quad + \frac{\partial H(x(t'), v(t'), t')}{\partial v(t')} (v(t) - v(t')) + \dots \\ \Psi(v(t), t) = \Psi(v(t'), t') + \frac{\partial \Psi(v(t'), t')}{\partial v(t')} (v(t) - v(t')) + \dots \end{cases} \quad (9)$$

To derive these Kramers-Moyal expansion coefficients, we substitute Eq.9 into Eq.1(b), and obtain

$$\begin{aligned} v(t + \tau) - v(t) &= \int_t^{t+\tau} H(x, v, t') dt' + \int_t^{t+\tau} \frac{\partial H}{\partial x} (x(t) - x(t')) dt' + \dots \\ &\quad + \int_t^{t+\tau} G(v) \xi(t') dt' + \int_t^{t+\tau} \frac{\partial G}{\partial v} (v(t) - v(t')) \xi(t') dt' + \dots \end{aligned} \quad (10)$$

where τ is a time interval.

The integration terms in Eq.10, $x(t) - x(t')$ and $v(t) - v(t')$, can be replaced by the first iteration of the equation itself, which leads to

$$\begin{aligned} v(t + \tau) - v(t) &= \int_t^{t+\tau} H(x(t'), v(t'), t') dt' + \int_t^{t+\tau} \frac{\partial H}{\partial x} \int_t^{t'} H(x, v, t'') dt'' dt' \\ &\quad + \int_t^{t+\tau} \frac{\partial H}{\partial v} \int_t^{t'} H(x, v, t'') dt'' dt' + \int_t^{t+\tau} \frac{\partial H}{\partial v} \int_t^{t'} G(v) \xi(t'') dt'' dt' + \dots \\ &\quad + \int_t^{t+\tau} \Psi(v, t') \xi(t') dt' + \int_t^{t+\tau} \frac{\partial \Psi}{\partial v} \xi(t') \int_t^{t'} H(x, v, t'') dt'' dt' \\ &\quad + \int_t^{t+\tau} \frac{\partial G}{\partial v} \xi(t') \int_t^{t'} G(v) \xi(t'') dt'' dt' + \dots \end{aligned} \quad (11)$$

Thus, the first-order and second-order moment are respectively given by

$$\begin{aligned} \langle v(t + \tau) - v(t) \rangle &= \int_t^{t+\tau} H(x(t'), v(t'), t') dt' + \int_t^{t+\tau} \frac{\partial H}{\partial x} \int_t^{t'} H(x, v, t'') dt'' dt' \\ &\quad + \int_t^{t+\tau} \frac{\partial H}{\partial v} \int_t^{t'} H(x, v, t'') dt'' dt' + \dots \\ &\quad + \int_t^{t+\tau} \frac{\partial G}{\partial v} \int_t^{t'} G(v) \delta(t'' - t') dt'' dt' + \dots \end{aligned} \quad (12)$$

and

$$\begin{aligned} \langle (v(t + \tau) - v(t))^2 \rangle &= \int_t^{t+\tau} H(x, v, t') dt' \int_t^{t+\tau} H(x, v, t') dt' \\ &\quad + \int_t^{t+\tau} \frac{\partial H}{\partial x} \int_t^{t'} H(x, v, t'') dt'' dt' + \dots \\ &\quad + \int_t^{t+\tau} G(v) \int_t^{t'} G(v) \delta(t'' - t') dt'' dt' + \dots \end{aligned} \quad (13)$$

Because of $\int_t^{t'} G(v(t'')) \delta(t'' - t') dt'' = G(v(t'))$, when $\tau \rightarrow 0$ the Kramers-Moyal expansion coefficients can be written as the following

$$\begin{cases} D_b^{(1)}(x, v, t) = \frac{1}{1!} \lim_{\tau \rightarrow 0} \frac{1}{\tau} \langle v(t + \tau) - v(t) \rangle \\ \quad = H(x, v, t) + \frac{\partial G(v)}{\partial v} G(v) \\ D_b^{(2)}(x, v, t) = \frac{1}{2!} \lim_{\tau \rightarrow 0} \frac{1}{\tau} \langle (v(t + \tau) - v(t))^2 \rangle \\ \quad = (G(v))^2 \\ D_b^{(n)}(x, v, t) = 0, \text{ for } n \geq 3 \end{cases} \quad (14)$$

Similarly, the Kramers-Moyal expansion coefficients, corresponding to Eq.1(a), can be written as

$$\begin{cases} D_a^{(1)}(x, v, t) = v \\ D_a^{(n)}(x, v, t) = 0, \text{ for } n \geq 2 \end{cases} \quad (15)$$

So we obtain the Fokker-Planck equation

$$\begin{aligned} \frac{\partial P(x, v, t)}{\partial t} = & -\frac{\partial}{\partial v} D_b^{(1)}(x, v, t) P(x, v, t) + \frac{1}{2} \frac{\partial^2}{\partial v^2} D_b^{(2)}(x, v, t) P(x, v, t) \\ & - \frac{\partial}{\partial x} D_a^{(1)}(x, v, t) P(x, v, t) \end{aligned} \quad (16)$$

Substituting Eqs.14-15 into Eq.16, it becomes

$$\begin{aligned} \frac{\partial P(x, v, t)}{\partial t} = & -\frac{\partial}{\partial v} \left(\frac{\gamma_0}{\Gamma(v)} t^v * \gamma(v) v + \nabla U(x) + \frac{\partial G(v)}{\partial v} G(v) \right) P \\ & + \frac{1}{2} \frac{\partial^2}{\partial v^2} (G(v))^2 P - v \frac{\partial P}{\partial x} \end{aligned} \quad (17)$$

Eq.17 can be simplified as

$$\dot{P}(x, v, t) = L_{FP} P(x, v, t) \quad (18)$$

Where $L_{FP} = -\frac{\partial}{\partial v} \left(\frac{\gamma_0}{\Gamma(v)} t^v * \gamma(v) v + \nabla U(x) + \frac{\partial G(v)}{\partial v} G(v) \right) + \frac{1}{2} \frac{\partial^2}{\partial v^2} (G(v))^2 - v \frac{\partial}{\partial x}$. In order to solve Eq.18, this operator can be decomposed into two parts: the time-independent operator L_{st} and the time-dependent operator L_{ext} . So we have

$$\begin{cases} L_{FP}(x, v, t) = L_{st}(x, v) + L_{ext}(x, v, t) \\ L_{st}(x, v) = -\frac{\partial}{\partial v} \left(\nabla U(x) + \frac{\partial G(v)}{\partial v} G(v) \right) + \frac{1}{2} \frac{\partial^2}{\partial v^2} (G(v))^2 - v \frac{\partial}{\partial x} \\ L_{ext}(x, v, t) = -\frac{\partial}{\partial v} \left(\frac{\gamma_0}{\Gamma(v)} t^v * \gamma(v) v \right) \end{cases} \quad (19)$$

To obtain the stationary solution of Eq.18, the time-independent operator L_{st} can be further divided into reversible flux operator L_{st}^{rev} and irreversible collision operator L_{st}^{ir} , that is

$$\begin{cases} L_{st} = L_{st}^{rev} + L_{st}^{ir} \\ L_{st}^{rev} = -\nabla U(x) \frac{\partial}{\partial v} - v \frac{\partial}{\partial x} \\ L_{st}^{ir} = \frac{G(v)^2}{2} \frac{\partial^2}{\partial v^2} \end{cases} \quad (20)$$

The stationary solutions of these operators are given by

$$\begin{cases} P_{\text{st}}^{\text{rev}} = \exp\left(-\frac{\omega_0^2 x^2}{2\rho} - \frac{v^2}{2}\right) \\ P_{\text{st}}^{\text{ir}} = v \end{cases} \quad (21)$$

Therefore, the stationary solution of operator L_{st} can take the following form

$$P_{\text{st}}(x, v) = v \exp\left(-\frac{\omega_0^2 x^2}{2\rho} - \frac{v^2}{2}\right) \quad (22)$$

Any solution of the Fokker-Planck equation can be split into the stationary solution and a time-dependent one, that is

$$P(x, v, t) = P_{\text{st}}(x, v) + P_{\text{ext}}(x, v, t) \quad (23)$$

The Fokker-Planck equation can be given by

$$\dot{P}(x, v, t) = \dot{P}_{\text{ext}}(x, v, t) = (L_{\text{st}} + L_{\text{ext}}) \times (P_{\text{st}}(x, v) + P_{\text{ext}}(x, v, t)) \quad (24)$$

Here term $L_{\text{ext}}(x, v, t)P_{\text{ext}}(x, v, t)$ can be neglected, the linear terms can be retained, that is

$$\dot{P}_{\text{ext}}(x, v, t) = L_{\text{st}}P_{\text{ext}}(x, v, t) + L_{\text{ext}}P_{\text{st}}(x, v) \quad (25)$$

A formal solution of this equation is given by

$$P_{\text{ext}}(x, v, t) = \int_{-\infty}^t e^{L_{\text{st}}(x, v)(t-t')} L_{\text{ext}}(x, v, t') P_{\text{st}}(x, v) dt' \quad (26)$$

To solve Eq.26, we introduce the transition probability density $P_{\text{ext}}(\sigma, t | \sigma', t')$ expressed by the delta function to substitute for the integration. Where $\sigma = (x, v)$ denotes a generalized vector space, which is composed of the generalized displacement and the velocity at time $t = t' + \tau$. Similarly, $\sigma' = (x', v')$ denotes a vector space at time t' . And then, we now introduce the representation of the δ function in terms of a Fourier integral, the solution of Eq.26 for small τ can be written as

$$\begin{aligned} P_{\text{ext}}(x, v, t | x', v', t') &= L_{\text{ext}}P_{\text{st}}(x, v) e^{L_{\text{st}}(x, v)(t-t')} \delta(\sigma(x, v) - \sigma(x', v')) \\ &= L_{\text{ext}}P_{\text{st}}(x, v) e^{L_{\text{st}}(x, v)(t-t')} \int_{-\infty}^{+\infty} e^{ik_1(x-x')} dk_1 \int_{-\infty}^{+\infty} e^{ik_2(v-v')} dk_2 \\ &= \frac{K(x, v)}{4\pi^2} \int_{-\infty}^{+\infty} e^{(-ik_2\tau(\nabla U(x) + \frac{\partial G(v)}{\partial v}G(v)) - \frac{1}{2}k_2^2\tau G(v)^2 + ik_2(v-v'))} dk_2 \\ &\quad \times \int_{-\infty}^{+\infty} e^{(ik_1(-v\tau + x-x'))} dk_1 \\ &= \frac{K(x, v)}{\sqrt{2\pi\tau G(v)^2}} \exp\left(-\frac{(v-v'-\tau(\nabla U(x) + \frac{\partial G(v)}{\partial v}G(v)))^2}{2\tau G(v)^2}\right) \end{aligned} \quad (27)$$

where $K(x, v, t) = \frac{\gamma_0}{\Gamma(v)} P_{\text{st}}(x, v) (vt^v * (3v^2 - \alpha) + (1 - v^2)t^v * (v^3 - \alpha v))$.

Therefore, the probability function of Eq.23 can be written as

$$P(x, v, t) = P_{\text{st}}(x, v) \left(1 - \frac{K(x, v)}{\sqrt{2\pi\tau G(v)^2}} e^{-\frac{(v-v'-\tau(\nabla U(x) + \frac{\partial G(v)}{\partial v}G(v)))^2}{2\tau G(v)^2}} \right) \quad (28)$$

3 Diffusion characteristic of particles

Now let's present the discussions on the transport characteristics based on the solution, Eq. 28. The parameters are selected as $\omega_0 = 5.0 \times 10^{-2}$, $r_0 = 1.0 \times 10^{-2}$, $\theta = 0.9$, $\alpha = 0.3$, $\nu = -0.2$. Based on the expression of probability density, we investigate the transport characteristics via calculating relevant statistical quantities, such as diffusion coefficient, viscosity coefficient and thermal conductivity coefficient.

3.1 The characteristics of the velocity distribution

Fig. 1 shows that the Gaussian distribution dominating the steady state. The results are different from Refs.[37,38] and our previous work without memory[1] where there is non-equilibrium spike, appearing on the background of such steady distribution, caused by the correlation between noise and space. Fig. 1(a) shows, the peak of the Gaussian distribution moves in the direction of velocity increasing and transforms into the flattening pattern with decrease of correlation intensity of noise. The shift of the peak indicates that the expected value of velocity and the standard deviation will increase. The flattening of the distribution curve implies that the decrease of the correlation of noise will homogenize the velocity distribution to weaken the correlation between particles. The reason may be that the memory maintains the state of every particle and restrains the correlation among them. It is also proved by Fig. 1(b), where the probability density gradually flattens with increase of the characteristic parameter of memory, ν (of which the value -1 and 0 respectively corresponds to the ideal memory and the absence of memory). The results indicate that the second moment of velocity increases, which also implies that the randomness of the system is strengthened. This is due to the fact that the memory can increase the independence of the particles, so the randomness of the system is wholly strengthened although the randomness of each particle is singly restrained. Compared with the situation without memory, one may draw a conclusion that memory can restrain the correlation between the noise and space. As shown in Fig. 1(c), the correlation between the noise and the space gradually decreases with increase of the parameter of memory. Meanwhile, the difference between dissipative region and pumping region disappears. This implies that the energy captured from environment in the pumping region decreased, and stochastic collision in the dissipative region is weakened by the memory effects. In fact, it has been stressed above that the memory can enhance the ability of the system keeping its state but weaken the response of the system to the stochastic fluctuation.

3.2 The characteristics of the diffusion

As is well known, if a macroscopic system is at a non-equilibrium state, a transport process will often occur to establish a new equilibrium. Generally, the transport process is irreversible, and the establishment of the new equilibrium will take some

Fig. 1 The velocity distribution under different parameter combinations. (a) The Gaussian distribution, the steady distribution of velocity flattens with the decrease of the correlation intensity denoted by D and Q ; (b) The steady distribution flattens with the characteristic parameter v as D, Q is $2.0 \times 10^{-5}, 1.0 \times 10^{-5}$, respectively; (c) The dependence of the correlation between the noise and the space on time with different parameters of memory.

limited time, named relaxation time. The characteristics of the process can be described by the transport coefficients such as diffusion, viscosity and thermal conductivity coefficient. Based on the statistical correlation of Green-Kubo[39–43], these coefficients can be given by

$$\begin{cases} D_{\text{eff}} = \frac{1}{2dt} \langle (x(t) - \langle x \rangle)^2 \rangle \\ \eta = \frac{1}{2dLt} \langle (\rho vx - \langle \rho vx \rangle)^2 \rangle \\ \kappa = \frac{1}{2dT^2Lt} \langle (xe - \langle xe \rangle)^2 \rangle \end{cases} \quad (29)$$

where $D_{\text{eff}}, \eta, \kappa$ denote diffusion, viscosity and thermal conductivity coefficient, respectively. L represents the spatial scale, d the spatial dimension, which is selected as $d = 1$ in the model, e the statistical kinetic energy and T the thermodynamic temperature. The thermodynamic temperature is classically defined as

$$T = k_B^{-1} \langle \rho v^2 \rangle \quad (30)$$

where k_B is the Boltzmann constant.

Fig. 2 shows the dependence of the mean square displacement and diffusion coefficient on time. Fig. 2(a) demonstrates that the mean square displacement depends on time partly in the power law with exponent $k_{\text{MSD}} = 0.003$. Obviously, the exponent is very closed to 0, and therefore the mean square displacement tends to a constant ($\langle (x - \langle x \rangle)^2 \rangle \approx 0.17$). Fig. 2(b) shows that the diffusion is also related to time in power law with exponent $k_D = -0.997$ which is very closed to -1 . The exponents for both of the mean square displacement and diffusion coefficient imply that the diffusion is a sub-diffusion[44–51]. Compared this diffusion process with that without memory[52–54], it is obvious that the non-equilibrium behaviors induced by correlation between the noise and the space do not appear. These results support the conclusion above that the memory enhances the randomness of the system.

As shown in Fig. 3, the dependence of the viscosity coefficient on statistical temperature obeys a power law with a plus exponent $k_\eta = 0.947$. This result might be consistent with the property of a sparse particle system in which the viscosity mainly originates from the collision among the particles. With increase of temperature, the

Fig. 2 The scaling relations of (a) the mean square displacement and (b) the diffusion coefficient relate to time.

collision frequency and the velocity increase, so the exchange of mass and momentum among the particles strengthens, which leads to the increase of the viscosity coefficient. In fact, the similar results are observed in the sparse systems such as Fermi gas[55–57], ideal fluid[58–62], etc.

Fig. 3 The scaling law between the viscosity coefficient and statistical temperature.

Fig. 4 shows the dependence of the thermal conductivity on temperature is dominated by the power law with scaling exponent $k_K = 2.1$. The similar relations are observed in some systems such as amorphous solid, ideal fluid and sparse clathrate hydrate[63–65]. Especially, the simulation results in a one-dimensional molecular chain indicate that the thermal conductivity increases monotonously with temperature as the chain density is less than 1[66]. So we can conclude that there is the same mechanism, collision/interaction among the particle, governing the relation between the viscosity and temperature and that between the thermal conductivity and temperature.

4 The correspondence of the diffusion characteristics to social behaviors

A social system can be regarded as a classical open system composed of the so-called quasi-particles describing public opinion, ideology, etc. It is evident that there is some

Fig. 4 Scaling law between the thermal conductivity coefficient and temperature.

similarities between the physical systems and the social systems, as well as that between the corresponding units, for instance, the particles and opinion particles[67, 68]. So the regularities dominating the behaviors of a physical system can be paralleled to a corresponding social system to interpret or understand the social behaviors. The change of a sociology is generally deeply influenced by its historical state, then the discussions above on the diffusion behaviors of the system with memory can be naturally used to describe the propagation of the opinion, idea, belief, etc.

The prerequisite for doing so is that we need to re-define the variables and parameters of the physical system so as to describe the social system reasonably. The variables in Eq.1 might become the "generalized" ones, and the corresponding new definitions are suggested in our previous work[.]. The generalized displacement is defined as the quantity of information carried by a quasi-particle in a social system. The velocity, the changing rate of the generalized displacement/the quantity of information, is defined as the sensitivity of the quasi-particle to an event. The effect of noise is that caused by the influence of the inter and/or outer environment that is introduced into the system via the coupling of it with the generalized displacement.

Now we might understand the information propagation in the process such as disseminations of opinion, idea, belief, etc. by means of the characteristic coefficients describing the diffusion process with memory. The overall characteristics of the system is depicted by Fig. 1. As shown by Fig. 1(a), the mean velocity of information propagation increases when the random force, originating from the social environment, weakens. As shown by Fig. 1(b), the memory enhances the randomness via decreasing the coupling between the individuals by comparing with that the memory strengthens the independence of the individuals. The scaling exponent, $k_D \rightarrow -1$, for the power law governing the variation of the diffusion coefficient with time as shown in Fig. 2 indicates that the information propagation process is an extreme sub-diffusion. The corresponding dynamic mechanism can be interpreted as that the information propagation is achieved mainly via colliding(message exchanging) between the quasi-particles due to the fact presented by Fig. 3 and Fig. 4. The two figures show that the viscosity coefficient and thermal conductivity coefficient respectively depend on temperature in power law manner. The plus scaling exponents indicate that the information propagate mainly by means of colliding between quasi-particles. It is a natural result that the disseminations of opinion mainly depends on the information

exchange between individuals. And then, one may draw a conclusion that the memory can increase the velocity of the information propagation (the Gaussian distribution curve moving rightward), enhance the individualism and restrain disturbing from the outer noise (the distribution curve flattening), enhance the stability for the structure of the social group (keeping the Gaussian distribution), to a certain extent.

5 Conclusion and Discussion

In this article, the transport behaviors in the open particle system with memory are investigated by the Fokker-Planck equation transformed from a generalized Langevin equation. The solution brings to light the details of the transport behaviors, which can be paralleled to the social system formed by quasi-particles to describe the propagation of idea, opinion, belief, etc..

In the particle system, the curve of the steady distribution of velocity moves in the direction of velocity increasing and transforms into the flattening pattern with decrease of correlation intensity of noise, and also gradually flattens with increase of the memory. These imply that the memory can enhance the ability of the system keeping its state but weaken the response of the system to the stochastic fluctuation. The minus scaling exponent for the power law governing the variation of the diffusion coefficient with time indicates that the process is a sub-diffusion. The plus scaling exponents for the power laws governing the relation between the viscosity coefficient and temperature as well as that between the thermal conductivity coefficient and temperature reveal that the corresponding transport mechanism can be mainly attributed to the collision between particles. The similar relations are observed in some systems such as sparse clathrate hydrate [63, 64, 69] and single crystal [65, 70].

Based on the similarity between the diffusion in physical system and the propagation of idea, opinion, belief, etc., in social system, the results and discussions in the former are paralleled to the latter. The mean velocity of information propagation increases when the random force, random disturbance originating from the social environment, weakens. The memory enhances the randomness via decreasing the coupling between the individuals. In other words, the memory increases the independence of the individuals, the collision/interaction, among them weakens, and therefore the transport process is sub-diffusion. Generally speaking, the memory can enhance the structure stability of a social group. It may be an important inspiration for us.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability statement

My manuscript has no associated data.

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