Dynamics of Spatial Epidemic Model with Infected Population Repulsion

Meng Yan

Henan Institute of Science and Technology

Qingshan Zhang (✉ qingshan11@yeah.net)

Henan Institute of Science and Technology <https://orcid.org/0000-0003-2729-9818>

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Dynamics of spatial epidemic model with infected population repulsion

Meng Yan · Qingshan Zhang

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Abstract In this paper, we are concerned with the spatial epidemic model with infected-taxis in which the susceptible individuals could avoid the infected ones. The spatial pattern for the resulted model is investigated under homogeneous Neumann boundary condition. We gain the condition for spatial pattern induced by diffusion term and infected-taxis term. Moreover, we obtain the condition for the occurrence of pattern formations induced by infected-taxis, in which the diffusion-driven Turing instability case is excluded. We give numerical examples to support the theoretical scheme.

Keywords Epidemic model · Cross-diffusion · Turing instability · Pattern formation

1 Introduction

Infectious diseases have seriously affected the development of human society, especially the coronavirus disease 2019 (COVID-19) poses a serious health threat to humans since its first case outbreaks in 2019. Globally, a great deal of efforts have focused on taking measures to contain the outbreak and prevent the further spread of virus. In the past decades, various ordinary differential equation (ODE) models are increasing used to study the mechanism of epidemic transmission in spatially homogeneous environment, for example one may refer to [1, 2, 4, 7, 14, 15] and the references therein. On

the other hand, it is known that spatial heterogeneity of the environment and individual motility could affect the spread of many diseases because of the localized nature of transmission or other forms of interaction [29–34]. Many spatial epidemic models have been developed by incorporating random diffusion terms in the ODE models. For these spatial models, Many researchers capture some new dynamics phenomenon in disease spread caused by incorporation of spatial heterogeneity and diffusion [16–28].

In the real world, the susceptible individuals always have the ability to recognize the infective ones and have the tendency to move away from the direction of higher infected density [11]. Given this feature, the following spatial epidemic model with self and cross diffusion has been investigated in [11, 13]:

$$\begin{cases} S_t = d_1 \Delta S + d \Delta I + F(S, I), \\ I_t = d_2 \Delta I + G(S, I). \end{cases} \quad (1.1)$$

The spatial pattern formation and dynamics of (1.1) with different incidence rate have been investigated in [3, 11, 13]. On the other hand, the repulsive tendency of healthy people to infected ones can be defined by the cross-diffusion term $\alpha \nabla \cdot (S \nabla I)$, which is derived in a similar way to chemotaxis [8]. We use this cross-diffusion term in model (1.1) to obtain the following epidemic reaction-diffusion system with infected-taxis:

$$\begin{cases} S_t = d_1 \Delta S + \alpha \nabla \cdot (S \nabla I) + F(S, I), \\ I_t = d_2 \Delta I + G(S, I). \end{cases} \quad (1.2)$$

Compared with (1.1), as far as we know there are few researches on pattern formation of model (1.2) and little is known how far the dynamic behavior be influenced by the cross-diffusion term. The goal of the present work is to study pattern formation for model (1.2) and show how the infected-taxis affects the Turing pattern.

Meng Yan
Department of Mathematics, Henan Institute of Science and Technology, Xinxiang 453003, China.

Qingshan Zhang (✉)
Department of Mathematics, Henan Institute of Science and Technology, Xinxiang 453003, China.
E-mail: qingshan11@yeah.net

The paper is organized as follows. In Section 2, we introduce a spatial epidemic model with infected population repulsion. In Section 3, we summarize some results on spatial epidemic model without infected population repulsion. In Section 4, through the standard linearization analysis, we investigate the dynamic behavior of spatial epidemic model with infected population repulsion. In Section 5, we give numerical stimulations to support the theoretical results. Finally, conclusions and discussion are presented in Section 6.

2 A spatial epidemic model with infected population repulsion

To take into account the inhomogeneous distribution of the susceptible and infections individuals in the two-dimensional space domain $\Omega = [0, L_x] \times [0, L_y]$, both species have the trend of spreading to the area with low population density. We assume the random diffusion to low density areas according to Fourier's law. On the other hand, the susceptible individuals have the consciousness of avoiding diseased individuals. Their repulsion actions are assumed to follow Fick's law which is proportional to the density gradient of the infected ones. Considering that susceptible individuals have the consciousness of avoiding diseased individuals and the random diffusion themselves, the diffusion and cross-diffusion term are incorporated into the model proposed by Liu et al. [5, 6], and we are led to the following reaction-diffusion equation

$$\begin{cases} S_t = d_1 \Delta S + \alpha \nabla \cdot (S \nabla I) - \beta S^p I^q + R - dS, \\ I_t = d_2 \Delta I + \beta S^p I^q - (d + \mu)I, \end{cases} \quad (2.1)$$

where S and I stand for are the densities of the susceptible and infected individuals, respectively, d_1 and d_2 represent their diffusion coefficients. The symbol $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator in two-dimensional space and ∇ is the two-dimensional gradient operator, R is the population recovery rate, d represents the natural mortality rate of the population, μ represents the fatality rate of infected persons due to disease, β represents the disease infection rate, α represents the rejection coefficient of susceptible individuals to infected individuals. According to the actual situation, all the parameters in the equation are positive.

We consider model (2.1) along with the initial conditions

$$S(x, y, 0) = S_0(x, y) > 0, \quad I(x, y, 0) = I_0(x, y) > 0, \quad (2.2)$$

where $(x, y) \in \Omega$ and impose the homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad (x, y) \in \partial\Omega, \quad t > 0, \quad (2.3)$$

where $\frac{\partial}{\partial n}$ is the differentiation with respect to outward normal on the boundary $\partial\Omega$. The boundary condition implies the susceptible and infections individuals locate within the fixed domain Ω and they are isolated from outside.

3 Some results on spatial epidemic model without cross-diffusion

Before studying the model, we first recall some results about pattern formation of the spatial epidemic model without cross-diffusion term [10, 12]. Let $\alpha = 0$, $p = 1$ and $q = 2$ in (2.1), then the model is shown as follow

$$\begin{cases} S_t = d_1 \Delta S - \beta SI^2 + R - dS, \\ I_t = d_2 \Delta I + \beta SI^2 - (d + \mu)I. \end{cases} \quad (3.1)$$

The corresponding non-spatial model of (3.1) is

$$\begin{cases} S_t = -\beta SI^2 + R - dS, \\ I_t = \beta SI^2 - (d + \mu)I. \end{cases} \quad (3.2)$$

For the convenience of subsequent calculations, we denote

$$F(S, I) = R - \beta SI^2 - dS, \quad G(S, I) = \beta SI^2 - (d + \mu)I.$$

Let $F(S, I) = 0$ and $G(S, I) = 0$, we get system (6.1) has three equilibrium points:

(a) Extinction of infected individual

$$E_0 = \left(\frac{R}{d}, 0\right);$$

(b) Coexistence of susceptible and infected individuals

$$E_1 = \left(\frac{R\beta + \sqrt{R^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d + \mu)}{R\beta + \sqrt{R^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}} \right);$$

it is easy to know that this equilibrium point is an unstable saddle point.

(c) Coexistence of susceptible and infected individuals

$$E^* = \left(\frac{R\beta - \sqrt{R^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d + \mu)}{R\beta - \sqrt{R^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}} \right),$$

the equilibrium point is a stable node.

The equilibrium point $E^* = (S^*, I^*)$ is positive if and only if $R^2\beta > 4d(d + \mu)^2$. By linearized method, the Turing instability condition [10, 12] of (3.1) concerning E^* is given by

1. $F_S^* + G_I^* < 0$;
2. $F_S^*G_I^* - F_I^*G_S^* > 0$;
3. $d_2F_S^* + d_1G_I^* > 0$;
4. $(d_2F_S^* + d_1G_I^*)^2 > 4d_1d_2\det(J)$,

where F_S^* , F_I^* and G_S^* , G_I^* are the first derivations of F and G to be evaluated at E^* . The Hopf bifurcation appears when

$$\text{Im}(\lambda_k) \neq 0, \quad \text{Re}(\lambda_k) = 0, \quad k = 0,$$

and Turing bifurcation occurs when

$$\text{Im}(\lambda_k) = 0, \quad \text{Re}(\lambda_k) = 0, \quad k = \sqrt{\frac{\det(J)}{d_1d_2}},$$

where λ_k are solutions of the corresponding characteristic equation

$$\lambda^2 + B(k^2)\lambda + C(k^2) = 0,$$

with

$$B(k^2) = (d_1 + d_2)k^2 - (F_S^* + G_I^*)$$

and

$$C(k^2) = d_1d_2k^4 - (d_2F_S^* + d_1G_I^*)k^2 + (F_S^*G_I^* - F_I^*G_S^*).$$

4 General analysis of cross-diffusion epidemic spatial model

In this section we investigate the dynamic behavior of the following general partial differential equation with cross diffusion

$$\begin{cases} S_t = d_1\Delta S + \alpha\nabla \cdot (S\nabla I) + F(S, I), \\ I_t = d_2\Delta I + G(S, I), \end{cases} \quad (4.1)$$

where $F(S, I)$ and $G(S, I)$ are general kinetic terms concerning S and I . Let S and I have the small space disturbances $\varepsilon S_1(x, y, t)$ and $\varepsilon I_1(x, y, t)$ around the equilibrium point $E^* = (S^*, I^*)$, namely,

$$(x, y, t) = S^* + \varepsilon S_1(x, y, t), \quad I(x, y, t) = I^* + \varepsilon I_1(x, y, t).$$

After standard linearization analysis [9] of equation (4.1), we obtain

$$\begin{cases} S_{1t} = d_1\Delta S_1 + \alpha S^*\Delta I_1 + F_S^*S_1 + F_I^*I_1, \\ I_{1t} = d_2\Delta I_1 + G_S^*S_1 + G_I^*I_1, \end{cases} \quad (4.2)$$

where

$$\begin{aligned} F_S^* &= F_S(S^*, I^*), & F_I^* &= F_I(S^*, I^*), \\ G_S^* &= G_S(S^*, I^*), & G_I^* &= G_I(S^*, I^*). \end{aligned}$$

Let the spatiotemporal perturbations $S_1(x, y, t)$ and $I_1(x, y, t)$ be of the form

$$S_1 = c_1 e^{\lambda t + i(k_1 x + k_2 y)} \quad \text{and} \quad I_1 = c_2 e^{\lambda t + i(k_1 x + k_2 y)}, \quad (4.3)$$

where c_1 and c_2 are amplitudes of the perturbations, λ represents the time growth rate of the disturbance at the equilibrium point E^* , $k = (k_1, k_2)$ represents the spatial wave number. Substituting (4.3) into equation (4.1), we obtain the matrix equation

$$\begin{pmatrix} \lambda + d_1k^2 - F_S^* & \alpha S^*k^2 - F_I^* \\ -G_S^* & \lambda + d_2k^2 - G_I^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

According to Cramer's Rule, the necessary and sufficient condition for homogeneous linear equations to have non-zero solutions is that the coefficient determinant is zero, i.e.

$$\begin{vmatrix} \lambda + d_1k^2 - F_S^* & \alpha S^*k^2 - F_I^* \\ -G_S^* & \lambda + d_2k^2 - G_I^* \end{vmatrix} = 0.$$

After simplifying the above determinant, we reach the characteristic equation of the system (4.1)

$$\lambda^2 + P(k^2)\lambda + Q_\alpha(k^2) = 0, \quad (4.4)$$

where

$$P(k^2) = (d_1 + d_2)k^2 - (F_S^* + G_I^*)$$

and

$$\begin{aligned} Q_\alpha(k^2) &= d_1d_2k^4 - (d_2F_S^* + d_1G_I^* - \alpha S^*G_S^*)k^2 \\ &\quad + (F_S^*G_I^* - F_I^*G_S^*). \end{aligned}$$

Therefore, the roots of (4.4) are given by

$$\lambda_{1,2}(k^2) = \frac{-P(k^2) \pm \sqrt{[P(k^2)]^2 - 4Q_\alpha(k^2)}}{2}.$$

4.1 Instability induced by diffusion and cross diffusion

In this subsection we discuss the instability driven by diffusion and cross-diffusion term in equation (4.1). Let

$$J = \begin{pmatrix} F_S^* & F_I^* \\ G_S^* & G_I^* \end{pmatrix}$$

be the Jacobian matrix of the non-spatial equation concerning (4.1). Linear stability of the equilibrium point

E^* for the non-spatial equation is ensured under the condition [9]

$$\text{tr}(J) = F_S^* + G_I^* < 0$$

and

$$\det(J) = F_S^* G_I^* - F_I^* G_S^* > 0.$$

For the steady state E^* to be unstable, we require at least one of the condition $P(k^2) > 0$ or $Q_\alpha(k^2) > 0$ is not true for some $k > 0$. Since $F_S^* + G_I^* < 0$, $d_1 > 0$ and $d_2 > 0$, we have

$$P(k^2) = (d_1 + d_2)k^2 - (F_S^* + G_I^*) > 0$$

for all $k > 0$. Therefore, we require $Q_\alpha(k^2) < 0$ for some $k > 0$ to ensure instability, which is

$$d_1 d_2 k^4 - (d_2 F_S^* + d_1 G_I^* - \alpha S^* G_S^*) k^2 + (F_S^* G_I^* - F_I^* G_S^*) < 0.$$

Given that $d_1 > 0$, $d_2 > 0$ and $\det(J) = F_S^* G_I^* - F_I^* G_S^* > 0$, from the Cauchy inequality we have

$$\begin{aligned} d_2 F_S^* + d_1 G_I^* - \alpha S^* G_S^* &> d_1 d_2 k^2 + \frac{\det(J)}{k^2} \\ &\geq 2\sqrt{d_1 d_2 \det(J)}, \end{aligned}$$

and " $=$ " is obtained at $k_c^2 = \sqrt{\frac{\det(J)}{d_1 d_2}}$. This shows that the parameters involving in equation (4.1) satisfies the above inequality, the equilibrium point E^* is unstable. Therefore, the condition of spatial pattern formation in (4.1) are that

1. $F_S^* + G_I^* < 0$;
2. $F_S^* G_I^* - F_I^* G_S^* > 0$;
3. $d_2 F_S^* + d_1 G_I^* - \alpha S^* G_S^* > 2\sqrt{d_1 d_2 \det(J)}$.

4.2 The influence of cross-diffusion on pattern formations

In this subsection we concentrate on the occurrence of pattern formations induced by the cross-diffusion term $\alpha \nabla \cdot (S \nabla I)$ without diffusion-driven Turing instability. Hence, we assume that

$$P(k^2) = (d_1 + d_2)k^2 - (F_S^* + G_I^*) > 0$$

and

$$\begin{aligned} Q_0(k^2) &= d_1 d_2 k^4 - (d_2 F_S^* + d_1 G_I^*) k^2 \\ &\quad + (F_S^* G_I^* - F_I^* G_S^*) > 0, \end{aligned}$$

which assure the equilibrium point E^* is stable for non-cross-diffusion model (namely, equation (4.1) with $\alpha = 0$). Now we are in the position to consider the instability of the equation (4.1) with $\alpha > 0$. Because the $P(k^2)$

in the characteristic equation (4.4) is not changed after adding the cross-diffusion term, the condition of instability for (4.1) is that the $Q_\alpha(k^2) > 0$ is violated. If $G_S^* > 0$, the condition $Q_\alpha(k^2) > 0$ is valid for all $\alpha > 0$, then the equation (4.1) remains stable and the cross-diffusion term does not affect the stability of the equation. Therefore, in order to make the equation with cross diffusion unstable, it is necessary to satisfy $Q_\alpha(k^2) < 0$ for some k in the case of $G_S^* < 0$, namely

$$d_1 d_2 k^4 - (d_2 F_S^* + d_1 G_I^* - \alpha S^* G_S^*) k^2 + (F_S^* G_I^* - F_I^* G_S^*) < 0$$

for some $k > 0$. Keep α on the left side of the inequality and sort it out

$$\begin{aligned} \alpha &> \frac{d_1 d_2 k^2 - (d_2 F_S^* + d_1 G_I^*) + \frac{(F_S^* G_I^* - F_I^* G_S^*)}{k^2}}{-S^* G_S^*} \\ &\geq \frac{2\sqrt{d_1 d_2 (F_S^* G_I^* - F_I^* G_S^*)} - (d_2 F_S^* + d_1 G_I^*)}{-S^* G_S^*} \stackrel{\text{def}}{=} \alpha_c. \end{aligned}$$

The above estimates obtain the equal sign at $k_c^2 = \sqrt{\frac{\det(J)}{d_1 d_2}}$. This is the critical wave number k_c^2 and it is the same as the case non-cross-diffusion case [10]. This shows that cross-diffusion does not change the critical wave number k_c^2 , but the root λ of the characteristic equation is changed. This shows that under the premise of $G_S^* < 0$, when α exceeds the above specified value α_c , the system is unstable.

5 Numerical simulation

In this section, we perform numerical simulations to visualize the conclusions obtained by the finite difference method. In order to get a fascinating insight of the analytical result, we have to discretize the proposed model (2.1)-(2.3) with $p = 1$ and $q = 2$. In other words, the equation needs to be converted from infinite-dimensional (continuous) form to finite-dimensional (discrete) form. We deal with the system (2.1) in a discrete domain with $M \times N$ lattice sites. The time derivative is discretized by the Euler method with the step size Δt and the Laplacian operator is discretized by five-point difference scheme with the step size Δh . Some fictitious grid points outside the discrete domain are imposed to solve the Neumann boundary condition.

In our simulation we set $M = N = 100$, $\Delta t = 0.01$ and $\Delta h = 1$. Different types of dynamics are observed and we have found that the distributions of susceptible individual and infected individual are always of the same type. Therefore, we can restrict our simulation of pattern formation to infected individuals. We choose $R = 1$, $\mu = 1.8$, $d = 1$, $\beta = 32$, $D_1 = 6$ and $D_2 = 1$. The initial density distributions are random spatial distributions of the species. We run the simulations until

Fig. 1 Spatial distributions developed by infected individual at different instants with $R = 1, \mu = 1.8, d = 1, \beta = 32, D_1 = 6, D_2 = 1, \alpha = 0.02$. (a): 0 iteration; (b): 2000 iterations; (c): 5000 iterations and (d): 8000 iterations.

they reach a stationary state or until they show a behavior that does not seem to change its characteristics anymore. Set $\alpha = 0.02$. We show the evolution of the spatial pattern of infected individual at 0, 2000, 5000 and 8000 iterations, with random perturbation of the steady states $E^* = (S^*, I^*)$ and the result shown in the Fig. 1. Furthermore, we perform 0, 2000, 5000 and 8000 iterations on the infected individuals with $\alpha = 0.8$. The results are shown in Fig. 2.

6 Conclusion and discussion

The pattern formation in the spatial epidemic model has important practical significance in ecology. It is shown that environmental heterogeneity has great influence on pattern formation of disease transmissions in the spatial epidemic model. In the real world, the susceptible individuals always have the ability to identify the infected ones and stay away from them. Such directional movement is not random, and the susceptible individuals move opposite to the gradient of infected distribution (sometime called infected-taxis). Taking into account the direct movement and random diffusion in the epidemic model, we consider the following spatial

epidemic model with infected population repulsion

$$\begin{cases} S_t = d_1 \Delta S + \alpha \nabla \cdot (S \nabla I) + F(S, I), \\ I_t = d_2 \Delta I + G(S, I), \end{cases} \quad (6.1)$$

By theoretical analysis, we found necessary conditions for pattern formation of (3.1) are that

1. $F_S^* + G_I^* < 0$;
2. $F_S^* G_I^* - F_I^* G_S^* > 0$;
3. $d_2 F_S^* + d_1 G_I^* - \alpha S^* G_S^* > 2\sqrt{d_1 d_2 \det(J)}$.

Moreover, we investigate the spatial pattern formation induced by infected-taxis term instead of diffusion-driven Turing instability. Hence we assume the model (3.1) without infected-taxis term ($\alpha = 0$ in (3.1)) is stability, and we have if $G_S^* > 0$, the infected-taxis term can not reduce instability in (3.1) and if $G_S^* < 0$ the infected-taxis term can drive instability when α is greater than critical value.

In summary, we have developed a general approach for pattern formation caused by cross-diffusion and (or) diffusion in the spatial epidemic model. The method we use in this paper is introduced in a general form, so it can be widely applied to models with a series of interacting populations. The method and result used in this

Fig. 2 Spatial distributions developed by infected individual at different instants with $R = 1, \mu = 1.8, d = 1, \beta = 32, D_1 = 6, D_2 = 1, \alpha = 0.8$. (a): 0 iteration; (b): 2000 iterations; (c): 5000 iterations and (d): 8000 iterations.

paper can not only enrich the research on pattern formation in spatial epidemic models with infected-taxis, but also explain the practical significance of some fields.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

All data generated or analysed during this study are included in this manuscript.

References

1. Anderson, R.M., Ma, R.M.: Infectious diseases of humans: dynamics and control. Oxford university press. (1992)
2. Chen, Y., Liu, F., Yu, Q., Li, T.: Review of fractional epidemic models. *Applied Mathematical Modelling*. (2021)
3. Duan, M., Chang, L., Jin, Z.: Turing patterns of an si epidemic model with cross-diffusion on complex networks. *Physica A: Statistical Mechanics and its Applications*, **533**(122023), pp. 122023–122026(2019)
4. Hethcote, H.W., The mathematics of infectious diseases. *SIAM review*. **42**(4), pp. 599–653(2000)
5. Liu, W.M., Hethcote, H.W., Levin, S.A.: Dynamical behavior of epidemiological models with nonlinear incidence rates. *J. Math. Biol.* **25**(4), pp. 359–380(1987)
6. Liu, W.M., Levin, S.A., Iwasa, Y.: Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models. *J. Math. Biol.* **23**(2), pp. 187–204(1986)
7. McCallum, H., Barlow, N., Hone, J.: How should pathogen transmission be modelled?. *Trends Ecol. Evol.* **16**(6), pp. 295–300(2001)
8. Murray, J.D.: Mathematical biology: I. An introduction. *Interdisciplinary applied mathematics*. Springer-Verlag. (2002)
9. Murray, J.D.: Mathematical Biology II: Spatial Models and Biomedical Applications. vol. **18** of *Interdisciplinary Applied Mathematics*, Springer New York. (2003)
10. Sun, G.-Q.: Pattern formation of an epidemic model with diffusion. *Nonlinear Dynam.* **69**(3), pp. 1097–1104(2012)
11. Sun, G.-Q., Jin, Z., Liu, Q.-X., Li, L.: Spatial pattern in an epidemic system with cross-diffusion of the susceptible. *Journal of Biological Systems*. **17**(01), pp. 54–25(2009)
12. Wang, Y., Cao, J., Sun, G.-Q., Li, J.: Effect of time delay on pattern dynamics in a spatial epidemic model. *Phys. A*. **412**, pp. 137–148(2014)

13. Wang,Y., Wang,J., Zhang,L.: Cross diffusion-induced pattern in an SI model. *Appl. Math. Comput.* **217**(5), pp. 1965–1970(2010)
14. Kermack,W.O., McKendrick,A.G.: A contribution to the mathematical theory of epidemics. *Proc. R. Soc. Lond.* **115**(772), pp. 700–721(1927)
15. Capasso,V., Serio,G.: A generalization of the Kermack-McKendrick deterministic epidemic model. *Mathematical Biosciences.* **42**(1–2), pp. 43–61(1978)
16. Abramson,G., Kenkre,V.M.: Spatiotemporal patterns in hantavirus infectionl. *Phys. Rev. E.* **66**(1), pp. 011912 (2002)
17. Abramson,G., Kenkre,V.M., Yates,T.L., Parmenter,R.R.: Traveling waves of infection in the hantavirus epidemics. *Bull. Math. Biol.* **65**(3), pp. 519–534(2003)
18. Lewis,M.A.,Renclawowicz,J.,van den Driessche,P.: Traveling waves and spread rates for a West Nile virus model. *Bull. Math. Biol.* **68**(1), pp. 3–23(2006)
19. Tarboush,A.K., Lin,Z., Zhang,M.: Spreading and vanishing in a West Nile virus model with expanding fronts. *Sci. China Math.* **60**(5), pp. 841–860(2017)
20. Takahashi,L.T., Maidana,N.A., Ferreira,W.C., Pulino,P., Yang,H.M.: Mathematical models for the Aedes aegypti dispersal dynamics: travelling waves by wing and wind, *Bull. Math. Biol.* **67**(3), pp. 509–528(2005)
21. Källén,A.: Thresholds and travelling waves in an epidemic model for rabies. *Nonlinear Anal.* **8**(8), pp. 851–856(1984)
22. Källén,A., Arcuri,P., Murray,J.D.: A simple model for the spatial spread and control of rabie. *J. Theoret. Biol.* **116**(3), pp. 377–393(1985)
23. Lou,Y., Zhao,X.-Q.: The periodic Ross-Macdonald model with diffusion and advection. *Appl. Anal.* **89**(7), pp. 1067–1089(2010)
24. Lou,Y., Zhao,X.-Q.: A reaction-diffusion malaria model with incubation period in the vector population. *J. Math. Biol.* **62**(4), pp. 543–568(2011)
25. Murray,J.D., Stanley,E.A., Brown,D.L.: On the spatial spread of rabies among foxes. *Proc. R. Soc. Lond. Ser. B.* **229**(1255), pp. 111–150(1986)
26. Peng,R.: Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model. Part I, *J. Differential Equations,* **247**(4), pp. 1096–1119(2009)
27. Allen,L.J.S., Bolker,B.M., Lou,Y., Nevai,A.L.: Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model. *Discrete Contin. Dyn. Syst.* **21**(1), pp. 1–20(2008)
28. Deng,K., Wu,Y.: Dynamics of an SIS epidemic reaction-diffusion model. *Proc. Roy. Soc. Edinburgh Sect. A.* **146**(5), pp. 929–946(2016)
29. Smith,D.L.,Lucey,B.,Waller,L.A.,Childs,J.E., Real,L.A.: Predicting the spatial dynamics of rabies epidemics on heterogeneous landscape. *Proc. Natl. Acad. Sci. USA* **99**(6), pp. 3668–3672(2002)
30. Keeling,M.J., Woolhouse,M.E.J., May,R.M., Davies,G., Grenfell,B.T.: Modelling vaccination strategies against foot-and-mouth disease. *Nature* **421**(6919) , pp. 136–142(2003)
31. Hufnagel,L., Brockmann,D., Geisel,T.: Forecast and control of epidemics in a globalized world. *Proc. Natl. Acad. Sci. USA* **101**(42), pp. 15124–15129(2004)
32. Riley,S., Ferguson,N.M.: Smallpox transmission and control: Spatial dynamics in Great Britain. *Proc. Natl. Acad. Sci. USA* **103**(33), pp. 12637–12642(2006)
33. Hanski,I.: Coexistence of competitors in patchy environment. *Ecology* **64**(3), pp. 493–500(1983)
34. Hastings,A.: Spatialheterogeneity and ecological models. *Ecology* **71**(2), pp. 426–428(1990)