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Generating novel multi-scroll chaotic attractors via fractal transformation

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Abstract

The fractal and chaos are bound tightly, and their relevant researches are well-established. Few of them, however, concentrates on the research of the possibility of combining the fractal and the chaotic systems to generate multi-scroll chaotic attractors. This paper presents a novel non-equilibrium point chaotic system, exhibiting extremely rich and complex hidden behaviors including chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability and initial offset-boosting. The proposed system is combined with fractal transformation respectively, and a new class of multi-scroll attractors, such as multi-ring attractors and separated-scroll attractors, is observed. Particularly, swallow-shaped attractors for the first time is found. Moreover, another efficient method to generate a different class of chaotic attractors uses parabola transformation and triangle transformation. Additionally, the spectrum entropy (*SE*) complexity is employed to discuss the complexity of the proposed system before and after fractal, resulting in a chaotic sequences with fractal transformation that has higher complexity. Finally, we develop a hardware platform to implement the presented attractors before and after fractal in a way to confirm the accuracy of the numerical simulations, providing a theoretical basis for the next application in image encryption.

Keywords: hidden attractor; extreme multi-stability; multi-scroll attractors; fractal transformation; separated attractors; MCU implementation.

1. Introduction

From a computational point of view, the attractors in chaotic systems can be divided into self-excited and hidden attractors. A self-excited [1-5] attractor can be defined as the

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intersection of the attractor's basin of attraction with any open neighborhood of the stationary state. Otherwise, it is known as a hidden attractor [6-11]. The hidden attractor cannot be located by applying standard calculation program. Hence, it is difficult to predict their existence in the system. Hidden attractors have played an important role in theoretical problems and practical engineering applications [12], so it is of great significance to investigate hidden attractors. Meanwhile, hidden attractors are often encountered in dynamic systems with line equilibria [13], no equilibrium [14] or stable equilibria [15]. At present, many chaotic systems with hidden attractors have been reported. In 2020, Gong et al. found a novel 4D chaotic system with infinite equilibria using a linear state feedback controller in the Sprott C system [16]. That same year, Deng et al. presented four-wing hidden attractors with one stable equilibrium point [17]. In 2021, Jahanshahi et al. discussed a four-dimensional system with hidden attractors that lacks equilibria [18]. Gu et al. also discovered a novel non-equilibrium memristor-based system with multi-wing attractors and multiple transient transitions [19]. Motivated by research literatures on chaotic systems with hidden attractors, we present a novel non-equilibrium point chaotic system in this paper. Although there is no equilibrium point in the system, the attractive characteristics, such as chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability and initial offset-boosting, can be observed.

Compared with the general chaotic systems, multi-scroll chaotic systems carry much more complexities, which plays a significant role in chaos-based communication security [20], chaotic cryptanalysis [21], and image encryption [22]. In the existing methods for generating multi-scroll chaotic attractors, such as piecewise linear function[23], step function [24], sine function [25], switching manifold [26], time delay saturation sequence[27], these methods of generating multi-scroll chaotic attractors based on function method lead to non-smooth chaotic system. Fortunately, using fractal transformation makes up for this deficiency. Nowadays, the researches on the combination of chaos and fractal to generate multi-scroll attractors are still relatively few, which indicates the worthiness of further investigations. Bouallegue [28] et al. proposed a general multi-scroll and multi-wing attractors, using fractal process with a cascade of Lorenz system in 2011. Also, Bouallegue [29] et al. combined Lorentz attractor and Chua attractor with fractal process respectively, in which a large number of separated attractors were produced. Guo [30] et al. derived a multi-wing spherical chaotic system by fractal map based on Qi-3D four-wing chaotic system. In 2018, Xiao [31] et al. combined Julia fractal with memristor-based chaotic system to generate abundant multi-scroll attractors, providing a new method to construct multi-scroll attractors. In 2021, Yan [32] et al. designed a memristor-based chaotic system, which can be able to generate multi-scroll chaotic attractors. In particular, a large number of multi-scroll attractors with different topologies are produced by combining the system with fractal process. However, the above-mentioned papers only display multi-scroll attractors without considering both the dynamic analysis and hardware implementation of the fractal system. In 2019, Dai [33] et al. proposed a rotational compound chaotic system that undergoes fractal transformation and is based on simplified

Lorenz multi-wing and Chua multi-scroll chaotic systems. It is noteworthy that dynamic analysis and hardware implementation were given for the first time in this paper; however, separated attractors have been neglected. In these foundations, we attempt to combine the non-equilibrium point chaotic system with fractal transformation to generate more complex multi-ring attractors and separated-scroll attractors. In addition, triangular transformation and parabolic transformation are also employed in fractal systems to generate novel attractors.

In summary, the major contributions of this paper are as follows

1. The multi-scroll attractors are produced by merely modifying the proposed system parameters.
2. The complex dynamical phenomena of the proposed system, such as chaos, hyper-chaos, various coexisting multi-scroll chaotic attractors and initial offset boosting, are observed.
3. Combining the hidden attractor with the fractal transformation generates abundant multi-scroll attractors. Then, by choosing different fractal systems, multi-ring attractors and separated attractors are generated. Furthermore, swallow-shaped attractors are also observed for the first time.
4. Based on either parabola transformation or triangle transformation, another new class of attractors is discovered.
5. The system after fractal has higher SE complexity, which becomes more suitable for image encryption and information security.
6. Hardware implementations are also considered in order to prove the effectiveness of numerical simulation and theoretical analysis.

The rest of the paper is organized as follows. In Section 2, a novel four-dimensional non-equilibrium point system with hidden attractors is presented. Moreover, abundant and complex dynamic phenomena, including chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability and initial offset boosting, are observed via phase diagrams, Lyapunov exponent diagrams, bifurcation diagrams, and basin of attraction. In Section 3, abundant multi-scroll attractors are presented based on the fractal transformation. In Section 4, the microcontroller-based experimental implementation of the proposed system before and after fractal transformation is given to verify its physical existence. Finally, the conclusion is elaborated in Section 5.

2. Analytical study

In this section, we present the analytical study of a novel chaotic system without equilibrium. Firstly, the corresponding chaotic system model is proposed and described. Then the multi-scroll chaotic attractors are generated by analyzing the parameters of the system. Finally, the effect of initial state on the system is also discussed in detail.

2.1 Model of the new system without equilibrium

The mathematical model of a novel four-dimensional autonomous system without equilibrium is investigated in this paper. It is described by

$$\begin{cases} \dot{x} = ay + bxz, \\ \dot{y} = -cx + yzw, \\ \dot{z} = d - |x|, \\ \dot{w} = gz, \end{cases} \quad (1)$$

The model consists of four nonlinear terms, four linear terms and an absolute term. In the model presented in Eq. (1), x , y , z and w are the state variables while b , d , g are three positive parameters that are fixed as $b = 15$, $c = 1$, $d = 1$, and $g = 5$. At the same time, parameter a is a variable employed to generate multi-scrolls attractors, which will be analyzed in the next section.

It is important to note that absolute value terms are added to the system (1), owing to the nonlinear features of the absolute term which is a good candidate to design chaotic circuits with special characteristics. From a practical view, an absolute term is easy to be implement by means of electronic components such as resistors, diodes, and operational amplifiers. Additionally, symmetric systems may indicate the coexistence of attractors.

It is easy to verify that the system (1) has non-equilibrium, and the detailed analysis is as follows

$$\begin{aligned} ay + bxz &= 0, \\ -cx + yzw &= 0, \\ d - |x| &= 0, \\ gz &= 0, \end{aligned} \quad (2)$$

by observing the second and the fourth term in Eq. (2), it's obvious that

$$z = 0, x = 0 \quad (3)$$

and from the third term in Eq. (2), we have

$$x = \pm 1 \quad (4)$$

Obviously, Eq. (3) is inconsistent with Eq. (4). Thus, there is no equilibrium in system (1). In other words, there are hidden attractors in the system (1).

Furthermore, it is observed that the system (1) is invariant when coordinates are transformed as follows

$$P: R^4 \rightarrow R^4, X \rightarrow PX, P = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5)$$

Eq. (5) indicates that the dynamical system (1) has symmetry on the z - w plane. Therefore, any projection of the attractor has rotational symmetry on the z - w plane, and there may be coexisting attractors in the system (1).

2.2 Hidden multi-scroll chaotic attractors

This section focuses on chaotic attractors with changeable numbers of scrolls. The flexibility of scroll numbers implies that the system parameters need to be adjusted for generating multi-scroll chaotic attractors [34]. Firstly, the bifurcation diagram and the Lyapunov spectrum are employed to investigate the abundant dynamic behaviors of the system (1) by keeping parameter a varying. The bifurcation diagram of state variable x is displayed in Fig. 1 (a) with the initial state (0.1, 0.1, 0.1, 0.1), which corresponds to Lyapunov exponent spectrum in Fig. 1 (b). According to Lyapunov exponent spectrum, the system(1) always keeps chaotic state in the range of $a \in [0, 1]$. Moreover, it is worth-noting that there are mainly two positive Lyapunov exponents in the interval $a \in [0.131, 0.624) \cup (0.624, 1]$, which indicate that the system (1) is hyper-chaotic. Generally speaking, the main change of the system (1) is that the transformation between chaos and hyper-chaos depends on the variable a . Next, typical parameter a is able to generate double-scroll chaotic attractor and hyper-chaotic attractor with $a=0.09$ (LE1=0.0993, LE2=-0.0011, LE3=-0.0144, LE4=-0.0538) and $a=0.9$ (LE1=0.1376, LE2=0.0187, LE3=-0.0093, LE4=-0.1285), which is illustrated in Figs. 2(a)-2(b). Interestingly, the hyper-chaotic system (1) can not only generate double-scroll chaotic attractors but also produces multi-scroll attractors. After the parameter a is adjusted, four typical multi-scroll chaotic attractors can be obtained as shown in Figs. 3(a)-3(d). Furthermore, the corresponding Poincare sections when $z=0$ are shown in Fig.4 which illustrates the existence of multi-scroll chaotic attractors. In addition, Poincare mapping also proves that the system (1) has rich and complex dynamic behaviors due to the existence of dense points with fractal structure.

(a)

(b)

Fig.1. The dynamical behavior of the system along with parameter a varying. (a) the bifurcation diagram, (b) the Lyapunov exponents obtained by system (1).

Fig.2. Chaotic and hyper-chaotic orbits on x - z plane. (a) chaotic ($a=0.2$) with initial conditions (0.1, 0.1, 0.1, 0.1), (b) hyper-chaotic ($a=0.9$) with initial conditions (0.1, 0.1, 0.1, 0.1).

Fig.3. Multi-scroll chaotic attractors are generated by parameter a varying. (a) $a=0.02$, (b) $a=0.03$, (c) $a=0.05$, (d) $a=0.012$.

Fig.4. Poincare maps of the system (1) when $z=0$. (a) $a=0.02$, (b) $a=0.03$, (c) $c=0.05$, (d) $c=0.0112$.

2.3 Hidden coexisting attractors

Multi-stability [16-18] means the coexistence of multiple attractors, which is an inherent characteristic of many nonlinear dynamical systems. Concurrently, multi-stability can be applied in image processing or as a random source for information engineering applications. Furthermore, multi-stability sometimes leads to difficulties in engineering application; however, the ability to improve the system state by changing the initial conditions provides great flexibility. Additionally, when the number of coexisting attractors produced via a nonlinear dynamical system approaches infinity, this phenomenon is called extreme multi-stability. From Eq.(2), it has been proved that there is no equilibrium point in the system (1), that is, the attractors generated from the system (1) are all hidden. At present, hidden coexisting multiple chaotic attractors has been observed in many nonlinear circuits; however, hidden multiple coexisting multi-scroll chaotic attractors are rare. In order to analyze the extreme multi-stability of the system (1), the parameters are configured as $a = 0.2$ and $c=1$, and the initial conditions are considered as $(0.1, y(0), 0.1, 0.1)$. The Lyapunov exponent and bifurcation diagram are then used to analyze the different motion states of the system (1) along with the varying initial condition $y(0)$, which is shown in Fig. 5 (a) and Fig. 5 (b) respectively. When the Lyapunov exponent spectrum concerning the initial state $y(0)$ is observed, it is apparent that a positive Lyapunov exponent or two positive Lyapunov exponents alternately appear in the interval $[-1, 1]$,

which implies the existence of the hidden chaotic attractors or hidden hyper-chaotic attractors in the system (1). When the initial value $y(0)$ varies continuously, the system (1) will produce many hidden coexisting multi-scroll attractors. To avoid drawing similar graphs, we choose several typical initial values to represent the coexistence of multiple hidden multi-scroll attractors, which are illustrated in Figs. 6(a). Additionally, the dynamical behaviors are summarized in Table 1.

It is known that the basin of attraction [35] is a convenient tool to discuss coexisting attractors and hidden attractors, and the basin of attraction is defined as a set of points where the trajectory of the system converges to the corresponding attractor at the corresponding set of initial conditions. Therefore, the basins of attraction in $x(0) - y(0)$ initial plane with $z(0) = w(0) = 0.1$ is displayed in Fig. 6(b), where the light blue represents the hidden two-scroll attractor that corresponds to $y(0) = 0.1$. Moreover, the light blue represents the three-scroll hidden attractor at $y(0) = -0.9$, and the red and the light yellow regions of the basin represent the different multi-scroll hidden attractors corresponding to $y(0) = 0.43$ and $y(0) = -0.5$ respectively. The attraction regions of different colors demonstrate different initial states $y(0)$, exhibiting completely different topologies. From this attraction basin, the probability to obtain the double-scroll attractor is greater than the probability to obtain the other attractors since the set of initial conditions leading to the double-scroll attractor is rather large than the others. Based on the former analyses, the proposed system (1) has significant initial value sensitivity, which reveals complex extreme multi-stability phenomena.

Fig.5. The dynamic behavior corresponds to the initial value $y(0)$. (a) Lyapunov exponent spectrum with respect to $y(0)$, (b) bifurcation diagram with respect to $y(0)$.

Fig.6. Coexistence of hidden multiple-scroll attractors with different initial conditions and Corresponding basin of attraction in the $x(0)$ - $y(0)$ plane. (a) coexistence of hidden multiple-scroll attractors, (b) basin of attraction in the $x(0)$ - $y(0)$ plane, where $a = 0.2$, $b = 15$, $c=1$, $d = 1$, and $g = 5$, $z(0) = w(0) = 0.1$.

Table 1. The typical attractor types in differently colored regions for $a=2$

Colored	$y(0)$	LEs	Dynamics	Colored	$y(0)$	LEs	Dynamics
0.1	(0.035,0.020,-0.002,-0.029)	Hyper-chaos		0.43	(0.062,-0.004,-0.006,-0.020)	chaos
-0.9	(0.032,0.011,-0.013,-0.021)	Hyper-chaos		-0.5	(0.061,0.002,0,-0.006)	Hyper-chaos

Additionally, in the process of investigating the system (1)'s extreme multi-stability, an intriguing phenomenon – known as initial offset-boosting behavior ³⁶⁻³⁷ – has been observed. From the initial value $w(0) = 0, 0.5, 1, 1.5, 2$ and 2.4 , the phase diagrams of the six hidden chaotic attractors in different planes are shown in Figs.7 (a) - 7 (b).

Fig.7. Initial offset boosting behavior with different initial value $x(0)$. (a) coexisting six hidden chaotic attractors in x - w plane, (b) coexisting six hidden chaotic attractors in x - z - w plane.

3. Multi-scroll attractors generated by fractal

In this section, since the seed system exhibits a wealth of nonlinear dynamic behaviors such as chaos, hyper-chaos, the hidden extreme multi-stability, and the initial offset-boosting, it is extremely sensitive to system parameters and initial values. Therefore, multi-ring attractors, after fractal with different topologies, will be observed by simply modifying the parameters and initial values of the system (1). Thus, we first introduce the specific fractal transformation algorithm while regarding system (1) as the seed system. Then, by designing different fractal systems, we are able to generate a variety of multi-scroll attractors including multi-ring attractors, separated-scroll attractors. Meanwhile, not only do we observe swallow-shaped attractors for the first time, but we also find another form of attractors based on fractal transformation, the seed system, parabola transformation and triangle transformation. Finally, the SE complexity of the seed system before and after fractal is compared. The experiment proves that the multi-scroll chaotic attractors based on Julia fractal are not only better at adjusting the number of scrolls, but also makes up for the lack of smoothness of chaotic system caused by the function method.

3.1 The fractal algorithm

Boualallegue et al. [28] proposed the fractal algorithm to obtain abundant multi-scroll chaotic attractors where the number of scrolls is power of 2, so the proposed algorithm can easily generates 2^n scrolls chaotic attractors. The description of the algorithm, however, is extremely difficult to analyze. Thus, we derive and improve the algorithm based on Julia iterative mapping which is defined as a quadratic complex Z mapping, whose the expression is as follows

$$Z_{n+1} = Z_n^2 + Z_c \quad (6)$$

Let

$$\begin{cases} Z_{n+1} = x_{n+1} + iy_{n+1} \\ Z_n = x_n + iy_n \\ Z_c = 0 \end{cases} \quad (7)$$

Taking equation (7) into Eq. (6), we can obtain

$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 \\ y_{n+1} = 2x_n y_n \end{cases}, \quad (8)$$

and then the subscripts of variables in Eq. (8) are interchanged to obtain

$$\begin{cases} x_n = x_{n+1}^2 - y_{n+1}^2 \\ y_n = 2x_{n+1}y_{n+1} \end{cases} \quad (9)$$

Thus x_{n+1} and y_{n+1} can be represented by x_n and y_n as

$$\begin{cases} x_{n+1} = \pm \sqrt{\frac{\sqrt{x_n^2 + y_n^2} + x_n}{2}} \\ y_{n+1} = \frac{y_n}{2x_{n+1}} \end{cases} \quad (10)$$

Where x_n and y_n are seen as the system variables. The block diagram about fractal transformation is designed by means of evolving Eq. (10) into an algorithm, which is illustrated in Fig.8. The seed system is the input sequences, x_n and y_n while u_n and v_n are regarded as output sequences after the fractal transformation. The input sequences x_n and y_n can be obtained by the Runge-Kutta method to solve the non-equilibrium numerically in the seed system. Next, the algorithm calculates the output sequences u_n and v_n through different functions according to numerical value of the variable x_n . This process is called a fractal transformation defined as PJ . Finally, the output sequences u_n and v_n are viewed as the next input sequences x_n and y_n , which implies that in theory the fractal transformation can loop infinitely. Therefore, the fractal transformation can be written as

$$(u_n, v_n) = PJ(x_n, y_n) \quad (11)$$

Fig. 8. The principle block diagram of the fractal algorithm.

3.2 Multi-ring chaotic attractors

This subsection introduces a multi-ring chaotic attractors generated by investigating the combination of fractal transformation with the seed system. The seed system exhibits a wealth of nonlinear dynamic behaviors including chaos, hyper-chaos, the hidden multi-

stability, and the initial offset boosting, which means that the seed system is very sensitive to parameters and initial values. Therefore, multi-ring attractors, after fractal with different topologies, will be observed by simply modifying the parameters and initial values of the seed system. All above-mentioned advantages due to the combination of the seed system and the fractal transformation are of great significance for a variety of practical applications. In this part, applying the fractal algorithm derived above that generates more complex multi-scroll attractors is introduced. Firstly, the fractal system is represented by

$$\begin{aligned} (u_1, v_1) &= PJ(x_1, y_1) \\ (u_2, v_2) &= PJ(u_1, v_1) \\ (u_3, v_3) &= PJ(u_2, v_2) \end{aligned} \quad , \quad (12)$$

where u_1 and v_1 represent the sequences with the one-time fractal transformation, and x_1 and y_1 represent the input sequences by solving the seed system. Next, the output series u_1 and v_1 of the one-time fractal transformation are viewed as input sequences u_2 and v_2 of the twice fractal transformation; The Eq. (12) is a fractal system with three-time fractal transformation. Figs. 9(a)-9(c) respectively illustrate the multi-scroll chaotic attractors after once, twice, three-time fractal transformation when the seed system parameters are at $a=0.005$, $c=1$, and the initial conditions $(0.1, 0.1, 0.1, 0.1)$. Similarly, when the parameters of the seed system with initial states $(0.1, -0.1, -0.1, -0.1)$ are configured as $a=0.1$ and $c=0.1$, the multi-scroll chaotic attractors with different topologies can also be observed after all three-times fractal transformation of the fractal system (12), which are shown in Figs.10(a)-10(c). The relationship between the number of scrolls and the fractal process can be easily found from Fig. 9 and Fig. 10. Suppose the initial number of scrolls are C_1 , and the number of fractal process is n , then the number of scrolls after fractal process is $C_2 = C_1 \times 2^n$. Using this approach, it is easy to generate multi-scroll chaotic attractors.

Fig.9. Chaotic system (1) with fractal transformation for $a=0.005$, $c=1$. (a) one-time fractal transformation, (b) twice fractal transformation, (c) three-times fractal transformation.

Fig.10. Chaotic system (1) with fractal transformation for $a=0.1$, $c=0.1$. (a) one-time transformation, (b) twice fractal transformation, (c) three-times fractal transformation.

In addition, different complex multi-scroll chaotic attractors can be obtained by adding nonlinear functions into fractal system (12). A simple example is designed as follows

$$\begin{cases} (u_1, v_1) = PJ(\arctan(x_1), \arctan(y_1)) \\ (u_2, v_2) = PJ(x_1, y_1) \\ (u_3, v_3) = PJ(v_1 + v_2, u_1 - u_2) \end{cases} \quad (13)$$

Firstly, the output sequences u_1 and v_1 can be calculated by taking the input sequences x_1 and y_1 as the variables of the arctangent function in the first fractal transformation. Then the output sequences u_2 and v_2 , in the second fractal transformation, are obtained by the input sequences x_1 and y_1 . Finally, the third fractal transformation combines the behavior of the first two fractal transformation. The change of the seed system's parameters increase the shape controllability where it is possible to employ different parameters a and c for the different shapes while changes obtained from different chaotic behaviors are shown in Figs.11(a)-11(b).

Fig.11. The multi-scroll chaotic attractor with different parameters for initial conditions (0.1, 0.1, 0.1, 0.1): (a) $a = 0.005$, $c=2$, (b) $a=0.1$, $c=0.5$.

To explore the influence of fractal system parameters on fractal system, another fractal system (14) is designed and considered with parameters of the seed system designated as $a=0.005$, $c=0.1$. Typical parameters in a fractal system (14) yield different topologies

chaotic attractors as the following: $k=1.6$, $g=-1.7$ and $k=2$, $g=-1.5$, and those two sets are depicted in Figs. 12(a)-12(b) respectively. It is shown that only through adjusting the parameters in the fractal system (14), the dynamic behavior of the attractor can be changed significantly.

$$\begin{cases} (u_1, v_1) = PJ(x + \arctan(x), y + \arctan(y)) \\ (u_2, v_2) = PJ(x, y) \\ (u_3, v_3) = PJ(ku_1 + gv_2, kv_1 + gu_2) \end{cases} \quad (14)$$

Fig.12. The multi-scroll chaotic attractor with different fractal system parameters. (a) $k=1.6$, $g=-1.7$,
(d) $k=2$, $g=-1.5$.

3.3 Chaotic attractors with scrolls separated

It should be pointed out that chaotic attractors have many practical applications in random number generators, image encryptions, and communication security. In order to increase the complexity of chaotic dynamics and improve the security of the system, the chaotic attractors with simple topologies can be replaced by several separated chaotic attractors. In addition, this section will introduce the formation process of swallow-shaped attractors in detail.

First of all, a fractal system which can produce multiple separated chaotic attractors is designed as follows

$$\begin{cases} (u_1, v_1) = PJ(x + \arctan(x), y + \arctan(y)) \\ (u_2, v_2) = PJ(x + k, y - k) \\ (u_3, v_3) = PJ(u_2, v_2) \end{cases}, \quad (15)$$

where k is separation factor, with the increase of separation factor k , the chaotic attractors will be separated. For the sake of clarity, we can divide the chaotic attractor into three stages: the initial state, the intact state before reaching the threshold, and the separation state. Fig. 13 gives a detailed analysis of the separation process for parameters $a=0.2$, $c=0.02$. The chaotic attractor is at initial state with $k = 0$, then attractor begins to separate with the increase of k . When $k=0.5$, separation distance reaches the threshold where the chaotic

attractor is about to be completely separated. Finally, the attractors are completely separated with $k=1$, and four swallow-shaped attractors can be observed as shown in Fig.13(c). To further verify the effectiveness of fractal system (15), the simulation results are given for parameters $a=0.02$, $c=1$, which are illustrated in Figs.14 (a)-14(c). In summary, we can draw the conclusion that the formation of separable attractors follows the rule that the higher the separation distance is in the fractal system, the farther apart each attractor is.

Fig.13. Different states in the separation process: (a) initial state ($k=0$), (b) intact state before reaching the threshold ($k=0.5$), (c) separation state ($k=1$).

Fig.14. Different states in the separation process. (a) initial state ($k=0$), (b) intact state before reaching the threshold ($k=2$), (c) separation state ($k=3$).

3.4 A different class of chaotic attractors

In order to obtain more complex and abundant chaotic attractors, two approaches-parabola transformation and triangle transformation [38] -are described in this subsection, which produces chaotic attractors different from section 3.2. The new fractal system is designed as follows

$$\begin{cases} (u_1, v_1) = PJ(\arctan(x_1), \arctan(y_1)) \\ (u_2, v_2) = PJ(2x_1, 2y_1) \\ (u_3, v_3) = T(2v_1, 2u_2) \end{cases} \quad (16)$$

where T represents the triangle transformation or the parabolic transformation, and the triangle transformation expression is the following

$$\begin{aligned}\dot{x} &= 6x - 6(x^2 - y^2) \\ \dot{y} &= 6y - 12xy\end{aligned}\quad (17)$$

By setting the parameters of the seed system as $a = 0.005$, $c = 0.1$, Fig. 15(a) shows the result of the chaotic attractor after triangle transformation. Similarly, parabolic transformation is defined as

$$\begin{aligned}\dot{x} &= 6x(1-x) \\ \dot{y} &= 6y\end{aligned}\quad (18)$$

The attractor after parabolic transformation can be obtained as shown in Fig. 15 (b).

Fig.15. Parabolic attractor and triangular attractor: (a) chaotic attractor by triangular transformation, (b) chaotic attractor by parabolic transformation.

By employing different fractal systems and different seed system's parameters, another form of attractors can be obtained based on parabolic transformation and triangular transformation. Here, the parameters of the seed system are configured as $a=0.1$, $c=0.1$. The phase diagrams are depicted in Figs. 16 (a) -16 (b).

$$\begin{cases} (u_1, v_1) = PJ(x_1 + \arctan(x_1) - 1, y_1 + \arctan(y_1) - 1) \\ (u_2, v_2) = PJ(2x_1, 2y_1) \\ (u_3, v_3) = T(2v_2, 2u_1) \end{cases}\quad (19)$$

Fig.16. Parabolic attractor and triangular attractor. (a) chaotic attractor based on triangular transformation, (b) chaotic attractor based on parabolic transformation.

3.5 Complexity analysis

The goal of this subsection is to discuss the complexity of the seed system sequences before and after one-time fractal transformation mentioned above. Generally known, the higher the chaos degree of the chaotic sequence is, the greater the corresponding complexity holds. The complexity of the algorithm plays an essential role in the dynamic analysis of nonlinear systems. At present, there are several methods to measure the complexity, such as spectrum entropy (SE) [39], C_0 entropy [40], and permutation entropy (PE) [41]. Out of the three, SE algorithm is used to calculate the complexity of the chaotic sequence because it has the advantages of simplicity, robustness, and efficiency and without any coarse-grained preprocessing. Therefore, the complexity of the seed system with once fractal transformation is analyzed by spectrum entropy (SE) algorithm.

To make better comparison, the seed system parameter $c = 1$ is fixed. Figs. 17(a)-17(b) respectively illustrates the SE complexity spectrum of the seed system before and after fractal transformation with respect to the control parameter a and initial state (0.1, 0.1, 0.1). In Figs. 17(a)-17(b), it is clearly observed that the complexity of the seed system after one-time fractal transformation increases from approximately 0.55 to 0.95. At the same time, in order to compare the influence of the complexity of the initial value seed system before and after fractal transformation, the spectrums of SE concerning the initial value $y(0)$ and parameters $a=0.005$, $c=1$ are drawn accordingly, and the results are illustrated in Figs. 17(c)-17(d). From Figs. 17(c) - 17(d), we also observe that after once fractal transformation, the complexity of initial value $y(0)$ increases from roughly 0.6 to 0.955. The results display that the complexity of the seed system can be greatly improved by one-time fractal transformation, therefore, the seed system after fractal transformation is more compatible for image encryption and information security.

Fig.17. the complexity of initial value $y(0)$ and parameter a before and after one-time fractal transformation. (a) the complexity of parameter a before one-time fractal transformation, (b) the complexity of parameter a after one-time fractal transformation, (c) the complexity of initial value $y(0)$ before one-time fractal transformation, (d) the complexity of initial value $y(0)$ after one-time fractal transformation.

4. MCU implementation

Compared with analog hardware circuit [42], the system built upon digital hardware circuit has the advantages of strong control and good stability, which gives the possibility of direct application to the field of digital-voice communication security [43]. For these reasons, a programmable hardware platform based on STM32 is designed to implement different multi-scroll chaotic attractors of the seed system before and after fractal transformation. The flow diagram of MCU implementation is depicted in Fig.18. This hardware platform includes a microcontroller named STM32F407ZGT6 and a 16-bit two-channel digital-to-analog converters AD5689. The calculations are carried out on the MCU platform. Through D/A converter, the digital signals are converted into analog signals which are then input into oscilloscope, and the phase diagrams are recorded via the oscilloscope.

The first step is to consider the seed system before fractal. Firstly, Euler algorithm is employed to discretize the seed system (1) proposed in this paper. The expression is given

$$\begin{cases} x(n+1) = x(n) + h(ay(n) + bx(n)z(n)) \\ y(n+1) = y(n) + h(-cx(n) + y(n)z(n)w(n)) \\ z(n+1) = z(n) + h(d - x(n)) \\ w(n+1) = w(n) + h(gz(n)) \end{cases} \quad (23)$$

In the Eq. (23), the discretization step h is set to 0.0001 and the selection of initial conditions is (0.1, 0.1, 0.1, 0.1). Then, the discrete solutions can be obtained by Eq. (23). Next the solutions are further sent to D/A converter for time series conversion. Finally, the phase orbit diagram can be clearly captured on the digital storage oscilloscope (RIGOL

DS2102A). Specially, Figs. 21(a)-21(b) display two multi-scroll hidden attractors with respect to the parameters $(a, c) = (0.02, 1)$ and $(a, c) = (0.05, 1)$, corresponding to Fig. 3(a) and Fig. 3 (c) respectively.

Fig.18. Flow diagram of the MCU implementation.

The second step is to consider the seed system (1) after fractal. Here, we take fractal system (13) as an example. Schematic diagram of the fractal system (13) which has undergone three fractal processes is shown in Fig.19. The schematic diagrams of other fractal systems are similar to Fig.19. Thus, employing the hardware experimental platform, the above-mentioned typical multi-scroll attractors after fractal can be captured.

Fig.19. Schematic diagram of the fractal system (13)

A screen capture of the hardware platform that generates the ring-scroll chaotic attractor and undergoes three - times fractal transformation of the seed system (1) is shown in Fig. 20.

Fig.20. Screenshot of the microcontroller-based hardware experiment platform and captured multi-scroll chaotic attractor.

Concurrently, the multi-scroll attractors with once fractal transformation and three-time fractal transformation for the parameters $(a, c) = (0.005, 1)$ are exhibited in Figs. 21(c)-21(d), corresponding to Fig. 9 (a) and Fig. 9 (c) respectively. When $(a, c) = (0.2, 0.02)$ and $(a, c) = (0.02, 1)$, separated-scroll attractors for $k=1$ and $k=3$ in the fractal system (15) are

depicted in Figs. 21(e)-22(f). Similarly, Figs. 21(g)-21(h) correspond to Figs. 15(a)-15(b) respectively. It is noticed that the results of MCU hardware implementation are generally consistent with those of numerical simulations, proving the possibility of physical implementation of the multi-scroll attractors.

Fig.21. chaotic attractors of the seed system before and after fractal process are captured from hardware platform. (a) multi-scroll attractors of system (1) for $a=0.02$ and $c=1$, (b) multi-scroll attractors of system (1) for $a=0.05$ and $c=1$, (c) once fractal transformation of fractal system (12) for $a=0.005$ and $c=1$, (d) three-times fractal transformation of fractal system (12) for $a=0.005$ and $c=1$, (e) fractal system (15) for $k=1$ with $a=0.2$ and $c=0.02$, (f) fractal system (15) for $k=3$ with $a=0.02$ and $c=1$, (g) triangle

transformation with $a=0.005$ and $c=0.1$, (h) parabolic transformation with $a=0.005$ and $c=0.1$.

5. Conclusion

In this paper, a novel non-equilibrium point system with hidden attractors has been reported and analyzed, which displays rich hidden dynamical properties such as chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability and initial offset boosting. The complex hidden dynamical characteristics of the system are investigated by means of Lyapunov exponents, bifurcation diagrams, phase diagrams, and basin of attraction. The next step combines the fractal transformation with the seed chaotic system, so a large amount of multi-ring chaotic attractors and separated attractors are generated for different seed system parameters. Then, another new class of attractors are also discovered based on parabola transformation and triangle transformation. Additionally, spectrum entropy (SE) complexity is employed to discuss the complexity of the system before and after fractal, and the result indicates that the chaotic sequences with fractal transformation have higher complexity. As a result of the developed algorithm and fractal system's abilities to generate various multi-scroll chaotic attractors, it allows us to choose a number of attractors for image encryptions and communication security. Finally, the MCU experiments show a coherence with Matlab simulation. By modifying the fractal system, more complex chaotic attractors, such as nested chaotic attractors, may be observed, which are potential areas for the future study.

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Data Availability Statements

The data that support the findings of this study are available on request from the corresponding author.

Compliance with ethical standards

Conflict of interest we declare that we have no conflict of interest. All authors contribute equally to this work

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