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Optimal robust control with cooperative game theory for lower limb exoskeleton robot

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Abstract

For achieving trajectory tracking issue of the lower limb exoskeleton robot, a novel optimal robust control with cooperative game theory is proposed. The uncertainties are considered (possible time-varying, bounded and fast) and the fuzzy set theory is creatively adopted to describe the boundary. From the view of analytical mechanics, the trajectory tracking is treated as the constraints control problem, including holonomic and nonholonomic constraints. A robust control is designed with two adjustable parameters to guarantee the uniform boundedness and uniform ultimate boundedness. For obtaining the optimal selection of two adjustable parameters, a novel optimal control employed cooperative game theory is proposed. Combining the robust control and optimal design, optimal robust control is formulated. The Pareto optimal solution is obtained to guarantee the minimum control cost. In the simulation, the adaptive robust control of Sun is chosen as a comparison. The existence of Pareto optimality and the effectiveness of optimal robust control have been verified via simulation results.

Keywords

Lower limb, robust control, fuzzy set theory, cooperative game theory, Pareto optimality.

1. Introduction

The aging problem can no longer be ignored, which has brought a serious of social issues. The strokes have struck numerous elder people, which is a predominant source of lower limb paralysis [1]. Medical theory and clinical medicine have proved that rehabilitation training process significantly effect on relieving the symptoms [2]. However, traditional rehabilitation training wastes lots of manpower and resources, and the training effect is poorer than that of lower limb exoskeleton robot (LLER) [3]. Depending on the degree of paralysis, the motion modes of LLER are primarily classified as active and passive mode [4]. In the passive mode, the LLER need to following a certain desired trajectory, which is usually derived from clinical gait database. Passive training is essential for further rehabilitation, and the trajectory tracking issue plays an indispensable role, which is involved in this article.

In recently years, lots of excellent efforts have been

carried out, such as robust control [5], hierarchical control [6], adaptive control [7], fault-tolerant control [8] and neural network control [9] and so on. Campbell et al. [10] develop the assistance-as-needed control to help the patient with rehabilitation which means the robot will provide assistance only when the patient deviates from the desired gait. In order to overcome the unstable phenomenon, Sun et al. [11] construct a reduced adaptive fuzzy control with compensation and verify its effective on LLER system. Yang et al. [12] describe a repetitive learning control of LLER in the frame of backstepping control and demonstrate its convergence using Lyapunov method. An active disturbance rejection control is proposed to compensate for the disturbance and converge quickly with the help of fast terminal sliding mode control [13]. However, these control methods mentioned above are not available for nonholonomic constraints, which are in common in reality. The optimal robust control (ORC) we proposed is applicable not only to holonomic, but also to nonholonomic constraints.

Professors Udwadia and Kalaba proposed a novel equation of motion from the view of analytical mechanics in 1992, called Udwadia-Kalaba (U-K) theory [14,15]. U-K theory overcomes the shortcoming of unavailable to nonholonomic constraints. In addition, Lagrange multipliers and puasi variables are not required in U-K theory, which bring about little computation [16]. Liu et al. [17] propose a systematic method adopted second-order constraints in the frame of U-K theory. Sun et al. [18] apply the U-K theory to solve tracking control problem of mobile robots. Sun et al. [19] design an adaptive robust control using U-K theory to carry out the dual avoidance-arrival problem. However, the uncertainties are not considered in U-K theory, which is a limitation on the way of development.

In fact, the uncertainties cannot be negligible in trajectory tracking control. The description and management of uncertainty is similarly important, and many control methods have done related work, such as torque control [20], self-adjusting leakage type adaptive control [21], robust bounded control with inequality constraints [22], terminal sliding mode control [23]. Unfortunately, the bounds on the uncertainties involved in the control methods presented above are all deterministic and constant, which is not reasonable in reality. So, a better method to describe and manage the uncertainties is necessary.

Probability theory has captured the favour of many scholars on describing the uncertainties by adopting lots of observed data, and numerous achievements has obtained [24]. Stochastic dynamic system method combined the probability theory and dynamic model [25] has a well performance in tracking, which also needs lots of observed data. However, not all data are easily available for observation or cannot be accurately repeatable in large amount, such as flood data. In 1965, Zadeh [26] firstly proposed the fuzzy set theory, which has the ability to describe the degree rather than frequency. Zhen et al. [27-29] employ fuzzy set theory to describe the uncertainties in the dynamic model and call it fuzzy dynamic system (FDS), which is not based on IF-THEN rules. In order to avoid the shortcomings of lots of probability theory, the uncertainties in LLER are described by fuzzy set theory. Then, based on U-K theory and FDS, a novel robust control with two adjustable parameters is proposed. Different parameters can cause different system's control cost. How to select an optimal pair of two adjustable parameters is a question worth considering.

Remarkable studies have been conducted about optimal control of multi-parameters in recent years, such as ant colony [30], particle swarm optimization [31] and genetic algorithm [32]. Unfortunately, the realization on FDS is difficult and inconvenient. [33-35] have researched the optimization of FDS. But their research object is single objective optimization. Referring to previous work of Chen et al. [36,37], we employ the cooperative game theory to solve the problem of optimal parameter selection, which was introduced by Pareto. The cooperative game theory involves two players, whose goal is to find the Pareto optimal solution, in order to minimize both of their cost. More important, for a better interpretation about our work, someone should notice

that the FDS is quite different with the fuzzy logic control, the latter is based on IF-THEN rules.

In a conclusion, our main contributions are listed as follows:

(1) The fuzzy set theory is employed to describe the boundary of uncertainties in LLER, which obtain the degree of occurrence not frequency.

(2) The trajectory tracking problem is treated as constraints control problem, including holonomic and nonholonomic constraints. A robust control method is designed with two adjustable parameters to guarantee the uniform boundedness and uniform ultimate boundedness based on U-K theory.

(3) The optimal design with cooperative game theory is achieved to solve the selection of optimal parameters. The Pareto optimal solution obtained minimizes the system's control cost.

The rest of this article is arranged as following. Section 2 introduces the preliminary knowledge about fuzzy set theory, U-K theory and cooperative game theory. Robust control with two adjustable parameters is designed in Section 3. Optimal control with cooperative game theory is constructed in Section 4. Section 5 conducts the simulation experiments to verify the existence of Pareto optimality and the effectiveness of optimal robust control. Section 6 is a conclusion of the whole paper.

2. Preliminary knowledge

2.1 Fuzzy set theory

Some basic knowledge about fuzzy set theory needs to be reviewed for further development.

Fuzzy number: Let $\mathfrak{S}$ be a fuzzy set in $\mathbb{R}$, the real number. $\mathfrak{S}$ is named a fuzzy number if: (1) $\mathfrak{S}$ is normal and convex, (2) the support of $\mathfrak{S}$ is bounded and (3) all α -cuts are closed intervals in $\mathbb{R}$. Moreover, assume the universe of discourse of a fuzzy number to be its 0-cut.

Fuzzy arithmetic: Let $\mathfrak{S}$ and $\wp$ be two fuzzy numbers and $\mathfrak{S}_\alpha = [d_\alpha^-, d_\alpha^+]$, $\wp_\alpha = [g_\alpha^-, g_\alpha^+]$ be their α -cuts, $\alpha \in [0, 1]$. Moreover, the four fundamental rules of $\mathfrak{S}$ and $\wp$ are list as shown

$$\begin{aligned} (\mathfrak{S} + \wp)_\alpha &= [d_\alpha^- + g_\alpha^-, d_\alpha^+ + g_\alpha^+], \\ (\mathfrak{S} - \wp)_\alpha &= [\min(d_\alpha^- - g_\alpha^-, d_\alpha^+ - g_\alpha^+), \\ &\quad \max(d_\alpha^- - g_\alpha^-, d_\alpha^+ - g_\alpha^+)], \end{aligned} \quad (2)$$

$$\begin{aligned} (\mathfrak{S} \cdot \wp)_\alpha &= [\min(d_\alpha^- g_\alpha^-, d_\alpha^- g_\alpha^+, d_\alpha^+ g_\alpha^-, d_\alpha^+ g_\alpha^+), \\ &\quad \max(d_\alpha^- g_\alpha^-, d_\alpha^- g_\alpha^+, d_\alpha^+ g_\alpha^-, d_\alpha^+ g_\alpha^+)], \end{aligned} \quad (3)$$

$$\begin{aligned} (\mathfrak{S}/\wp)_\alpha &= [\min(d_\alpha^- g_\alpha^-, d_\alpha^- g_\alpha^+, d_\alpha^+ g_\alpha^-, d_\alpha^+ g_\alpha^+), \\ &\quad \max(d_\alpha^- g_\alpha^-, d_\alpha^- g_\alpha^+, d_\alpha^+ g_\alpha^-, d_\alpha^+ g_\alpha^+)]. \end{aligned} \quad (4)$$

Decomposition theorem: Define a fuzzy set $\tilde{H}_\alpha$ in U

whose membership function is $\mu_{\tilde{H}_\alpha} = \alpha I_{\tilde{H}_\alpha(x)}$, where $I_{\tilde{H}_\alpha(x)} = 1$ if $x \in \tilde{H}_\alpha$ and $I_{\tilde{H}_\alpha(x)} = 0$ if $x \in U - \tilde{H}_\alpha$. Then the fuzzy set H is obtained as

$$H = \bigcup_{\alpha \in [0,1]} \tilde{H}_\alpha \quad (5)$$

where $\bigcup$ is the union of the fuzzy sets (that is, sup over $\alpha \in [0,1]$). On the ground of decomposition theorem, we can formulate the membership function of the resulting fuzzy number with the operation between the two fuzzy numbers.

D-Operation [38]: Assume a fuzzy set

$$N = \{(\epsilon, \mu_N(\epsilon)) | \epsilon \in N\}. \quad (6)$$

For any function $f : N \rightarrow R$, the D-operation $D[f(v)]$ is formulated by

$$D[f(\epsilon)] = \frac{\int_N f(\epsilon) \mu_N(\epsilon) d\epsilon}{\int_N \mu_N(\epsilon) d\epsilon}. \quad (7)$$

To some extent, the D-operation $D[f(\epsilon)]$ takes an average value of $f(\epsilon)$ over $\mu_N(\epsilon)$. In the special case that $f(\epsilon) = \epsilon$, this is simplified to the commonly known center-of-gravity defuzzification method [38]. If N is crisp [i.e., $\mu_N(\epsilon) = 1$] for all $v \in N$, then $D[f(\epsilon)] = f(\epsilon)$.

Lemma 1: For any crisp constant $a \in \mathbb{R}$,

$$D[af(\epsilon)] = aD[f(\epsilon)]. \quad (8)$$

2.2 Udwadia-Kalaba theory

Establish the dynamic model of a mechanical system using Newtonian or Lagrangian method, which can be formulated as

$$\mathbf{M}(\mathbf{x}(t), \boldsymbol{\delta}(t), t) \ddot{\mathbf{x}} = \mathbf{Q}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\delta}(t), t) + \mathbf{B}\boldsymbol{\tau}(t) \quad (9)$$

where $t \in \mathbb{R}$ is the time, $\mathbf{x}(t), \dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t) \in \mathbb{R}^n$ stand for generalized coordinates, velocity and acceleration, separately. $\boldsymbol{\delta}(t) \in \mathcal{Z}$ denotes the uncertain parameters, which may be fast and time-varying. $\mathcal{Z}$ represents the boundary of the uncertainties. $\mathbf{M} \in \mathbb{R}^{n \times n} > 0$ denotes the inertial mass matrix; $\mathbf{Q} \in \mathbb{R}^n$ denotes the known force imposed on the control system. $\boldsymbol{\tau} \in \mathbb{R}^m (m \leq n)$ stands for the control input and $\mathbf{B}$ represents the corresponding coefficient. When the system is free of constrained, $\boldsymbol{\tau} = 0$. Furthermore, $\mathbf{M}(\cdot)$ and $\mathbf{Q}(\cdot)$ are of appropriate dimensions, continuous and Lebesgue measurable.

The coordinates $\mathbf{x}(t)$ can be chosen according to the specific situation, which do not necessarily to be generalized coordinates. Consider the mechanical system is subjected to the following constraints:

$$\sum_{i=1}^n \mathbf{B}_{li}(\mathbf{x}, t) = \mathbf{d}_l(\mathbf{x}, t) \quad l = 1, \dots, m. \quad (10)$$

where (10) can contain the holonomic and nonholonomic constraints. Determining the first-order derivative of (10) about time t , we can obtain:

$$\sum_{i=1}^n \mathbf{A}_{li}(\mathbf{x}, t) \dot{\mathbf{x}} = \mathbf{c}_l(\mathbf{x}, t) \quad (11)$$

where

$$\mathbf{A}_{li}(\mathbf{x}, t) = \sum_{k=1}^n \frac{\partial \mathbf{B}_{li}(\mathbf{x}, t)}{\partial \mathbf{x}_k} - \sum_{k=1}^n \frac{\partial \mathbf{d}_l(\mathbf{x}, t)}{\partial \mathbf{x}_k}, \quad (12)$$

$$\mathbf{c}_l(\mathbf{x}, t) = \frac{\partial \mathbf{d}_l(\mathbf{x}, t)}{\partial t} - \frac{\partial \mathbf{B}_{li}(\mathbf{x}, t)}{\partial t}. \quad (13)$$

Rewrite the constraints (11) as a matrix form:

$$\mathbf{A}(\mathbf{x}, t) \dot{\mathbf{x}} = \mathbf{c}(\mathbf{x}, t) \quad (14)$$

where $\mathbf{A} = [\mathbf{A}_{li}]_{m \times n}$ and $\mathbf{c} = [c_1, c_2, \dots, c_m]^T$ which contains the first derivative of $\mathbf{x}$.

The second derivative of (11) is obtained:

$$\sum_{i=1}^n \frac{d}{dt} \mathbf{A}_{li}(\mathbf{x}, t) \dot{\mathbf{x}}_i + \sum_{i=1}^n \mathbf{A}_{li}(\mathbf{x}, t) \ddot{\mathbf{x}}_i = \frac{d}{dt} \mathbf{c}_l(\mathbf{x}, t) \quad (15)$$

where

$$\frac{d}{dt} \mathbf{A}_{li}(\mathbf{x}, t) = \sum_{k=1}^n \frac{\partial \mathbf{A}_{li}(\mathbf{x}, t)}{\partial \mathbf{x}_k} \dot{\mathbf{x}}_k + \frac{\partial \mathbf{A}_{li}(\mathbf{x}, t)}{\partial t} \quad (16)$$

and

$$\frac{d}{dt} \mathbf{c}_l(\mathbf{x}, t) = \sum_{k=1}^n \frac{\partial \mathbf{c}_l(\mathbf{x}, t)}{\partial \mathbf{x}_k} \dot{\mathbf{x}}_k + \frac{\partial \mathbf{c}_l(\mathbf{x}, t)}{\partial t} \quad (17)$$

(15) can be rewritten as the matrix form:

$$\mathbf{A}(\mathbf{x}, t) \ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (18)$$

where $\mathbf{A} = [\mathbf{A}_{li}]_{m \times n}$ and $\mathbf{b} = [b_1, b_2, \dots, b_m]^T$. The acceleration is a linear function in the second-order constraint which is beneficial for further dynamic analysis. Almost all control problems can be written in the second-order constrained form, including stability, trajectory tracking and optimal control.

Assumption 1: The constraint in (18) is continuous and full rank, then $\mathbf{A}(\mathbf{x}, t) \mathbf{A}^T(\mathbf{x}, t)$ is invertible.

Remark 1: For a given constraint, $\mathbf{A}$ and $\mathbf{b}$ are determined. There is at least one solution for $\ddot{\mathbf{x}}$ in (18), then (18) can be seen as continuous.

Theorem 1 (U-K theory): Assumptions 1 is satisfied, an unconstrained mechanical system is subjected to constraints as shown in (11), then the constrain force $\mathbf{Q}^c$ is shown as [39]

$$\mathbf{Q}^c = \mathbf{M}^{1/2} (\mathbf{A} \mathbf{M}^{-1/2})^+ \times (\mathbf{b} - \mathbf{A} \mathbf{M}^{-1} \mathbf{Q}). \quad (19)$$

The mechanical system considered in U-K theory is free of uncertainties or the uncertainties are known, in this case $\boldsymbol{\tau} = \mathbf{Q}^c$. Nominal control, also called Udwadia control, can

be designed based on U-K theory. However, the uncertainties are inevitable and undetermined in practical applications.

2.3 Cooperative game theory

Assume there are Y participants in the game, and the cost function is formulated as

$$J_i(\cdot): \prod_i^Y \mathcal{H}_i \rightarrow \mathbb{R}, i = 1, 2, \dots, Y \quad (20)$$

where $J_i(\cdot)$ and $\mathcal{H}_i$ represent the cost function and the decision set for the player i , respectively.

Generally, for each participant, the least cost in the game is his goal. If there is $\mathbf{h}^* \in \prod_i^Y \mathcal{H}_i$ for all $i = \{1, 2, \dots, Y\}$, we have

$$J_i(\mathbf{h}^*) \leq J(\mathbf{h}) \quad \forall \mathbf{h} \in \prod_i^Y \mathcal{H}_i \quad (21)$$

where $\mathbf{h}^*$ is surely a desirable decision Y -tuple, which minimizes every participant's cost. However, the wonderful decision is not available in the reality and then how to find the optimal decision is a thorny problem.

In order to solve the problem, a novel thought was proposed by Pareto, which advises every participant to cooperate with each other to obtain the optimal decision. This is the cooperative game theory, which can coordinate the optimal solution problem between multiple parameters and objects. Moreover, Pareto defines the Pareto optimal decision Y -tuple as follows.

Definition: A decision Y -tuple $\mathbf{h}^* \in \prod_i^Y \mathcal{H}_i$ is Pareto optimal if and only if for every $\mathbf{h} \in \prod_i^Y \mathcal{H}_i$ either

$$J_i(\mathbf{h}) = J_i(\mathbf{h}^*) \quad \forall i \in \{1, 2, \dots, Y\} \quad (22)$$

or there is at least one $i \in \{1, 2, \dots, Y\}$ such that

$$J_i(\mathbf{h}) > J_i(\mathbf{h}^*). \quad (23)$$

In general, the Pareto optimal decision Y -tuple may be more than one, and the set of the cost outcomes $(J_1, J_2, \dots, J_Y)$ under a different Pareto optimal decision is called the Pareto front. Based on *Definition*, the following lemma is proposed to find the Pareto optimal solution.

Lemma 2: Decision Y -tuple $\mathbf{h}^* \in \prod_i^Y \mathcal{H}_i$ is Pareto optimal if there exists a $\mathbf{g} \in \mathbb{R}$ with $g_i > 0, i = 1, 2, \dots, M$, and $\sum_{i=1}^Y g_i = 1$, such that

$$J(\mathbf{h}^*) \leq J(\mathbf{h}) \quad \forall \mathbf{h} \in \prod_i^Y \mathcal{H}_i \quad (24)$$

where $J(\mathbf{h}) = \sum_{i=1}^Y g_i J_i(\mathbf{h})$.

Proof: Let $\mathbf{h} \in \prod_i^Y \mathcal{H}_i$. Then, equation (24) implies either $J(\mathbf{h}^*) < J(\mathbf{h})$ or $J(\mathbf{h}^*) = J(\mathbf{h})$. Suppose $J(\mathbf{h}^*) = J(\mathbf{h})$, then we have $J_i(\mathbf{h}^*) = J_i(\mathbf{h})$ for all $i \in \{1, 2, \dots, Y\}$ [then equation (22) is satisfied] or there exists $i, j \in$

$\{1, 2, \dots, Y\}$ such that $J_i(\mathbf{h}^*) < J_i(\mathbf{h})$ [then equation (23) is satisfied] and $J_j(\mathbf{h}^*) > J_j(\mathbf{h})$. If there exists $J(\mathbf{h}^*) < J(\mathbf{h})$, then there exists $i \in \{1, 2, \dots, Y\}$ such $J_i(\mathbf{h}^*) < J_i(\mathbf{h})$ [then equation (23) is satisfied].

Remark 2: From the perspective of cooperative game, we can transform the parameters and evaluation indicators to the participants and cost functions. The optimal solution can obtain by solving the Pareto optimal solution.

3. Robust control design

We propose an adaptive robust control to ensure the LLER can track the desired trajectory in this section. The control flow chart is shown as Figure 2.

Decompose the matrices/vectors $\mathbf{M}$ and $\mathbf{Q}$ of equation (9) into two parts which are the "nominal" parts and the uncertain parts as follows:

$$\mathbf{M}(\mathbf{x}, \delta, t) = \bar{\mathbf{M}}(\mathbf{x}, t) + \Delta \mathbf{M}(\mathbf{x}, \delta, t), \quad (25)$$

$$\mathbf{Q}(\mathbf{x}, \dot{\mathbf{x}}, \delta, t) = \bar{\mathbf{Q}}(\mathbf{x}, \dot{\mathbf{x}}, t) + \Delta \mathbf{Q}(\mathbf{x}, \dot{\mathbf{x}}, \delta, t), \quad (26)$$

Assume that $\bar{\mathbf{M}} > 0$. $\bar{\mathbf{M}}(\cdot)$, $\bar{\mathbf{Q}}(\cdot)$ and $\Delta \mathbf{Q}(\cdot)$ are all continuous and uninterrupted. In the ideal case, there are no uncertainty and the corresponding matrixes are equal to the corresponding "nominal" parts. Let

$$\mathbf{F}(\mathbf{x}, \delta, t) := \bar{\mathbf{M}}(\mathbf{x}, t) \mathbf{M}^{-1}(\mathbf{x}, \delta, t) - \mathbf{I}, \quad (27)$$

$$\mathbf{E}(\mathbf{x}, t) := \bar{\mathbf{M}}^{-1}(\mathbf{x}, t), \quad (28)$$

$$\Delta \mathbf{E}(\mathbf{x}, \delta, t) := \mathbf{M}^{-1}(\mathbf{x}, \delta, t) - \bar{\mathbf{M}}^{-1}(\mathbf{x}, t). \quad (29)$$

Then we can obtain

$$\mathbf{M}^{-1}(\mathbf{x}, \delta, t) = \mathbf{E}(\mathbf{x}, t) + \Delta \mathbf{E}(\mathbf{x}, \delta, t), \quad (30)$$

$$\Delta \mathbf{E}(\mathbf{x}, \delta, t) = \mathbf{E}(\mathbf{x}, t) \mathbf{F}(\mathbf{x}, \delta, t). \quad (31)$$

Assumption 2: When *Assumption 1* is satisfied, for a given $\mathbf{P} \in \mathbb{R}^{m \times m}, \mathbf{P} > 0$, let

$$\mathbf{W}(\mathbf{x}, \delta, t) := \mathbf{P} \mathbf{A}(\mathbf{x}, t) \mathbf{E}(\mathbf{x}, t) \mathbf{F}(\mathbf{x}, \delta, t) \mathbf{B} \times \\ (\mathbf{A}(\mathbf{x}, t) \bar{\mathbf{M}}^{-1}(\mathbf{x}, t))^{-1} \mathbf{P}^{-1}. \quad (32)$$

There exists a constant $\rho_E > -1$. For all $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}$, ρ_E satisfies the (34):

$$\frac{1}{2} \min_{\delta \in \Sigma} (\lambda_m(\mathbf{W}(\mathbf{x}, \delta, t) + \mathbf{W}^T(\mathbf{x}, \delta, t))) \geq \rho_E \quad (33)$$

where λ_m denotes eigenvalues of matrix.

Remark 3: The value of ρ_E , $\mathbf{W}$ and $\mathbf{E}$ depend on $\mathbf{W}$, $\mathbf{E}$ and the uncertainty bound Σ , respectively. Since the bound of uncertainty $\Sigma \subset \mathbb{R}^p$ is unknown, ρ_E is unknown. In the ideal situation, $\mathbf{E} = 0, \mathbf{W} = 0$ and $\rho_E = 0$.

Assumption 3: 1) There exists an unknown constant vector $\alpha \in (0, \infty)^k$ and a known function $\Pi(\cdot): (0, \infty)^k \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_+$, such that, for all $(\mathbf{x}, \dot{\mathbf{x}}, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \delta \in \Sigma$,

$$(1 + \rho_E)^{-1} \max(\|2\mathbf{P}\mathbf{A}(\mathbf{x}, t)\mathbf{E}(\mathbf{x}, t)\mathbf{Q}(\mathbf{x}, \dot{\mathbf{x}}, \delta, t) - \mathbf{P}\mathbf{A}(\mathbf{x}, t)\Delta\mathbf{E}(\mathbf{x}, \delta, t)\bar{\mathbf{Q}}(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{P}\mathbf{A}(\mathbf{x}, t)\Delta\mathbf{E}(\mathbf{x}, \delta, t)\mathbf{B} \times (\mathbf{w}_1(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{w}_2(\mathbf{x}, \dot{\mathbf{x}}, t))\|) \leq \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t). \quad (34)$$

2) For each $(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t)$, $\Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t)$ is a linearized function with respect to α ; there exists a function $\Pi(\cdot): \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) = \alpha^T \tilde{\Pi}(\mathbf{x}, \dot{\mathbf{x}}, t). \quad (35)$$

here, Σ which is unknown determines the value of α .

According to U-K theory and *Assumption 1-3*, the adaptive robust control $\tau(t)$ is proposed as follows:

$$\tau(t) = \mathbf{w}_1(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \mathbf{w}_2(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \mathbf{w}_3(\hat{\alpha}, \mathbf{x}(t), \dot{\mathbf{x}}(t), t) \quad (36)$$

where

$$\mathbf{w}_1(\mathbf{x}, \dot{\mathbf{x}}, t) := (\mathbf{A}(\mathbf{x}, t)\bar{\mathbf{M}}^{-1}(\mathbf{x}, t)\mathbf{B})^+ (\mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t) - \mathbf{A}(\mathbf{x}, t)\bar{\mathbf{M}}^{-1}(\mathbf{x}, t)\bar{\mathbf{Q}}(\mathbf{x}, \dot{\mathbf{x}}, t)) \quad (37)$$

$$\mathbf{w}_2(\mathbf{x}, \dot{\mathbf{x}}, t) := -k_1(\mathbf{A}(\mathbf{x}, t)\bar{\mathbf{M}}^{-1}(\mathbf{x}, t)\mathbf{B})^+ \times \mathbf{P}^{-1}\boldsymbol{\beta}(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (38)$$

$$\mathbf{w}_3(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) := -k_1^2(\mathbf{A}(\mathbf{x}, t)\bar{\mathbf{M}}^{-1}(\mathbf{x}, t)\mathbf{B})^+ \times \mathbf{P}^{-1}\boldsymbol{\chi}(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) \quad (39)$$

$$\boldsymbol{\chi}(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) = \gamma(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t)\boldsymbol{\mu}(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t)\Pi(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) \quad (40)$$

$$\gamma(n, \hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) = \frac{\vartheta^{n-1}}{\vartheta^{n-1} + \vartheta^{n-2}k_2 + \dots + \vartheta k_2^{n-2} + k_2^{n-1}} \quad (41)$$

$$\vartheta(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) = k_1^2 \|\boldsymbol{\mu}(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t)\|^2 \quad (42)$$

$$\boldsymbol{\mu}(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) = \boldsymbol{\beta}(\mathbf{x}, \dot{\mathbf{x}}, t)\Pi(\hat{\alpha}, \mathbf{x}, \dot{\mathbf{x}}, t) \quad (43)$$

$$\boldsymbol{\beta}(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{A}(\mathbf{x}, t) \dot{\mathbf{x}} - \mathbf{c}(\mathbf{x}, t) \quad (44)$$

where $k_1, k_2 > 0, k_1, k_2 \in \mathbb{R}$, $n = 1, 2, 3, \dots$ are scalar constants, $\boldsymbol{\beta}(\mathbf{x}, \dot{\mathbf{x}}, t)$ stands for the constrained first-order error.

The adaptive law can be designed to govern the parameter $\hat{\alpha}$ as follows:

$$\dot{\hat{\alpha}} = k_1(L_1 \tilde{\Pi}(\mathbf{x}, \dot{\mathbf{x}}, t)\|\boldsymbol{\beta}(\mathbf{x}, \dot{\mathbf{x}}, t)\| - (L_2 e^{-\|\boldsymbol{\beta}(\mathbf{x}, \dot{\mathbf{x}}, t)\|} + L_3)\hat{\alpha}) \quad (45)$$

$\hat{\alpha}_i(t_0) > 0$ (where $\hat{\alpha}_i$ is the i -th element of the vector $\hat{\alpha}$, $i = 1, 2, \dots, k$), $L_1, L_2, L_3 > 0, L_1, L_2, L_3 \in \mathbb{R}$. The adaptive parameter $\hat{\alpha}$ is a substitute for the unknown constant α , and adaptive law (45) can determine the value of $\hat{\alpha}$.

Theorem 2: Let $\boldsymbol{\varphi} := [\boldsymbol{\beta}^T (\hat{\alpha} - \alpha)^T]^T \in \mathbb{R}^{m+k}$. Assume that *Assumptions 1-3* are satisfied, under the system of (9), the adaptive robust control (37) renders the uniform boundness and uniform ultimate boundedness:

(1) Uniform boundedness: For any $r > 0$, there exists a $d(r) < \infty$, such that if $\|\boldsymbol{\varphi}(t)\| \leq r$, then $\|\boldsymbol{\varphi}(t)\| \leq d(r)$ for all $t \geq t_0$.

(2) Uniform ultimate boundedness: For any $r > 0$ and $\|\boldsymbol{\varphi}(t_0)\| \leq r$, there exists a $\underline{d} > 0$. Such that for any $\bar{d} > \underline{d}$ as $t > t_0 + T(\bar{d}, r)$, where $T(\bar{d}, r) < \infty$, there exists $\|\boldsymbol{\varphi}(t)\| \leq \bar{d}$.

The proof of Theorem 2 is given in Appendix A.

4. Optimal design

According to Rayleigh's principle

$$\lambda_{\min}(\mathbf{P})\|\boldsymbol{\beta}\|^2 \leq \boldsymbol{\beta}^T \mathbf{P} \boldsymbol{\beta} \leq \lambda_{\max}(\mathbf{P})\|\boldsymbol{\beta}\|^2 \quad (46)$$

where $\lambda_{\min}(\mathbf{P}) > 0$, because $\mathbf{P}$ is positive definite. So

$$\begin{aligned} -2k_1 \frac{\lambda_{\min}(\mathbf{P})}{\lambda_{\max}(\mathbf{P})} \|\boldsymbol{\beta}\|^2 &\geq -\frac{2k_1}{\lambda_{\max}(\mathbf{P})} \boldsymbol{\beta}^T \mathbf{P} \boldsymbol{\beta} \\ &\geq -2k_1 \|\boldsymbol{\beta}\|^2. \end{aligned} \quad (47)$$

If we let $\hat{\rho} := \min\{2/\lambda_{\max}(\mathbf{P}), L_3\}$, we can obtain

$$\begin{aligned} -2k_1 \|\boldsymbol{\beta}\|^2 - (1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 \\ \leq -\frac{2k_1}{\lambda_{\max}(\mathbf{P})} \boldsymbol{\beta}^T \mathbf{P} \boldsymbol{\beta} - k_1 L_3 (k_1 L_1)^{-1} (1 + \rho_E) \\ \times (\hat{\alpha} - \alpha)^T (\hat{\alpha} - \alpha) \leq -k_1 \hat{\rho} V. \end{aligned} \quad (48)$$

With (48) and (112),

$$\begin{aligned} \dot{V} &\leq -2k_1 \|\boldsymbol{\beta}\|^2 - (1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 + \Omega \\ &\leq -k_1 \hat{\rho} V(t) + \Omega \end{aligned}$$

where $V(t_0) = \boldsymbol{\beta}^T(t_0)\mathbf{P}\boldsymbol{\beta}(t_0) + (k_1 L_1)^{-1}(1 + \rho_E) \times$

$(\hat{\alpha}(t_0) - \alpha)^T (\hat{\alpha}(t_0) - \alpha)$. The analysis of (38) is presented next.

First, consider the differential equation

$$\dot{y}(t) = -k_1 \hat{\rho} y(t) + \Omega, \quad y(t_0) = V_0 = V(t_0). \quad (50)$$

The right-hand-side of (50) meets the global Lipschitz condition with

$$L = k_1 \hat{\rho} \quad (51)$$

Solve the differential (50) and obtain the result

$$y(t) = \left(V_0 - \frac{\Omega}{k_1 \hat{\rho}}\right) e^{-k_1 \hat{\rho}(t-t_0)} + \frac{\Omega}{k_1 \hat{\rho}}. \quad (52)$$

Then, based on the solution process of differential inequality, we can obtain

$$V(t) \leq y(t) \quad (53)$$

or

$$V(t) \leq \left(V_0 - \frac{\Omega}{k_1 \hat{\rho}}\right) e^{-k_1 \hat{\rho}(t-t_0)} + \frac{\Omega}{k_1 \hat{\rho}} \quad (54)$$

for all $t \geq t_0$. Similarly, for any t_s and $\Gamma \geq t_s$

$$V(\Gamma) \leq \left(V_s - \frac{\Omega}{k_1 \hat{\rho}}\right) e^{-k_1 \hat{\rho}(\Gamma - t_s)} + \frac{\Omega}{k_1 \hat{\rho}} \quad (55)$$

where $V_s = V(t_s) = \mathbf{P}^T(t_s) \mathbf{P} \mathbf{B}(t_s) + (k_1 L_1)^{-1} (1 + \rho_E) \times (\hat{\alpha}(t_s) - \alpha)^T (\hat{\alpha}(t_s) - \alpha)$. t_s is not necessary to be equal to zero, it's the time when control system starts.

To separate the variable, V_s is rewritten as

$$V_s = V_{s1} + \frac{V_{s2}}{k_1} \quad (56)$$

where $V_{s1} := \mathbf{P}^T(t_s) \mathbf{P} \mathbf{B}(t_s)$ and $V_{s2} := L_1^{-1} (1 + \rho_E) \times (\hat{\alpha}(t_s) - \alpha)^T (\hat{\alpha}(t_s) - \alpha)$. With (46), we have $V(\Gamma) \geq \lambda_{\min}(\mathbf{P}) \|\mathbf{B}\|^2 + (k_1 L_1)^{-1} (1 + \rho_E) \|\hat{\alpha} - \alpha\|^2$ and the right-hand-side of (55) provides an upper bound of $\lambda_{\min}(\mathbf{P}) \|\mathbf{B}\|^2 + (k_1 L_1)^{-1} (1 + \rho_E) \|\hat{\alpha} - \alpha\|^2$. This can be an upper bound of $\|\mathbf{B}\|^2$. For each $\Gamma \geq t_s$, let

$$\eta(\alpha, k_1, k_2, \Gamma, t_s) = \left(V_s - \frac{\Omega}{k_1 \hat{\rho}}\right) e^{-k_1 \hat{\rho}(\Gamma - t_s)} \quad (57)$$

$$\eta_\infty(\alpha, k_1, k_2) = \frac{\Omega}{k_1 \hat{\rho}}. \quad (58)$$

Remark 4: The uncertainty is described by fuzzy set theory, which can perfectly be solved adopting the corresponding membership function. Compared with probability theory, fuzzy set theory does not need numerous observation data. [40] has a more detailed description of fuzzy set theory.

The performance indicators for k_1 and k_2 are formulated as

$$J_1(k_1, k_2, t_s) := D \left[\int_{t_s}^{\infty} \eta^2(\alpha, k_1, k_2, \Gamma, t_s) d\Gamma \right] + k_1^2 \quad (59)$$

$$J_2(k_1, k_2, t_s) := D[\eta_\infty^2(\alpha, k_1, k_2)] + k_2^{-2} \quad (60)$$

It can be shown that

$$\begin{aligned} & \int_{t_s}^{\infty} \eta^2(\alpha, k_1, k_2, \Gamma, t_s) d\Gamma \\ &= \left(V_s - \frac{\Omega}{k_1 \hat{\rho}}\right)^2 \int_{t_s}^{\infty} e^{-2k_1 \hat{\rho}(\Gamma - t_s)} d\Gamma \\ &= \left(V_s - \frac{\Omega}{k_1 \hat{\rho}}\right)^2 \left(-\frac{1}{2k_1 \hat{\rho}}\right) e^{-2k_1 \hat{\rho}(\Gamma - t_s)} \Big|_{t_s}^{\infty} \\ &= \left(V_s - \frac{\Omega}{k_1 \hat{\rho}}\right)^2 \frac{1}{2k_1 \hat{\rho}} \\ &= \left(V_{s1} + \frac{V_{s2}}{k_1} - \frac{\Omega}{k_1 \hat{\rho}}\right)^2 \frac{1}{2k_1 \hat{\rho}} \end{aligned}$$

Considering the D-operation, we have

$$D \left[\int_{t_s}^{\infty} \eta^2(\alpha, k_1, k_2, \Gamma, t_s) d\Gamma \right]$$

$$\begin{aligned} &= D \left[\left(V_{s1} + \frac{V_{s2}}{k_1} - \frac{\Omega}{k_1 \hat{\rho}}\right)^2 \frac{1}{2k_1 \hat{\rho}} \right] \\ &:= \mathcal{L}_1 k_1^{-7} + \mathcal{L}_2 k_1^{-5} + \mathcal{L}_3 k_1^{-4} + \mathcal{L}_4 k_1^{-3} + \mathcal{L}_5 k_1^{-2} + \mathcal{L}_6 k_1^{-1} \\ &\quad + \mathcal{L}_7 k_1^{-3} k_2 + \mathcal{L}_8 k_1^{-3} k_2^2 + \mathcal{L}_9 k_1^{-5} k_2 + \mathcal{L}_{10} k_1^{-2} k_2 \end{aligned} \quad (62)$$

details for $\mathcal{L}_i, i = \{1, 2, \dots, 10\}$ are given in Appendix B.

Similarly, by D-operation, we can obtain

$$\begin{aligned} D[\eta_\infty^2(\alpha, k_1, k_2)] &= D \left[\frac{\Omega^2}{k_1^2 \hat{\rho}^2} \right] \\ &:= \mathcal{K}_1 k_1^{-6} + \mathcal{K}_2 k_1^{-4} + \mathcal{K}_3 k_1^{-2} + \mathcal{K}_4 k_1^{-2} k_2 \\ &\quad + \mathcal{K}_5 k_1^{-4} k_2 + \mathcal{K}_6 k_1^{-2} k_2^2 \end{aligned} \quad (63)$$

details for $\mathcal{K}_i, i = \{1, 2, \dots, 6\}$ are given in Appendix B.

Then, J_1 and J_2 can be rewritten as

$$\begin{aligned} J_1(k_1, k_2) &= \mathcal{L}_1 k_1^{-7} + \mathcal{L}_2 k_1^{-5} + \mathcal{L}_3 k_1^{-4} + \mathcal{L}_4 k_1^{-3} \\ &\quad + \mathcal{L}_5 k_1^{-2} + \mathcal{L}_6 k_1^{-1} + \mathcal{L}_7 k_1^{-3} k_2 + \mathcal{L}_8 k_1^{-3} k_2^2 \\ &\quad + \mathcal{L}_9 k_1^{-5} k_2 + \mathcal{L}_{10} k_1^{-2} k_2 + k_1^2 \end{aligned} \quad (64)$$

$$\begin{aligned} J_2(k_1, k_2) &= \mathcal{K}_1 k_1^{-6} + \mathcal{K}_2 k_1^{-4} + \mathcal{K}_3 k_1^{-2} + \mathcal{K}_4 k_1^{-2} k_2 \\ &\quad + \mathcal{K}_5 k_1^{-4} k_2 + \mathcal{K}_6 k_1^{-2} k_2^2 + k_2^{-2} \end{aligned} \quad (65)$$

The optimal design is obtained by the following cooperative game.

Find (k_1^*, k_2^*) such that for every $(k_1, k_2) \in \prod_{i=1}^2 \mathcal{H}_i$ either

$$J_i(k_1, k_2) = J_i(k_1^*, k_2^*) \quad \forall i \in \{1, 2, \dots, Y\} \quad (66)$$

or there is at least one $i \in \{1, 2\}$ such that

$$J_i(k_1, k_2) > J_i(k_1^*, k_2^*) \quad (67)$$

To find (k_1^*, k_2^*) , we adopt Lemma 3: the decision pair (k_1^*, k_2^*) is Pareto optimal if there exist weighting factors $\ell_1 > 0$ and $\ell_2 > 0$ with $\ell_1 + \ell_2 = 1$ such that

$$J(k_1^*, k_2^*) \leq J(k_1, k_2) \quad \forall k_1 \in \mathcal{H}_1, k_2 \in \mathcal{H}_2 \quad (68)$$

where $J(k_1, k_2) = \ell_1 J_1(k_1, k_2) + \ell_2 J_2(k_1, k_2)$

Consequently, we can solve the following constrained optimization problem.

For any t_s and given $D_{1,2}$

$$\min_{k_1, k_2} J(k_1, k_2) \text{ subject to: } k_1 \in \mathcal{H}_1, k_2 \in \mathcal{H}_2. \quad (69)$$

To obtain the solution, we apply a partial differential operator to $J(k_1, k_2)$ with respect to k_1

$$\begin{aligned} \frac{\partial J}{\partial k_1} &= \ell_1 (-7\mathcal{L}_1 k_1^{-8} - 5\mathcal{L}_2 k_1^{-6} - 4\mathcal{L}_3 k_1^{-5} - 3\mathcal{L}_4 k_1^{-4} \\ &\quad - 2\mathcal{L}_5 k_1^{-3} - \mathcal{L}_6 k_1^{-2} - 3\mathcal{L}_7 k_1^{-4} k_2 - 3\mathcal{L}_8 k_1^{-4} k_2^2 \end{aligned}$$

$$\begin{aligned}
& -5\mathcal{L}_9 k_1^{-6} k_2 - 2\mathcal{L}_{10} k_1^{-3} k_2 + 2k_1) \\
& + \ell_2(-6\kappa_1 k_1^{-7} - 4\kappa_2 k_1^{-5} - 2\kappa_3 k_1^{-3} - 2\kappa_4 k_1^{-3} k_2 \\
& - 4\kappa_5 k_1^{-5} k_2 - 2\kappa_6 k_1^{-3} k_2^2) \quad (70)
\end{aligned}$$

Similarly, applying partial differential operator to $J(k_1, k_2)$ with respect to k_2 gives

$$\begin{aligned}
\frac{\partial J}{\partial k_2} &= \ell_1(\mathcal{L}_7 k_1^{-3} + 2\mathcal{L}_8 k_1^{-3} k_2 + \mathcal{L}_9 k_1^{-5} + \mathcal{L}_{10} k_1^{-2}) \\
&+ \ell_2(\kappa_4 k_1^{-2} + \kappa_5 k_1^{-4} + 2\kappa_6 k_1^{-2} k_2 - 2k_2^{-3}). \quad (71)
\end{aligned}$$

Let

$$\frac{\partial J}{\partial k_1} = 0 \quad (72)$$

$$\frac{\partial J}{\partial k_2} = 0 \quad (73)$$

with $k_1 \in \mathcal{H}_1, k_2 \in \mathcal{H}_2$.

By analyzing the second-order derivative of J , we have

$$\Delta(k_1, k_2) = \begin{bmatrix} \frac{\partial^2 J}{\partial k_1^2} & \frac{\partial^2 J}{\partial k_1 \partial k_2} \\ \frac{\partial^2 J}{\partial k_2 \partial k_1} & \frac{\partial^2 J}{\partial k_2^2} \end{bmatrix} \quad (74)$$

Suppose the candidate solutions k_1^*, k_2^* can meet the sufficient condition

$$\Delta(k_1, k_2) > 0. \quad (75)$$

Then, (68) can be minimized by the solutions of (72) and (73) and the solutions are Pareto optimal.

Remark 5: The k_1^* and k_2^* can be determined with suitable choices of $\mathbf{P}$, ρ_E and $L_i (i = 1, 2, 3)$.

Theorem 3: Suppose the solutions $k_1^* > 0$ and $k_2^* > 0$ which satisfy (72), (73) and (75) exist. Then, they are the Pareto optimality.

Proof: The proof of *Theorem 3* has already been presented above in (59)-(75).

Based on the above analysis, the resulting minimal performance index is acquired by

$$J_{min} = \ell_1 J_1(k_1^*, k_2^*) + \ell_2 J_2(k_1^*, k_2^*) \quad (76)$$

The design procedure of optimal design is displayed in Figure 1.

Figure 1. The design procedure of optimal design.

5. Numerical Simulation

5.1 Description of LLER

Figure 2 has demonstrated the 3D prototype and simplified 2D model of LLER. We assume the LLER has two active degrees of freedom (DOFs) of one leg, containing the flexion and extension DOFs of hip and knee joints. This idea is inspired by [9, 11, 41]. $\mathbf{x} = [\theta_1 \ \theta_2]^T$ is chosen as the generalized coordinates, which denotes the rotation angle of hip and knee, respectively.

Employed the Lagrangian dynamics, the dynamic model of LLER is formulated

$$\mathbf{M}(\mathbf{x}(t), \boldsymbol{\delta}(t), t) \ddot{\mathbf{x}} = \mathbf{Q}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\delta}(t), t) + \mathbf{B}\boldsymbol{\tau}(t) \quad (77)$$

where

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (78)$$

$$\mathbf{Q} = [Q_{11} \ Q_{21}]^T \quad (79)$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (80)$$

$$M_{11} = m_1 d_1^2 + I_1 + m_2 l_1^2 \quad (81)$$

$$M_{12} = M_{21} = -m_2 l_1 d_2 \cos(\theta_1 + \theta_2) \quad (82)$$

$$M_{22} = m_2 d_2^2 + I_2 \quad (83)$$

$$\begin{aligned}
Q_{11} &= -m_2 l_1 d_2 \sin(\theta_1 + \theta_2) \ddot{\theta}_2 + m_2 l_1 d_2 \sin(\theta_1 + \theta_2) \\
&\times (\dot{\theta}_1 + \dot{\theta}_2) + m_1 g d_1 \sin \theta_1 + m_2 g l_1 \sin \theta_1 \quad (84)
\end{aligned}$$

$$\begin{aligned}
Q_{21} &= m_2 l_1 d_2 \sin(\theta_1 + \theta_2) (\dot{\theta}_1 + \dot{\theta}_2) - m_2 l_1 d_2 \\
&\times \sin(\theta_1 + \theta_2) \ddot{\theta}_1 + m_2 g d_2 \sin \theta_2 \quad (85)
\end{aligned}$$

Moreover, $m_i (i = 1, 2)$ stands for the mass of thigh and shank. g is gravity acceleration. $l_i = 2d_i (i = 1, 2)$

denotes the length of thigh and shank. $I_i (i = 1, 2)$ represents for the moment of inertia of thigh and shank, respectively.

Figure 2. The 3D and 2D models of LLER.

Apply the servo constraints to the LLER, which is obtained from Clinical Gait Analysis data [9]

$$\theta_1 = 0.8 + 0.2 \sin(2\pi t) \quad (86)$$

$$\theta_2 = 1.2 - 0.2 \cos(2\pi t) \quad (87)$$

Rewriting the equations (86)-(87) in the form of (10)

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{d}(\mathbf{x}, t) \quad (88)$$

where

$$\mathbf{B}(\mathbf{x}, t) = [\theta_1 \quad \theta_2]^T \quad (89)$$

$$\mathbf{d}(\mathbf{x}, t) = \begin{bmatrix} 0.8 + 0.2 \sin(2\pi t) \\ 1.2 - 0.2 \cos(2\pi t) \end{bmatrix} \quad (90)$$

We differentiate the equation (88) and rewrite them in the form of the equation (14) and (18)

$$\mathbf{A}(\mathbf{x}, t)\dot{\mathbf{x}} = \mathbf{c}(\mathbf{x}, t) \quad (91)$$

$$\mathbf{A}(\mathbf{x}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (92)$$

where

$$\mathbf{A}(\mathbf{x}, t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (93)$$

$$\mathbf{c}(\mathbf{x}, t) = \begin{bmatrix} 0.4\pi \cos(2\pi t) \\ 0.4\pi \sin(2\pi t) \end{bmatrix} \quad (94)$$

$$\mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t) = \begin{bmatrix} -0.8\pi^2 \sin(2\pi t) \\ 0.8\pi^2 \cos(2\pi t) \end{bmatrix} \quad (95)$$

Here, equations (91)-(92) denote velocity and acceleration constraints, respectively.

Uncertainties are inevitable in the LLER, and the masses are chosen as the uncertain parameters: $m_1 = \bar{m}_1 + \Delta m_1$, $m_2 = \bar{m}_2 + \Delta m_2$. When there are no uncertainties, $\Delta m_i (i = 1, 2) = 0$. The parameters are chosen as Table 1. $\Delta m_i (i = 1, 2)$ are supposed as follows: Δm_1 is "close to 0.2", Δm_2 is "close to 0.1". Using fuzzy set theory, the corresponding membership functions are obtained as follows

$$\mu_{\Delta m_1}(\epsilon) \begin{cases} 5\epsilon & 0 \leq \epsilon \leq 0.2 \\ -5\epsilon + 2 & 0.2 \leq \epsilon \leq 0.4 \end{cases} \quad (96)$$

$$\mu_{\Delta m_2}(\epsilon) \begin{cases} 10\epsilon & 0 \leq \epsilon \leq 0.1 \\ -10\epsilon + 2 & 0.1 \leq \epsilon \leq 0.2 \end{cases} \quad (97)$$

Table 1. The parameters of the LLER.

Parameter	Value	Unit
$\bar{m}_1$	28	kg
$\bar{m}_2$	23	kg
l_1	0.45	m
l_2	0.35	m
I_1	2.5	kg · m ²
I_2	1.0	kg · m ²
g	9.8	m/s ²

5.2 Optimal robust control

Choose the initial conditions as $\theta_1(0) = 1, \theta_2(0) = 0.8, \dot{\theta}_1(0) = 0.5, \dot{\theta}_2(0) = 0.5$ which don't satisfy the initial conditions in the ideal condition. Furthermore, the parameters $\mathbf{P} = I_{2 \times 2}$, $n = 5$, $L_i (i = 1, 2, 3) = 1$, $\rho_E = -0.1$ and $\hat{\alpha}(0) = 0.2$ are chosen. Then, Assumptions 1-2 can be easily satisfied. In order to meet Assumption 3, $\Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t)$ can be chosen as follow:

$$\begin{aligned} \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) &= \alpha_1 \|\dot{\mathbf{x}}\|^2 + \alpha_2 \|\dot{\mathbf{x}}\| + \alpha_3 \\ &= (\alpha_1 \quad \alpha_2 \quad \alpha_3) \begin{bmatrix} \|\mathbf{x}\|^2 \\ \|\dot{\mathbf{x}}\| \\ 1 \end{bmatrix} \end{aligned} \quad (98)$$

where $\alpha_i > 0 (i = 1, 2, 3)$. $\Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t)$ can also be chosen as follow to satisfy Assumption 3

$$\alpha_1 \|\dot{\mathbf{x}}\|^2 + \alpha_2 \|\dot{\mathbf{x}}\| + \alpha_3 \leq \alpha (\|\dot{\mathbf{x}}\| + 1)^2 = \alpha^T \tilde{\Pi}(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (99)$$

where $\alpha = \max\{\alpha_1, \frac{\alpha_2}{2}, \alpha_3\}$.

The task we should consider is to seek the optimal solution of k_1 and k_2 . Cooperative game theory is adopted, and we calculate that $\mathcal{L}_1 = 0.1013$, $\mathcal{L}_2 = -0.2403$, $\mathcal{L}_3 = -0.1534$, $\mathcal{L}_4 = 0.1362$, $\mathcal{L}_5 = 0.1780$, $\mathcal{L}_6 = 0.0581$, $\mathcal{L}_7 = -0.9396$, $\mathcal{L}_8 = 1.6200$, $\mathcal{L}_9 = 0.2025$, $\mathcal{L}_{10} = -0.6137$, $\kappa_1 = 0.2025$, $\kappa_2 = 0.0486$, $\kappa_3 = 0.0029$, $\kappa_4 = 0.1944$, $\kappa_5 = 1.6200$ and $\kappa_6 = 3.2400$.

According to equation (72)-(73), the Pareto optimal solution can be solved. Figure 4 also verifies the existence of that by showing the plots of $\partial J(k_1, k_2) / \partial k_1$, $\partial J(k_1, k_2) / \partial k_2$ and $\partial J / \partial k_i (i = 1, 2)$ in the case of $\ell_i (i = 1, 2) = 0.5$. After verification, the yellow point is the Pareto optimal solution. Figure 5 shows the relationship between J and $k_{1,2}$ under $\ell_i (i = 1, 2) = 0.5$, and the yellow point is the solution of minimizing the cost performance J , named the Pareto optimality. For a better interpretation of cooperative game theory, Table 2 gives five groups of $k_i (i = 1, 2)$ and corresponding minimal cost performance J_{min} under different $\ell_i (i = 1, 2)$.

Figure 3. The relationship between $\partial J(k_1, k_2)/\partial k_1$, $\partial J(k_1, k_2)/\partial k_2$ and k_1, k_2 ($\ell_1 = \ell_2 = 0.5$).

Figure 4. The relation between $J(k_1, k_2)$ and k_1, k_2 ($\ell_1 = \ell_2 = 0.5$).

Table 2. Optimal solutions and minimal cost performance under five groups of $\ell_i (i = 1, 2)$.

NO.	(ℓ_1, ℓ_2)	(k_1^*, k_2^*)	J_{min}
1	(0.1, 0.9)	(2.0000, 1.0300)	2.1685
2	(0.3, 0.7)	(1.7800, 0.9600)	2.5327
3	(0.5, 0.5)	(1.4100, 0.8300)	2.5118
4	(0.7, 0.3)	(1.1200, 0.6900)	2.1154
5	(0.9, 0.1)	(1.0000, 0.5800)	1.2942

5.3 Simulation results

The software is MATLAB R2019b. Figures 5-15 display the simulation results by using ode15i algorithm. The uncertainties are represented in the simulation as $\Delta m_1(t) = 0.05 \cos(t) \text{ kg}$, $\Delta m_2(t) = 0.01 \sin(t) \text{ kg}$.

Figures 5-9 demonstrate the trajectories of θ_1 , θ_2 , $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\alpha}$, when $\ell_1 = \ell_2 = 0.5$. The actual trajectories of ORC proposed by us can suppress the contribution of uncertainties and have an excellent tracking performance.

Figure 5. Trajectories of θ_1 about time ($\ell_1 = \ell_2 = 0.5$).

Figure 6. Trajectories of θ_2 about time ($\ell_1 = \ell_2 = 0.5$).

Figure 7. Trajectories of $\dot{\theta}_1$ about time ($\ell_1 = \ell_2 = 0.5$).

Figure 8. Trajectories of $\dot{\theta}_2$ about time ($\ell_1 = \ell_2 = 0.5$).

Figure 9 reveals the relationship between $\dot{\alpha}$ and time. Firstly, the value of $\dot{\alpha}$ increases from initial value 0.2 to the maximum 0.34 in order to suppress the impact of uncertainties. As the deviation is compensated (about 1s), the value of $\dot{\alpha}$ decreases rapidly and gradually stabilized. This phenomenon also implies ORC method possesses well robustness in trajectory tracking of LLER.

Figure 9. Trajectory of $\dot{\alpha}$ about time ($\ell_1 = \ell_2 = 0.5$).

For better illustrate the effect of $\ell_i (i = 1, 2)$ in Table 2, Figures 10-13 compare the trajectories of τ_1 , τ_2 , error1 and error2 under five groups of $k_i (i = 1, 2)$ to explore its impact for system performance. A significant event is discovered by us: The smaller the peak value of $\tau_i (i = 1, 2)$, the slower the error1 and error2's speed converge to zero. We need to endure the larger torque cost to achieve the faster error convergence, which means more expensive motors. In conclusion, the value of $\ell_i (i = 1, 2)$ should be chosen according to the specific situation. The curves of error1 and error2 also prove the ORC method has the uniform boundedness and uniform ultimate boundedness

Adaptive robust control (ARC) of Sun [18] is chosen as a comparative control method, and the corresponding simulation results are also shown in Figures 10-13. The trajectories tracking errors are larger than that of ORC, which means the $\theta_i's (i = 1, 2)$ tracking effect of is worse. And the absolute torque value of hip is larger than that of ORC, although that of knee is smaller than that of ORC. Considering the overall system, the performance of ORC is

better than that of ARC of Sun.

Figure 10. Trajectories of τ_1 about time.

Figure 11. Trajectories of τ_2 about time.

Figure 12. Trajectories of error1 about time.

Figure 13. Trajectories of error2 about time.

Figures 14-15 illustrate the absolute accumulative error (AAE) of $\theta_i (i = 1, 2)$ under five groups of $k_i (i = 1, 2)$ and ARC of Sun. AAE of $\theta_i (i = 1, 2)$ with ARC of Sun is significantly greater than that with ORC under five groups of $k_i (i = 1, 2)$. In summary, the tracking performance of ORC proposed by us is better than that of ARC of Sun, and people can choose the appropriate pair of $\ell_i (i = 1, 2)$ to meet the practical requirements of LLER.

Figure 14. Absolute accumulative error of θ_1 .

Figure 15. Absolute accumulative error of θ_2 .

6. Conclusion

In this paper, the optimal robust control with cooperative game theory is designed to solve the trajectory

tracking problem of LLER, including the uncertainties. The uncertainties are time-varying and bounded. In order to a better description and control of uncertainties, fuzzy set theory is employed, which is essentially different from the traditional fuzzy logic theory. Afterwards, the robust control with two adjustable parameters is proposed in the frame of U-K theory, which guarantees the uniform boundedness and uniform ultimate boundedness. Furthermore, the optimal design with cooperative game theory is formulated to find the Pareto optimal solution of two adjustable parameters. Finally, the simulation results verify the existence of Pareto optimal solution and the effectiveness of the optimal robust control proposed in this paper.

Last but not least, the fuzzy set theory describes the uncertainties first, which are embedded into the robust control. Then the cooperative game theory is used to optimal design. Our work extends the application of U-K and cooperative game theory, has a great significance on trajectory tracking control of LLER with uncertainties. The experiment will be another important work in the next step.

Author contributions

Jin Tian involved in writing - original draft, validation and software. Liang Yuan and Li He involved in writing-review & editing and supervision. Wendong Xiao, Teng Ran and Jianbo Zhang took part in methodology, investigation and formal analysis.

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Availability of data and materials

All data available upon request at the authors' email address.

Code availability

Custom code available upon request at Liang Yuan email address.

Declarations

Conflicts of interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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APPENDIX A

PROOF OF THEOREM 2

Proof: Choose a legitimate Lyapunov function candidate as follow [42]:

$$V(\beta, \hat{\alpha} - \alpha) = \beta^T P \beta + k_1^{-1} L_1^{-1} (1 + \rho_E) (\hat{\alpha} - \alpha)^T (\hat{\alpha} - \alpha). \quad (100)$$

For a specific mechanical system with uncertainty and desired constraints. The $\dot{V}$ is shown as:

$$\dot{V} = 2\beta^T P \dot{\beta} + 2k_1^{-1} L_1^{-1} (1 + \rho_E) (\hat{\alpha} - \alpha)^T \dot{\hat{\alpha}}. \quad (101)$$

The first term of (101) can be analyzed as:

$$\begin{aligned} 2\beta^T P \dot{\beta} &= 2\beta^T P (A\ddot{x} - b) \\ &= 2\beta^T P (AM^{-1}(Q + \tau) - b) \\ &= 2\beta^T P (AM^{-1}(Q + Bw_1 + Bw_2 + Bw_3) - b) \end{aligned} \quad (102)$$

Based on (25) - (26),

$$\begin{aligned} &AM^{-1}(Q + Bw_1 + Bw_2 + Bw_3) - b \\ &= A(E + \Delta E)(\bar{Q} + \Delta Q + Bw_1 + Bw_2 + Bw_3) - b \\ &= AE\bar{Q} + AE\Delta Q + A\Delta EQ + AEBw_1 + AEBw_2 \\ &\quad + A\Delta EB(w_1 + w_2) + A(E + \Delta E)Bw_3 - b. \end{aligned} \quad (103)$$

According to (37),

$$AE\bar{Q} + AEBw_1 - b = 0. \quad (104)$$

According to *Assumption 3-I*,

$$\begin{aligned} &2\beta^T P(AE\Delta Q + A\Delta E(Q + Bw_1 + Bw_2)) \\ &\leq 2\|\beta\| \|PAE\Delta Q + PA\Delta E(Q + Bw_1 + Bw_2)\| \\ &\leq 2(1 + \rho_E) \|\beta\| \Pi(\alpha, x, \dot{x}, t). \end{aligned}$$

Based on (38)

$$\begin{aligned} 2\beta^T PAEBw_2 &= 2\beta^T PAEB(-k_1(A\bar{M}^{-1}B)^{-1}P^{-1}\beta) \\ &= -2k_1\beta^T \beta = -2k_1\|\beta\|^2. \end{aligned} \quad (106)$$

Based on (31), (39) and (43),

$$\begin{aligned} &2\beta^T PA(E + \Delta E)Bw_3 \\ &= 2\beta^T PAEBw_3 + 2\beta^T PAEFBw_3 \\ &= 2\beta^T PAE(-k_1^2(A\bar{M}^{-1}B)^+P^{-1}\chi(\hat{\alpha}, x, \dot{x}, t)) \\ &\quad + 2\beta^T PAEFBw_3 \\ &= -2k_1^2\beta^T \chi(\hat{\alpha}, x, \dot{x}, t) + 2\beta^T PAEFBw_3 \\ &= -2k_1^2\mu^T \gamma \mu + 2\beta^T PAEFBw_3 \\ &= -2k_1^2\gamma \|\mu\|^2 + 2\beta^T PAEFBw_3 \end{aligned}$$

According to *Assumption 3* and Rayleigh's principle,

$$\begin{aligned} &2\beta^T PAEFBw_3 \\ &= 2\beta^T PAEFB(-k_1^2(A\bar{M}^{-1}B)^+P^{-1}\chi(\hat{\alpha}, x, \dot{x}, t)) \\ &= -2k_1^2\gamma \frac{1}{2}\mu^T (PAEFB(A\bar{M}^{-1}B)^+P^{-1} + P^{-1}((A\bar{M}^{-1}B)^+)^T \times \\ &\quad B^T F^T E A^T P) \mu = -2k_1^2\gamma \frac{1}{2}\mu^T (W + W^T) \mu \\ &\leq -2k_1^2\gamma \frac{1}{2}\lambda_m(W + W^T) \|\mu\|^2 \leq -2k_1^2\gamma \rho_E \|\mu\|^2. \end{aligned} \quad (108)$$

Combining (107) and (108),

$$2\beta^T PA(E + \Delta E)Bw_3 \leq -2k_1^2\gamma(1 + \rho_E) \|\mu\|^2. \quad (109)$$

By (41), we have

$$\begin{aligned} &2\beta^T P \dot{\beta} \\ &\leq -2k_1\|\beta\|^2 + 2(1 + \rho_E) \|\beta\| \Pi(\alpha, x, \dot{x}, t) \\ &\quad - 2k_1^2\gamma(1 + \rho_E) \|\mu\|^2 \\ &= -2k_1\|\beta\|^2 + 2(1 + \rho_E) \|\beta\| \Pi(\alpha, x, \dot{x}, t) - 2(1 + \rho_E) \\ &\quad \times \frac{\vartheta^{n-1}}{\vartheta^{n-1} + \vartheta^{n-2}k_2 + \dots + \vartheta k_2^{n-2} + k_2^{n-1}} \\ &\leq -2k_1\|\beta\|^2 + 2(1 + \rho_E) \|\beta\| \Pi(\alpha, x, \dot{x}, t) - 2(1 + \rho_E) \end{aligned}$$

$$\begin{aligned}
& \times \frac{\vartheta^n - k_2^n}{\vartheta^{n-1} + \vartheta^{n-2}k_2 + \dots + \vartheta k_2^{n-2} + k_2^{n-1}} \\
& = -2k_1 \|\boldsymbol{\beta}\|^2 + 2(1 + \rho_E) \|\boldsymbol{\beta}\| \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) - 2(1 + \rho_E) \\
& \times \frac{(\vartheta - k_2)(\vartheta^{n-1} + \vartheta^{n-2}k_2 + \dots + \vartheta k_2^{n-2} + k_2^{n-1})}{\vartheta^{n-1} + \vartheta^{n-2}k_2 + \dots + \vartheta k_2^{n-2} + k_2^{n-1}} \\
& = -2k_1 \|\boldsymbol{\beta}\|^2 + 2(1 + \rho_E) \|\boldsymbol{\beta}\| \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) + 2(1 + \rho_E)k_2 \\
& \quad - 2(1 + \rho_E)k_1^2 \|\boldsymbol{\beta}\|^2 \Pi^2(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) \\
& = -2k_1 \|\boldsymbol{\beta}\|^2 + 2(1 + \rho_E) \|\boldsymbol{\beta}\| \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) + 2(1 + \rho_E)k_2 \\
& \quad + 2(1 + \rho_E) \|\boldsymbol{\beta}\| \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) - 2(1 + \rho_E) \|\boldsymbol{\beta}\| \Pi(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) \\
& \quad - 2(1 + \rho_E)k_1^2 \|\boldsymbol{\beta}\|^2 \Pi^2(\alpha, \mathbf{x}, \dot{\mathbf{x}}, t) \\
& \leq -2k_1 \|\boldsymbol{\beta}\|^2 + 2(1 + \rho_E) \|\boldsymbol{\beta}\| (\alpha - \hat{\alpha})^T \tilde{\Pi}(\mathbf{x}, \dot{\mathbf{x}}, t) \\
& \quad + 2(1 + \rho_E)k_2 + \frac{1}{2}(1 + \rho_E)k_1^{-2}. \quad (110)
\end{aligned}$$

For the second term of the right-hand-side of (101), by using adaptive law (45) and $e^{-\|\boldsymbol{\beta}\|} \leq 1$, we can obtain

$$\begin{aligned}
& 2k_1^{-1}L_1^{-1}(1 + \rho_E)(\hat{\alpha} - \alpha)^T \hat{\alpha} \\
& = 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T (\tilde{\Pi} \|\boldsymbol{\beta}\| - L_1^{-1}(L_2 e^{-\|\boldsymbol{\beta}\|} - L_3) \hat{\alpha}) \\
& = 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| - 2(1 + \rho_E)L_1^{-1} \times \\
& \quad (L_2 e^{-\|\boldsymbol{\beta}\|} - L_3)(\hat{\alpha} - \alpha)^T (\hat{\alpha} - \alpha + \alpha) \\
& \leq 2(1 + \rho_E)((\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| - 2(1 + \rho_E)L_1^{-1} \times \\
& \quad (L_2 + L_3)(\|\hat{\alpha} - \alpha\|^2 - \|\hat{\alpha} - \alpha\| \|\alpha\|)) \\
& = 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| - 2(1 + \rho_E)L_1^{-1}L_2 \times \\
& \quad (\|\hat{\alpha} - \alpha\|^2 - \|\hat{\alpha} - \alpha\| \|\alpha\|) - 2(1 + \rho_E)L_1^{-1}L_3 \times \\
& \quad (\|\hat{\alpha} - \alpha\|^2 - \|\hat{\alpha} - \alpha\| \|\alpha\|) \\
& \leq 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| + \frac{1}{2}(1 + \rho_E)L_1^{-1}L_2 \times \\
& \quad \|\alpha\|^2 - 2(1 + \rho_E)L_1^{-1}L_3(\|\hat{\alpha} - \alpha\|^2 - \|\hat{\alpha} - \alpha\| \|\alpha\|) \\
& = 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| + \frac{1}{2}(1 + \rho_E)L_1^{-1}L_2 \|\alpha\|^2 \\
& \quad - 2(1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 + \\
& \quad 2(1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\| \|\alpha\| \\
& \leq 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| + \frac{1}{2}(1 + \rho_E)L_1^{-1}L_2 \|\alpha\|^2 \\
& \quad - 2(1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 + (1 + \rho_E)L_1^{-1}L_3 \|\alpha\|^2
\end{aligned}$$

$$\begin{aligned}
& + (1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 \\
& = 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| + \frac{1}{2}(1 + \rho_E)L_1^{-1}L_2 \|\alpha\|^2 \\
& \quad - (1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 + (1 + \rho_E)L_1^{-1}L_3 \|\alpha\|^2. \quad (111)
\end{aligned}$$

With (101) and (110) (notice that $\|\boldsymbol{\varphi}\|^2 = \|\boldsymbol{\beta}\|^2 + \|\hat{\alpha} - \alpha\|^2$), we have

$$\begin{aligned}
\dot{V} & \leq -2k_1 \|\boldsymbol{\beta}\|^2 + 2(1 + \rho_E) \|\boldsymbol{\beta}\| (\hat{\alpha} - \alpha)^T \tilde{\Pi} \\
& \quad + 2(1 + \rho_E)k_2 + \frac{1}{2}(1 + \rho_E)k_1^{-2} + \frac{1}{2}(1 + \rho_E) \times \\
& \quad L_1^{-1}L_2 \|\alpha\|^2 + 2(1 + \rho_E)(\hat{\alpha} - \alpha)^T \tilde{\Pi} \|\boldsymbol{\beta}\| + \\
& \quad (1 + \rho_E)L_1^{-1}L_3 \|\alpha\|^2 - (1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 \\
& = -2k_1 \|\boldsymbol{\beta}\|^2 - (1 + \rho_E)L_1^{-1}L_3 \|\hat{\alpha} - \alpha\|^2 + 2(1 + \rho_E)k_2 \\
& \quad + \frac{1}{2}(1 + \rho_E)k_1^{-2} + \frac{1}{2}(1 + \rho_E)L_1^{-1}L_2 \|\alpha\|^2 \\
& \quad + (1 + \rho_E)L_1^{-1}L_3 \|\alpha\|^2 \\
& \leq -\rho \|\boldsymbol{\varphi}\|^2 + \Omega
\end{aligned}$$

where $\rho := \min\{2k_1, (1 + \rho_E)L_1^{-1}L_3\}$, $\Omega := (1/2) \times (1 + \rho_E)k_1^{-2} + 2(1 + \rho_E)k_2 + (1/2)(1 + \rho_E)L_1^{-1}L_2 \|\alpha\|^2 \times (1 + \rho_E)L_1^{-1}L_3 \|\alpha\|^2$.

The function $d(r)$ in Theorem 2(1) is shown as

$$d(r) = \begin{cases} \sqrt{\frac{\Psi_2}{\Psi_1}} R & \text{if } r \leq R \\ \sqrt{\frac{\Psi_2}{\Psi_1}} r & \text{if } r > R \end{cases} \quad (113)$$

$$R = \sqrt{\frac{\Omega}{\rho}} \quad (114)$$

where $\Psi_1 = \min\{\lambda_{\min}(\mathbf{P}), (k_1 L_1)^{-1}(1 + \rho_E)\}$ and $\Psi_2 = \max\{\lambda_{\max}(\mathbf{P}), (k_1 L_1)^{-1}(1 + \rho_E)\}$.

In addition, uniform ultimate boundedness in Theorem 2(2) is also obedient to

$$\underline{d} = \sqrt{\frac{\Psi_2}{\Psi_1}} R \quad (115)$$

$T(\bar{d}, r) =$

$$\begin{cases} 0, & \text{if } r \leq \bar{d} \\ \frac{\Psi_2 r^2 - (\Psi_1^2 / \Psi_2) \bar{d}^2}{\rho \bar{d}^2 (\Psi_1 / \Psi_2) - \Omega}, & \text{otherwise} \end{cases}$$

APPENDIX B

$$\begin{aligned}
\mathcal{L}_1 &= \frac{(1 + \rho_E)^2}{8\hat{\rho}^2} \\
\mathcal{L}_2 &= \frac{(1 + \rho_E)^2 L_1^{-1} L_2}{4\hat{\rho}^3} D[\|\alpha\|^2] + \frac{(1 + \rho_E)^2 L_1^{-1} L_3}{2\hat{\rho}^3} D[\|\alpha\|^2] \\
&\quad - \frac{(1 + \rho_E)}{2\hat{\rho}^2} D[V_{s2}] \\
\mathcal{L}_3 &= -\frac{(1 + \rho_E)}{2\hat{\rho}^2} D[V_{s1}] \\
\mathcal{L}_4 &= \frac{D[V_{s2}^2]}{2\hat{\rho}} + \frac{(1 + \rho_E)^2 L_1^{-1} L_2^2}{8\hat{\rho}^3} D[\|\alpha\|^4] \\
&\quad + \frac{(1 + \rho_E)^2 L_1^{-2} L_2^3}{2\hat{\rho}^3} D[\|\alpha\|^4] + \frac{(1 + \rho_E)^2 L_1^{-2} L_2 L_3}{2\hat{\rho}^3} D[\|\alpha\|^4] \\
&\quad - \frac{(1 + \rho_E) L_1^{-1} L_2}{2\hat{\rho}^2} D[V_{s2} \|\alpha\|^2] - \\
&\quad \frac{(1 + \rho_E) L_1^{-1} L_3}{\hat{\rho}^2} D[V_{s2} \|\alpha\|^2] \\
\mathcal{L}_5 &= \frac{D[V_{s1} V_{s2}]}{\hat{\rho}} - \frac{(1 + \rho_E) L_1^{-1} L_2}{2\hat{\rho}^2} D[V_{s1} \|\alpha\|^2] \\
&\quad - \frac{(1 + \rho_E) L_1^{-1} L_3}{\hat{\rho}^2} D[V_{s1} \|\alpha\|^2] \\
\mathcal{L}_6 &= \frac{D[V_{s1}^2]}{2\hat{\rho}^2} \\
\mathcal{L}_7 &= \frac{(1 + \rho_E)^2 L_1^{-1} L_2}{\hat{\rho}^3} D[\|\alpha\|^2] + \frac{2(1 + \rho_E)^2 L_1^{-1} L_3}{\hat{\rho}^3} \times \\
&\quad D[\|\alpha\|^2] - \frac{2(1 + \rho_E)}{\hat{\rho}^2} D[V_{s2}] \\
\mathcal{L}_8 &= \frac{2(1 + \rho_E)^2}{\hat{\rho}^3} \\
\mathcal{L}_9 &= \frac{(1 + \rho_E)^2}{\hat{\rho}^3} \\
\mathcal{L}_{10} &= -\frac{2(1 + \rho_E)}{\hat{\rho}^2} D[V_{s1}] \\
\kappa_1 &= \frac{(1 + \rho_E)^2}{4\hat{\rho}^2} \\
\kappa_2 &= \frac{(1 + \rho_E)^2 L_1^{-1} L_2}{2\hat{\rho}^2} D[\|\alpha\|^2] + \frac{(1 + \rho_E)^2 L_1^{-1} L_3}{\hat{\rho}^2} D[\|\alpha\|^2] \\
\kappa_3 &= \frac{(1 + \rho_E)^2 L_1^{-1} L_2^2}{4\hat{\rho}^2} D[\|\alpha\|^4] + \frac{(1 + \rho_E)^2 L_1^{-1} L_3^2}{\hat{\rho}^2} D[\|\alpha\|^4]
\end{aligned}$$

$$\begin{aligned}
&\quad + \frac{(1 + \rho_E)^2 L_1^{-1} L_2 L_3}{\hat{\rho}^2} D[\|\alpha\|^4] \\
\kappa_4 &= \frac{2(1 + \rho_E)^2 L_1^{-1} L_2}{\hat{\rho}^2} D[\|\alpha\|^2] + \\
&\quad \frac{4(1 + \rho_E)^2 L_1^{-1} L_3}{\hat{\rho}^2} D[\|\alpha\|^2] \\
\kappa_5 &= \frac{2(1 + \rho_E)^2}{\hat{\rho}^2} \\
\kappa_6 &= \frac{4(1 + \rho_E)^2}{\hat{\rho}^2}
\end{aligned}$$

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