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Symmetry breaking Paradigm In Typical Laminar-Turbulence Transition System

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Abstract

A stationary cylindrical vessel containing a rotating plate near the bottom surface is partially filled with liquid. With the bottom rotating, the shape of the liquid surface would become polygon-like. This polygon vortex phenomenon is a ideal system to demonstrate the Laminar-Turbulent transition process. Within the framework of equilibrium statistical mechanics, a profound comparison with Landau's phase transition theory was applied in symmetry breaking aspect to derive the evolution equation of this system phenomenologically. Comparison between theoretical prediction and experimental data is carried out. We concluded a considerably highly matched result, while some exceptions are viewed as the natural result that the experiment break through the up-limit of using equilibrium mechanics as a effective theory, namely breaking through the Arnold Tongue. Some extremely complex Non-equilibrium approaches was desired to solve this problem thoroughly in the future. So our method could be viewed as a linear approximation of this theoretical framework.

1 Introduction

The polygonal vortex phenomenon refers to the phenomenon that the free surface of a partially filled cylindrical container with a rotating plane near the bottom appears a polygonal shape with the rotation of the plane. This phenomenon could be divided into dry polygon and wet polygon by whether there is water or not in the core region upon the bottom plane. With the varying initial height of the liquid and rotating frequency, these two different forms of polygon vortex could be seen separately.

Polygon-like free surface formation was first reported by Vatistas(1990) [1]. In that paper, the researchers reported a periodic sloshing phenomena with the polygon formation. After this qualitative experiment, Some more detailed and quantitative experiments also were carried out with other collaborators [2][3][4]. Furthermore, some experimental research even broadened the boundary of this phenomenon in extremely huge scale and non-viscosity system[5][6][7].

On the other hand, some theoretical researches also be carried out by other groups. One of this works named Point Vortex model, which inheriting the work of Lord Kelvin [8][9]. However, some researchers doubt the strong assumption and limitations of this model in the further development[10]. Another candidate is Wave Resonance model which is regarded as the most successful model of the this field nowadays, first developed by Tophøj(2013)[11]. After that, J Mougel went further and solved the sloshing and switching problem during the evolution process within the framework of this model[12]. Then, they promote this model even further to solve the global stability problem in a common parameter space.[13]

However, the beneath symmetry breaking idea of this system reminds us that the introduction of standard phase transition paradigm in equilibrium statistical mechanics could be promising. This view point would be almost purely non-hydrodynamical, macroscopic and phenomenological while also catches the physical significance as this ideal Laminar-Turbulence transition process generally is a non-equilibrium mechanical problem.

In this writing, we will first derive a profound promotion of Landau's phase transition theory to get the evolution equation of hand-set order parameter in a transcendental way in Section.2, then we will utilize the

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most successful Wave Resonance framework to recognize a physically solid base for the hand-set order parameter in Section.3. Some fascinating outlook about this topic with non-equilibrium statistical viewpoint are putted forward in Section.4.

2 Transcendental Promotion

During the evolution history of this Polygon Vortex system, two different stages would be seen. One of them is the axisymmetric stage. In this stage, the general shape of the free surface is symmetric with respect to the central axis of the cylindrical vessel. While within the spin-up process, this geometric symmetry would be broken, and get into the non-axisymmetric stage, in which, the polygon vortex phenomenon would appear so that we conclude that the system gets through a symmetry breaking process phenomenologically, very similar with the phase transition in equilibrium statistical mechanics, and we can similarly name the different stages as *Axisymmetric phases* and *Non-axisymmetric phases*. Then the evolution history of this system would be simplified to a phase transition process, and the dominant factor is the symmetry of the free surface. This symmetries broken phenomenon characterizes the occurrence of phase transition.

The free energy G in Landau theory was replaced by the total gravitational potential E in this system. Inspired by the phenomenon that every time the pattern transfer from one to the other, the drop of the liquid height would be seen in the container[14], we consider that at that point must exist a local minima in gravitational potential, which has the similar behavior compared with the free energy in the phase transition theory.

Then a hand-set order parameter, named *Maxima frequency*, was defined as in the polar coordinates the oscillation frequency of the intersection between the polygon pattern and the given polar angle radius. To characterize the different rotation orientation, we defined it as the absolute value of a vector which has a same orientation with the angular momentum. The mathematical form of this parameter is $\mathbf{M} = f_M \hat{L}$, $\hat{L}$ is the unit vector along the angular momentum direction. More specifically, we can rigorously define this order parameter as

$$\mathbf{M} = \frac{n\omega}{4\pi} \quad (1)$$

which is also concise in form. where n is the sides number, ω is the angular velocity. This definition apparently satisfies the sufficient conditions of being a order parameter, which means it remain 0 when the system is still highly symmetric while the number jumps to a non-zero value after a symmetry breaking process happened. Besides, frequency f_M is proportional to the sides number n , which is a theoretical formulation for the experimental results of frequency locking[15]. The physical origin analysis would be shown in next section, by far, we just view it as a hand-set pragmatic parameter.

Following is the symmetry analysis part. First expanding the gravitational potential with Taylor series (near the $\mathbf{M} = 0$) $E = \sum_{j=0}^{\infty} L_j \cdot (\mathbf{M})^j$, in which L_j are the expanding coefficients. Expanding gravitational potential near the $\mathbf{M} = 0$ points means that this operation is done in the neighbouring region of the symmetry breaking point.

Due to the rotation symmetry of the system and the formula must be a scalar, all the odd terms should be canceled out. So the Taylor series reduced to

$$E = \sum_{j=0}^{\infty} L_{2j} \cdot (\mathbf{M})^{2j} \quad (2)$$

When $f \leq f_c$, $E = L_0(f)$, where f is rotation frequency, f_c is critical phase transition frequency. Calculating the L_0^{wet} term out of Euler equation for wet polygon. Then for some super shallow situation it could still be axisymmetric even after the dry area appearing. Thus we derive the L_0^{dry} term for dry polygon by volume conservation and Navier-Stokes equation(see Figure 1). Actually, when we take the derivation procedure to get the minimas, the messy L_0 can always be omitted, resulting to the irrelevance of the final results.(detailed derivation about L_0 would be found in Supplementary material.1)

After that, we could omit all the items of E above forth order as a low order approximation,

$$E = L_0(f) + Cu \cdot \nu (f_c - f)^{-1} \mathbf{M}^2 + H_{in} R (f_c - f)^{-2} \mathbf{M}^4 \quad (3)$$

in this formula, f_c is the critical frequency of phase transition, ν is kinematic viscosity of the fluid, H_{in} is the initial height of the liquid in the container, Cu is a dimensionless quantity which is originated from some solid physical considerations:

Figure 1: Sketch of the system set with parameters (Cross-sectional view), featuring the bottom plate and side wall, with different fluid patterns demonstrating both dry polygon and wet polygon

1. The coefficient of the quadratic term illustrates the dynamical constraints for the system and the dimensionless number Cu combine the effect of Coriolis force, gravitational force, centrifugal force, viscous force and momentum force.

$$Cu = \frac{(\text{Coriolis force})^3 (\text{viscous force})}{(\text{gravitational force})^2 (\text{momentum force})^2} \quad (4)$$

2. The coefficient of the fourth order term reflects the spatial restriction of the evolution of the polygon. For both H_{in} and R are the main spatial scale dominators of this fluid system.
3. The overall tool to determine the exponent is dimensional analysis.

After getting the expression of E with respect to the order parameter $\mathbf{M}$. The basic picture would become clear: with the increase of rotation frequency, the local minima, which initially is located at the $\mathbf{M} = 0$, would split to other points, as the physical meaning of local minima is referred to the physically realisable phase of the system. So accompanied with the splitting behavior, the phase transition happens. To find the local minima and get the dynamic information of them with respect to rotation frequency, with the external condition $\partial E / \partial \mathbf{M} = 0$, we could get

$$\mathbf{M} = 0 \quad \mathbf{M} = \pm \left[-\frac{L_2(f)}{2L_4(f)} \right]^{\frac{1}{2}} \hat{L} \quad (5)$$

These two solutions demonstrate us the accessible local extremas. Then, with $\partial^2 E / \partial \mathbf{M}^2 > 0$, we can gain the information that in which frequency interval, E would be the local minima.

$$\mathbf{M}(f) = \begin{cases} 0 & (f < f_c) \\ \left[\frac{1}{2} \frac{Cu \cdot \nu}{H_{in} R} (f - f_c) \right]^{\frac{1}{2}} \hat{L} & (f \geq f_c) \end{cases} \quad (6)$$

So that means when the frequency is lower than the critical phase transition value, the order parameter would remain zero. When the frequency is larger than the critical value, the transition happens. The value of $\mathbf{M}$ would be determined by the next formula, and as the $\mathbf{M} \propto n$ in (16), the sides number of the polygon would be uniquely specified.

Taking (16) into (21), the formulation of the sides number of the polygon n could be derived out with respect to the rotation frequency of the bottom plane.

$$n = \frac{1}{2} T_{po} \left[\frac{Cu \cdot \nu}{2H_{in} R} (f - f_c) \right]^{\frac{1}{2}} \quad (7)$$

where T_{po} is the rotating period of the polygon-like free surface, this formula would reflect how the sides number would evolve as the frequency increases. The dimensionless number Cu , initial height H , critical frequency and container radius R are the vital quantities which determine the evolution of the polygon vortex.

Using the experimental data of B.Bach's group[14], we preformed the comparison between experiment and the prediction of our model in Figure 2. Therefore we conclude that, with only one adjustable parameter in the

Figure 2: Thanks to the data of DOI: 10.1017/jfm.2014.568. In this phase diagram, we label experimental datas as different point parten. And the different sides numbers could be predicted by our model, labeled by vairous colored lines, where n is the sides number

formula and all the result comes from the general symmetry feature analysis of this system, the performance of our model is incredibly excellent. What's more, as the initial value could be uniquely determined by (19), we find out the following numerical value of Cu could be written by a concise form $Cu = kn$ after analyzing the experimental datas, which may also be true under different initial conditons with coresponding constant k . For we don't know exactly why it is so effectual especially with a hand-set order parameter, some further analysis from the fluid dynamics origin aspect would be benificial and neccessary.

3 Physical Origin

In this section, the solid base of our physical idea would be explored, which is originated from the Wave Resonance theory, in which we recognized a vital physical quantity named dispersion relation that has the same structure of satisfying the conditions of being an order parameter. Furthermore, their similar polygonal system evolution behaviors reval the same physical significance between them. So we assume it as the pround origin of this model.

To analyze the free surface generated wave motion, a velocity potential pertubations would be added by $\vec{v}_p = \nabla\Phi$ with the method inheriting from Tophøj(2013)[16]. Obviously, this potential would satifies the Laplace equation.

With the free surface boundary condition in a cylindrical coordinates, with its definition, the flow velocity of animuthal direction can be expressed by $U(r) = (\Gamma/2\pi r)\hat{\theta}$, where Γ is the circulation.

By solving Laplace equation, we got two different solutions which we identified them with gravity waves family and centrifugal waves family separately.

$$\begin{aligned}\Phi_c &= \left[K_1 \left(\frac{r}{R} \right)^m + K_2 \left(\frac{r}{R} \right)^{-m} \right] e^{i(m\theta - \omega t)} \Rightarrow \text{gravity waves family} \\ \Phi_g &= \left(K_3 e^{mz/R} + K_4 e^{-mz/R} \right) e^{i(m\theta - \omega t)} \Rightarrow \text{centrifugal waves family}\end{aligned}\tag{8}$$

Where m is the wave number same as n in our model(other geometrical parameter could be found in Figure 1). Then with the free surface boundary condition, we could constrain these solutions to a governing equation of the waves families

$$\left[\left(\omega - m \frac{V(R_d)}{R_d} \right)^2 - g_c m F / R_d \right] \left[\left(\omega - m \frac{V(R)}{R} \right)^2 - g m F / R \right] = \frac{m^2 g g_c}{R_d R} (F^2 - 1)\tag{9}$$

in which g_c is the centrifugal acceleration $g_c = \Gamma^2/(4\pi^2 R_d^3)$, where F is

$$F = \frac{1 + e^{-2m\zeta/R} \left(\frac{R_d}{R}\right)^{2m}}{1 - e^{-2m\zeta/R} \left(\frac{R_d}{R}\right)^{2m}} \quad (10)$$

As was mentioned in Tophøj(2013)[18], for $(F^2 - 1)$ is comparable small quantity in large m limit, we recognize the dispersion relations for the two families of waves that

$$\begin{aligned} D_g(\omega) &= \left[\left(\omega - m \frac{V(R)}{R} \right)^2 - gmF/R \right] \Rightarrow \text{Gravity waves dispersion relations} \\ D_c(\omega) &= \left[\left(\omega - m \frac{V(R_d)}{R_d} \right)^2 - g_c mF/R_d \right] \Rightarrow \text{Centrifugal waves dispersion relation} \end{aligned} \quad (11)$$

With a more careful identification, we can clarify the characteristics of the dispersion relations.

1. They are separately related with different wave families and the product of these quantity could be the dominator of this system. The remaining processdure is to solve the equation (24).
2. When the free surface remain axisymmetric, their value all would be zero, while symmetry is broken, they will get some non-zero values. The sides number of the polygon vortex is directly reflected in these quantities with the wave number m .
3. The dimensionality of these dispersion relations was $[T]^{-2}$, which is the square of the dimensionality of our hand-set order parameter.

For the 1-st and 2-nd feature we can conclude that the product of these dispersion relations could be seen as a order parameter, and with 3-rd characteristic we can conclude that the dispersion relations are power-correlated with our order parameter. So the main governing equation from purly Hydrodynamics aspect could be wirrten as this form

$$D_s(\omega) \equiv D_g(\omega)D_c(\omega) = \frac{m^2 g g_c}{R_d R} (F^2 - 1) \quad (12)$$

In which, we defined a dominate order parameter of this whole system with the notation $D_s(\omega)$. Compared with our hand-set order parameter $\mathbf{M}$ by the way of dimensional analysis, we can find the relation:

$$[D_s(\omega)]^{\frac{1}{4}} \propto m^{\frac{1}{2}} \quad (13)$$

Then with the relation $Cu = kn$ mentioned before, we can also find the relation in $\mathbf{M}$

$$|\mathbf{M}| \propto n^{\frac{1}{2}} \quad (14)$$

Thus, it's obvious that the evolution of $[D_s(\omega)]^{\frac{1}{4}}$ and $\vec{M}$ is subject to changes in $m^{\frac{1}{2}}$ and $n^{\frac{1}{2}}$, respectively. While the wave number m is the same as sides number n in our model, we can draw a conclusion that the Wave Resonance theory of Tophøj can be simplified as the work of seeking main order parameter in this system. So we just determine the dominant order parameter in two different ways. Compared with these two order parameters, the hand-set one even is almost purly based on the physical intuition of ourselves, while still own its irreplaceable advantages. For instance, as a vector, the rotating orientation could be reflected but on the other side this direction information wouldn't influent the final result which illustrates the symmetry of the system with respect to the direction of that rotation. Furthermore, as the hydrodynamics direvation is based on potential flow hypothesis and Bernoulli principle, the physical assumptions would be regarded as extremly strong confinement to the power of this order parameter from the Wave Resonance framework.

4 Conclusion

As all the previous researchs in this field are hydrodynamical-focusing, inspired by Landau's second order phase transition theory, one of the motivations of this work is to fill the blank of the modelling which is constructed systematically based on the symmetry transformation behaviors. However, with the advance of investigation, we generally realized the intrinsic focus of this model we built as it is following the paradigm of equilibrium statistical mechanics. Since this system could be identified as the classical non-equilibrium states

in *polygon vortex* phase space, the model we derived always could be seen as using equilibrium phase transition paradigm to investigate the non-equilibrium system. Some clues imply this belief clearly: multi-stability [17] as the time-inversion & phase-space-inversion symmetry violation due to the small perturbations during the evolution process; The stable states, Kelvin equilibrium, could be seen as the Limit cycle's attraction points during the evolution. The phase point in the phase space will follow the phase track absorbed in the limit cycle and falls into the attraction points. Then with increasing the perturbation level, this phase will be transformed into other stable states. And by definition, the diversity of the non-equilibrium system behaviors could be highly determined by initial setting of the parameter space volume. When the outer parameter reached some specific value, the Arnold tongue would be broken through, and the quasi-periodical behaviors that the system demonstrated before will be violated to the totally undeterminable chaos. The applicable scale of our model is limited in the periodical and quasi-periodical phase space. In the chaotic system, multiple order parameters could be helpful as Haken suggested in 1980s[18]. Or a more valid order parameter intrinsic to this system and even on Laminar-Turbulence transition would be called as complexity science suggested. In fact, these fancy concepts already existed in this some intelligent experimentists' data analysis[19], referring to the same structure as we mentioned. Some also attempted to apply them in much less complicated system modelling[20][21][22]. But for chaos and complexity, no research ever been implemented in this direction. The model in this writing could inspire the following research of the ultimate model. Along the direction of our model, a more powerful and elegant unified theory could be completed.

5 Supplementary Calculation

As we explained in the text before, when the $\vec{M} = 0$, the equation $E = L_0$ is set up. That means before the system has a symmetry breaking process, L_0 reflects the initial total gravitational potential of the symmetric system. Thus the definition of L_0 is:

$$L_0 = \frac{\int_{fluid} gz \cdot dV}{\int_{fluid} dV}$$

In this formula, Z is the height at which the cylinder is the zero potential point and V is the volume of fluid, which clearly reflects the change of gravitational potential in the system.

5.1 The calculation of L_0^{wet}

For wet polygon, firstly, we describe the system using cylindrical coordinates (r, θ, z) . Out of Euler equation, we have a relationship for the surface shape of the fluid:

$$z(r) = \frac{\Omega^2 r^2}{2g} + C(f) \quad (15)$$

Ω is the rotate speed of the cylinder. $C(f)$ is a first order constant with respect to frequency. Thus we transform the integral:

$$\int gz \cdot dV \Rightarrow \int d\theta \int r dr \int_0^{\frac{\Omega^2 r^2}{2g} + C(f)} gz \cdot dz$$

By solving this equation, we can get the result:

$$\int gz \cdot dV = 2\pi \left(\frac{1}{4} C(f)^2 g R^2 + \frac{1}{8} C(f) R^4 \Omega^2 + \frac{R^6 \Omega^4}{48g} \right) \quad (16)$$

In a similar way:

$$\begin{aligned} \int dV &\Rightarrow \int d\theta \int dr \int_0^{\frac{\Omega^2 r^2}{2g} + C(f)} dz \\ \int dV &= 2\pi \left(\frac{C(f) R^2}{2} + \frac{R^4 \Omega^2}{8g} \right) \end{aligned} \quad (17)$$

Divide(18)into(17):

$$L_0^{wet} = C(f)g + \frac{R^4 \Omega^4 - 12C(f)^2 g^2}{24C(f)g + 6R^2 \Omega^2} \quad (18)$$

Calling H_{in} the initial height of the liquid level in the cylinder, it's obviously that $\pi R^2 H_{in} = \int dV$ is the fluid volume, where $C(f)$ stay unsolven. So using the conservation of liquid, the value of $C(f)$ is uniquely determined as:

$$H_{in}\pi R^2 = 2\pi \left(\frac{C(f)R^2}{2} + \frac{R^4\Omega^2}{8g} \right)$$

$$C(f) = H_{in} - \frac{R^2\Omega^2}{4g} \quad (19)$$

Substitute $\Omega = 2\pi f$ and (19) into (18):

$$L_0^{wet} = \frac{gH_{in}}{2} + \frac{f^4\pi^4 R^4}{6gH_{in}} \quad (20)$$

5.2 The calculation of L_0^{dry}

Because The shape of the fluid body in the unperturbed equilibrium state is determined by the balance between the gravitational acceleration g and the circulation Γ [1], we can get:

$$z(r) = \frac{1}{2g} \left(\frac{\Gamma}{2\pi R} \right)^2 \left(\frac{R^2}{R_d^2} - \frac{R^2}{r^2} \right) \quad (21)$$

where R is the radius of the cylinder, R_d is the radius of the dry core. Then by the conservation of volume, we can express ζ and Γ in terms of H_{in} , R_d , and R :

$$\pi R^2 H_{in} = \int_{R_d}^R dr 2\pi r z(r)$$

Outer height of the fluid volume ζ :

$$\zeta = H_{in} \left(1 - 2 \frac{R_d^2 \ln \frac{R}{R_d}}{R^2 - R_d^2} \right)^{-1} \quad (22)$$

Circulation Γ :

$$\frac{\Gamma}{2\pi R} = \frac{R_d \sqrt{2gH_{in}}}{\sqrt{R^2 - R_d^2 - 2R_d^2 \ln \frac{R}{R_d}}} \quad (23)$$

Substitute (23) into (21):

$$z(r) = \frac{(r^2 - R_d^2)}{R^2 - R_d^2 - 2R_d^2 \ln \frac{R}{R_d}} \left(\frac{R^2}{r^2} \right) H_{in} \quad (24)$$

Take

$$\kappa(r) \equiv \frac{(r^2 - R_d^2)}{R^2 - R_d^2 - 2R_d^2 \ln \frac{R}{R_d}} \left(\frac{R^2}{r^2} \right)$$

as a fractional non-dimensional factor. Apart from the simplification of formula, it demonstrates the change of fluid surface due to the different rotation frequency and initial container radius in the cylinder. So the factor can be also seen as the compression or amplification of the H_{in} . Then we describe the system using cylindrical coordinates (r, θ, z)

$$\int gz \cdot dV \Rightarrow \int d\theta \int r dr \int_0^{\kappa \cdot H_{in} \cdot (R^2/r^2)} gz \cdot dz$$

By solving this equation, we can get the result:

$$\int gz \cdot dV = 2\pi \frac{gR^2 H_{in}^2 \left(R^4 - 4R^2 \ln \left[\frac{R}{R_d} \right] R_d^2 - R_d^4 \right)}{4 \left(R^2 - \left(1 + 2 \ln \left[\frac{R}{R_d} \right] \right) R_d^2 \right)^2} \quad (25)$$

We define $\eta = R/R_d$ (Hollow Core Ratio). As a dimensionless factor, η is varied with the rotating frequency that affects the value of R_d . So we can simplify this equation:

$$\int gz \cdot dV = 2\pi \frac{gR^2 H_{in}^2 [1 + 4\eta^2 \ln(\eta) - \eta^4]}{4 [1 - (1 - 2\eta^2 \ln(\eta)) \eta^2]^2} \quad (26)$$

For $V = \pi R^2 H_{in}$:

$$L_0^{dry} = \frac{gH_{in}}{2} \frac{[1 + 4\eta^2 \ln(\eta) - \eta^4]}{[1 - (1 - 2\eta^2 \ln(\eta)) \eta^2]^2} \quad (27)$$

Take

$$\chi(\eta) \equiv \frac{[1 + 4\eta^2 \ln(\eta) - \eta^4]}{[1 - (1 - 2 \ln(\eta)) \eta^2]^2} \quad (28)$$

Which is a dimensionless parameter controlling the dynamical evolution of the dry polygon. Then L_0^{dry} is:

$$L_0^{dry} = \frac{1}{2} g \chi(\eta) \cdot H_{in}$$

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