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Mathematical Modeling of the Impact of Rehabilitation Centers on Illicit Drug Usage

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Abstract

This paper presents a mathematical model that studies the importance of treatment at rehabilitation centers intending to show the impact of the rehabilitation centers in minimizing illicit drug usage. We consider the global stability of endemic equilibrium using the properties of Volterra-Lyapunov matrices. Numerical simulations were carried out to show the impact of rehabilitation centers on illicit drug users.

Keywords: Illicit drugs; rehabilitation centers; global stability; Volterra – Lyapunov stability; numerical simulations.

1. Introduction

Illicit drugs are organic substances that are illegally consumed in order to relieve or induce human system to act beyond natural examples of such drugs are cocaine or amphetamines, Tramadol and heroine. Illicit drugs cause hallucinogenic effect to the central nervous system which leads to trauma or complete collapse of human organs that is why, it is declared prohibited globally [1]. Based on the forensic science examination, illicit drugs represent the largest volume of criminal cases in the world. They can occur naturally as with marijuana or cocaine, or they can be prepared from naturally occurring substances such as the case with heroin, or they can be totally synthetic as in the case with amphetamines and most other prescribed drugs [2]. Despite many clinical and theoretical studies and tremendous educational campaigns, illicit drugs remain a significant threat to public health all over the world [3].

There are four major types of illicit drugs: stimulants, opioids, hallucinogens, and depressants. The effects of illicit drugs depend on the type of drug. Drugs are classified into categories based on their effects: Stimulants; consist of cocaine or methamphetamines. They pervasively cause hyperactivity and increase heart rate and brain activity [4]. Opioids; these are painkillers that also affect chemicals that regulate mood in the brain. They can also depress or slow down the central nervous system and affect breathing [4]. Hallucinogens; Marijuana, psilocybin mushrooms, and LSD are all considered hallucinogens. They alter the user's perception of space, time, and reality [4]. Depressants or sedatives; these drugs are not always illicit, but people may get addicted to prescription of medications of all kinds [4].

Illicit drugs in the United States are controlled both by the Federal and all the 50 state of the federation. The model laws adopted by the United State embodies in the Uniform Controlled Substances Act, which put drugs in one of 5 Schedules, according to potential for abuse and approved medical use. Illicit drugs rarely occur as pure substances and therefore must be separated from the cutting agents. This can be

accomplished by liquid extraction or by some form of chromatography. Almost all illicit drug cases must have a confirmatory test, such as GC/MS in order to be presented in court [2].

Globally, an estimated 183,000 drug-related deaths were reported in 2012. It is estimated that, in 2012, between 162 million and 324 million people aged between 15 and 64 had used an illicit drug [5].

In Nigeria, the commander, National Drug Law Enforcement Agency (NDLEA), Kano command, says the agency has intercepted 501.984 kilogrammes of different illicit drugs in the state in August 2019. He said the command intercepted 332.657 kilogrammes of Cannabis Savita, 167.322 Psychotropic Substance, 0.05 grams of cocaine and two gram of Khat, a drug imported from South Africa [6].

Rehabilitation is the treatment designed to revive the victims of illicit drugs through the use of relief measures and constant guidance by the experts to their normal condition. The purpose of rehabilitation; is to restore the patients mental capacity that were lost due to addictive usage of illicit drugs. Rehabilitation includes assisting the patient to regain his/her physical, social and mental being which cannot be restored medically [7]. There are three types of rehabilitations; Physical, Occupation and Speech therapy. Each rehabilitation program is to the individual patients need and can include one or more of the therapies [7].

Mathematical Model is a mathematical description of an entity or state of affairs. There are many ways in which device or behavior can be described. We can use words, drawings or sketches, physical models and mathematical formula. The use of mathematical models to explore the dynamics of drug use has been an interesting topic for a couple researchers [8-10], [19 – 22].

In 2007, white and Comiskey [10] proposed a mathematical model to evaluate the role of treatment and relapse in the dynamics of heroine. Their work exposed among others relapse of individuals who would have quit heroine. Also evidence from [11-15] shows that the four classes of drugs (opioids, amphetamines, cocaine, and cannabis) are used in most countries, but quantities estimate of such use are scarcer. This is especially so for the estimate of drug dependence. The classification of harmful drug users causes physical (e.g. organ damage) or psychological harmful (e.g. drugs – indeed psychosis). So far in literature there is no research that studies the use of mathematical modeling for rehabilitation centers in the control of illicit drug.

In this paper we construct a mathematical model that studies the impact of rehabilitation centers in the control of illicit drug usage. The population of the illicit drug users is classified into four compartments: mild illicit drug users (M), extreme illicit drug users (E), rehabilitation Centre (R) and Quitters of illicit drug (Q). Our goal is to find the impact of rehabilitation center in minimizing illicit drug usage.

The content of this paper are as follows. In section 3, we construct the mathematical model describing the classes of illicit drug users. In section 4 and section 5, we have the optimal control problem for the model and we characterize these optimal controls using pontryagin's maximum principle. In section 6, we give numerical simulations and we conclude the paper.

2. Preliminary Definitions and Theorems

Definition 1 [17]: (Basic reproduction number): Denoted as R_0 , in a susceptible population, R_0 is the measurement of average number of secondary illicit drug users produced by a typical case of an illicit drug user.

Definition 2 [18]: We write a matrix $B > 0$ (< 0) if B is symmetric positive (negative) definite.

Lemma 1 [18]: Let B be an $n \times n$ real matrix. Then all the eigen values of B have negative (positive) real parts if and only if there exist a matrix $D > 0$ such that $DB + B^T D^T < 0$ (> 0).

Definition 3 [18]: We say a non singular $n \times n$ matrix B is volterra Lyapunov stable if there exists a positive diagonal $n \times n$ matrix M such that $MB + B^T M^T < 0$.

Lemma 2 [18]: A matrix A defined as, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is volterra- Lyapunov stable if and only if

$$a_{11} < 0, a_{22} < 0 \text{ and } \text{Det}(A) = a_{11} a_{22} - a_{12} a_{21} > 0.$$

3. Model Formulation

We consider a mathematical model of the SIR form and divide the population into four compartments M, E, R and Q.

The compartment M contains mild illicit drug users, and this compartment is increased at the rate π and some of those that leave rehabilitation center at the rate $(1-\theta)$ and it is decreased when mild illicit drug users move to extreme at the rate α , decreased when mild illicit drug users decided to quit on their own at the rate γ also decreased by natural death μ .

The compartment E comprises of extreme illicit drug users. This compartment increase at rate α and decreases when some of them decide to stop it on their own at rate τ , also decrease at rate β when extreme illicit drug users join rehabilitation center and at rate δ when death occur due to extreme illicit drug usage, so also natural death at rate μ .

The compartment R encompasses those extreme illicit drug users who join rehabilitation center. It is increased by the rate β when extreme join rehabilitation center and decreased when illicit drug users quit after rehabilitation at the rate θ , and when R moves to mild illicit drug users at the rate $(1-\theta)$ and natural death μ .

The compartment Q contains those individuals that quit the use of illicit drug permanently. It is increased by those individuals who undergo treatment at the rate θ , also increased by those extreme illicit drug users who quitted without undergoing treatment at the rate τ and by mild illicit drug users who also quitted without undergoing treatment at the rate γ . It is decreased by natural death at rate μ . Tab. 1. gives the meaning of the parameters and Fig. 1. gives the schematic diagram of the model.

The model equations are derived as follow:

$$\frac{dM}{dt} = \pi + (1 - \theta)R - (\alpha + \gamma + \mu)M, \quad (1)$$

$$\frac{dE}{dt} = \alpha M - (\beta + \tau + \mu + \delta)E, \quad (2)$$

$$\frac{dR}{dt} = \beta E - (1 + \mu)R, \quad (3)$$

$$\frac{dQ}{dt} = \theta R + \tau E + \gamma M - \mu Q, \quad (4)$$

with the initial conditions, $M(0) \geq 0, E(0) \geq 0, R(0) \geq 0, Q(0) \geq 0$.

Table 1: Meaning of Parameters in model 1

S/N	Parameter	Meaning
1.	π	Recruitment rate
2.	α	Rate at which people in mild class move to extreme class
3.	β	Rate at which people in extreme class move to rehabilitation center
4.	θ	Rate at which people in rehabilitation center move to quitters class
5.	τ	Rate at which people in extreme class move to quitters class
6.	γ	Rate at which people in mild class move to quitters class
7.	δ	Death rate due to illicit drug intake
8.	μ	Natural death rate

Figure 1: Schematic diagram of the model

4. Boundedness and Equilibrium solutions

4.1 Boundedness

Let

$$N = M + E + R + Q, \quad (5)$$

Then

$$\begin{aligned} \frac{dN}{dt} &= \frac{dM}{dt} + \frac{dE}{dt} + \frac{dR}{dt} + \frac{dQ}{dt}, \\ &= \pi + (1-\theta)R - (\alpha + \gamma + \mu)M + \alpha M - (\beta + \tau + \mu + \delta)E + \beta E - (1-\theta)R - \theta R - \mu R + \theta R + \tau E + \gamma M - \mu Q, \end{aligned}$$

$$\frac{dN}{dt} = \pi - \mu N - \delta E.$$

This implies,

$$\frac{dN}{dt} \leq \pi - \mu N. \quad (6)$$

By solving the differential inequality at $N(t_0) = N_0$ we get,

$$N(t) \leq \frac{\pi}{\mu} + ke^{-\mu t}. \quad (7)$$

At $t = 0$,

$$N(0) \leq \frac{\pi}{\mu} + k,$$

$$N(t) \leq \frac{\pi}{\mu} + [N(0) - \frac{\pi}{\mu}]e^{-\mu t}.$$

as $t \rightarrow \infty$, we obtain

$$N(t) \leq \frac{\pi}{\mu}. \quad (8)$$

Hence $N(t)$ is bounded above.

4.2 Equilibrium solutions

To find the equilibrium solutions we equate system (1) to (3) to zero and solve the system simultaneously,

$$\pi + (1 - \theta)R - (\alpha + \gamma + \mu)M = 0, \quad (1^*)$$

$$\alpha M - (\beta + \tau + \mu + \delta)E = 0, \quad (2^*)$$

$$\beta E - (1 + \mu)R = 0, \quad (3^*)$$

We obtained the rehabilitation free equilibrium (E_0) as;

$$E_0 = (M^0, E^0, R^0),$$

$$M^0 = \frac{\pi}{\alpha + \gamma + \mu}, \quad (9)$$

$$E^0 = \frac{\alpha \pi}{(\alpha + \gamma + \mu)(\beta + \tau + \mu + \delta)}, \quad (10)$$

$$R^0 = 0. \quad (11)$$

The endemic equilibrium is obtained by solving equation (1*) to (3*)

Where $E_1 = (M^*, E^*, R^*)$,

$$M^* = \frac{(\beta + \tau + \mu + \delta)(1 + \mu)\pi}{(\alpha + \gamma + \mu)(\beta + \tau + \mu + \delta)(1 + \mu) - (1 - \theta)}, \quad (12)$$

$$E^* = \frac{(1 + \mu)\pi\alpha}{(\alpha + \gamma + \mu)(\beta + \tau + \mu + \delta)(1 + \mu) - (1 - \theta)}, \quad (13)$$

$$R^* = \frac{\pi\alpha\beta}{(\alpha + \gamma + \mu)(\beta + \tau + \mu + \delta)(1 + \mu) - (1 - \theta)}, \quad (14)$$

Hence we obtained two equilibrium solutions; Rehabilitation free equilibrium (E_0) and endemic equilibrium (E_1).

5. Stability Analysis of the Equilibrium Solutions of the Model

In this section we carry out the global stability analysis of the Rehabilitation free equilibrium and Endemic equilibrium.

Lemma 3: Consider a model system written in the form

$$\frac{dR}{dt} = G_1(R, I), \quad (15)$$

$$\frac{dI}{dt} = G_2(R, I), \quad (16)$$

$$G_2(R, 0) = 0. \quad (17)$$

Where $R \in \mathfrak{R}^m$, $I \in \mathfrak{R}^n$ indicate the illicit drug free and illicit drug user's population. And $E_0 = (M^0, E^0, 0)$ is the rehabilitation free equilibrium of the model.

We further establish the following conditions:

- $\frac{dR}{dt} = G_1(R, 0)$, R is globally asymptotically stable,
- $G_2(R, I) = AI + C$, with $G_2(R, I) \geq 0$.

THEOREM 1: E_0 is globally asymptotically stable.

Proof:

$$\frac{dR}{dt} = \beta E - (1 + \mu)R, \quad (18)$$

$$R(t) = \frac{\beta E}{1 + \mu} + k e^{-(1 + \mu)t}. \quad (19)$$

As t approaches ∞ ,

$R(t) \leq \frac{\beta E}{1 + \mu}$, and this confirms condition 1 holds for this model.

$$I = \begin{pmatrix} M \\ E \end{pmatrix} = \begin{pmatrix} \pi + (1 - \theta)R - (\alpha + \gamma + \mu)M \\ \alpha M - (\beta + \tau + \mu + \delta)E \end{pmatrix},$$

$$\text{But } R = \frac{\beta E}{1 + \mu},$$

$$\text{So, } I = \begin{pmatrix} \pi + \frac{\beta(1 - \theta)E}{1 + \mu} - (\alpha + \gamma + \mu)M \\ \alpha M - (\beta + \tau + \mu + \delta)E \end{pmatrix},$$

$$= \begin{pmatrix} -(\alpha + \gamma + \mu) & \frac{\beta(1-\theta)}{1+\mu} \\ \alpha & -(\beta + \tau + \mu + \delta) \end{pmatrix} \begin{pmatrix} M \\ E \end{pmatrix} + \begin{pmatrix} \pi \\ 0 \end{pmatrix} = AI + C.$$

$$\text{Where } A = \begin{pmatrix} -(\alpha + \gamma + \mu) & \frac{\beta(1-\theta)}{1+\mu} \\ \alpha & -(\beta + \tau + \mu + \delta) \end{pmatrix}, I = \begin{pmatrix} M \\ E \end{pmatrix} \text{ and } C(R, I) = \begin{pmatrix} \pi \\ 0 \end{pmatrix}.$$

If $I(0) > 0$, and $I(t) \geq 0$. Since A is an M -matrix therefore, using Variation formula, we have

$$0 \leq I(t) = e^{At} I(0) - \int_0^t e^{A(t-r)} C(R(r), I(r)) dr \leq e^{At} I(0). \quad (20)$$

Since A is M -matrix and A has a dominant eigenvalue $m(A) < 0$ and we have

$$\lim_{t \rightarrow \infty} \|e^{At}\| = 0.$$

Thus,

$$\lim_{t \rightarrow \infty} I(t) = 0. \quad (21)$$

Therefore, E_0 is globally stable.

THEOREM 2: E_1 is globally asymptotically stable.

Proof: Consider the following Lyapunov function;

$$\lambda = z_1 (R - R^*)^2 + z_2 (M - M^*)^2 + z_3 (E - E^*)^2, \quad (22)$$

Where z_1, z_2, z_3 are positive constants.

Then,

$$\begin{aligned} \frac{d\lambda}{dt} &= 2z_1 (R - R^*) \frac{dR}{dt} + 2z_2 (M - M^*) \frac{dM}{dt} + 2z_3 (E - E^*) \frac{dE}{dt}, \\ &= 2z_1 \beta (R - R^*) (E - E^*) + 2z_1 (1 + \mu) (R - R^*)^2 + 2z_2 \pi (M - M^*) + 2z_2 (1 - \theta) (M - M^*) (R - R^*) \\ &\quad - 2z_2 (\alpha + \gamma + \mu) (M - M^*)^2 + 2z_3 \alpha (E - E^*) (M - M^*) - 2z_3 (\beta + \tau + \mu + \delta) (E - E^*)^2 \\ &= Y(ZQ + Q^D Z^T) Y^T, \end{aligned} \quad (23)$$

Where $Y = [R - R^*, M - M^*, E - E^*]$, $Z = \text{diag}(z_1, z_2, z_3)$,

Now, choosing

$$Q = \begin{bmatrix} -(1+\mu) & 0 & 2\beta \\ 2(1-\theta) & -(\alpha+\gamma+\mu) & 0 \\ 0 & 2\alpha & -(\beta+\tau+\mu+\delta) \end{bmatrix}, \quad (24)$$

Then,

$$Q^D = \begin{bmatrix} -(1+\mu) & 0 & 0 \\ 0 & -(\alpha+\gamma+\mu) & 0 \\ 0 & 0 & -(\beta+\tau+\mu+\delta) \end{bmatrix}.$$

Clearly $-Q_{33} = \beta + \tau + \mu + \delta > 0$.

Reducing the matrix $-Q$ to $Q_{2 \times 2}$, named $-\tilde{Q}$,

$$-\tilde{Q} = \begin{bmatrix} 1+\mu & 0 \\ 2\theta-2 & \alpha+\gamma+\mu \end{bmatrix}, \quad (25)$$

We have $-\tilde{Q}_{11} > 0$ and $-\tilde{Q}_{22} > 0$ and the determinant

$$\det(-\tilde{Q}) = (1+\mu)(\alpha+\gamma+\mu) > 0.$$

Therefore $-\tilde{Q}$ is diagonal stable. By Lemma 2, the diagonal stability of $-\tilde{Q}^{-1}$ is determined. Now, we reduce the matrix Q^{-1} and define the matrix $H = -\tilde{Q}^{-1}$. We see that

$$H = -\tilde{Q}^{-1} = \frac{1}{\det(-Q)} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

Where

$$H_{11} = (\beta + \tau + \mu + \delta)(\alpha + \gamma + \mu),$$

$$H_{12} = -4\alpha\beta,$$

$$H_{21} = (2\theta - 2)(\beta + \tau + \mu + \delta),$$

$$H_{22} = (1 + \mu)(\beta + \tau + \mu + \delta),$$

Clearly,

$$H_{11} > 0 \text{ and } H_{22} > 0,$$

Also $\det(-Q) = (1 + \mu)(\beta + \tau + \mu + \delta)(\alpha + \gamma + \mu) + 4\alpha\beta(2\theta - 2) > 0$, (26)

Hence, $\det(-\tilde{Q}^{-1}) = \frac{1}{\det(-Q)}(H_{11}H_{22} - H_{12}H_{21}) > 0$,

$$-\tilde{Q}_{11}^{-1} = \frac{1}{\det(-Q)}H_{11} > 0,$$

$$-\tilde{Q}_{22}^{-1} = \frac{1}{\det(-Q)}H_{22} > 0,$$

Therefore, we conclude that $-\tilde{Q}^{-1}$ is diagonal stable.

6. NUMERICAL SIMULATIONS

In this section, we shall solve numerically the MERQ model. The numerical solution of model equation (1) to equation (4) is executed using Matlab2014 ODE solver with the following parameter values:

$$\pi=0.5, \alpha=0.01, \beta=0.0002, \theta=1, \tau=0.5, \gamma=0.2, \delta=0.7, \mu=0.2.$$

Figure 2: Time evolution depicting the dynamical behavior of the systems

Fig. 2. shows that the population of Mild illicit drug users decreases with respect to time. The Extreme illicit drug user's population doesn't last long, because people in the population move to rehabilitation centers, die or become mentally affected. The population of the people in Rehabilitation center class R, also decreases rapidly as its members move to Quitters class. Quitters of illicit drug usage Q class, which was initially zero increases rapidly and decreases slowly with respect to time possibly due to natural death.

Figure 3: Dynamics of the populations in the absence of Rehabilitation centers

In Fig. 3., it can be seen that in the absence of rehabilitation centers, population of both the Mild and Extreme illicit drug users decreases slower than in Fig. 2. Population of Quitters increases and later on decreases clearly, but do not increase much as there is no rehabilitation.

Figure 4: Dynamics of the populations when there are no Extreme illicit drug users

In Fig. 4. Mild class decreases slower compared to Fig. 2. but same as in Fig. 3. Rehabilitation center population moves to zero at a very short period of time and Quitters class rises and starts falling within a short period of time. This is because mild illicit drug users used illicit drugs occasionally and may quit at any moment without facing difficulty or going to rehabilitation centers.

Figure 5: Dynamics of the Populations when there are no Mild illicit Drug Users

In Fig. 5. Extreme class decreases over a very short period of time. Rehabilitation class decreases swiftly over a short time, as its members join Q class. Q class increases rapidly in a short period of time and decreases slowly reaching zero population over a long period of time.

Figure 6: Dynamics of the populations when there are no rehabilitation centers and Quitters of illicit drug usage

In Fig. 6. Mild class decreases and reaches zero after a long period of time while Extreme class decreases rapidly in a short period of time but reaches zero after some time, showing that extreme class does decrease rapidly even without going to rehabilitation center or Quitting. This is due to high death rate and psychological effect (madness).

7. CONCLUSION

This paper presents a mathematical model that describes the interaction between classes of illicit drug users. We name them as; mild illicit drug users (M), extreme illicit drug users (E), rehabilitation centers (R) and quitters of illicit drug usage (Q). The model explains the resulting outcome of treatment at rehabilitation centers intending to find the impact of rehabilitation center in minimizing illicit drug usage. We use the properties of Volterra-Lyapunov matrices to show the global stability of endemic equilibrium. We carry out numerical simulation as it plays a vital role in exhibiting the impact of rehabilitation centers on illicit drug users. Use of rehabilitation centers helps in extirpation of illicit drug users especially those that reach extreme class as they are bound to bring negative impact to the society when neglected.

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