Fractional-order LCL filters: principle, frequency characteristics, and their analysis

Junhua Xu (✉ minghuxjh@126.com)

Guangxi University <https://orcid.org/0000-0003-3959-9842>

Xiaocong Li

Guangxi University

Xueli Luo

Guangxi University

Liliang Hou

Guangxi University

Jianbo Qin

Guangxi University

Xuanren Meng

Electric Power Research Institute of Guangxi Power Grid Corporation

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Fractional-order LCL filters: principle, frequency characteristics, and their analysis

Junhua Xu ^a, Xiaocong Li ^{a,b}, Xueli Luo ^a, Liliang Hou ^a, Jianbo Qin ^a, Xuanren Meng ^c

^a College of Electrical Engineering, Guangxi University, Nanning 530004, China

^b College of Mechanical and Electrical and Quality Technology Engineering, Nanning University, Nanning 530200, China

^c Electric Power Research Institute of Guangxi Power Grid Corporation, Nanning, 530023, China

Abstract— In this paper, a fractional-order LCL (FOLCL) filter is constructed by introducing fractional-order inductors (FOIs) and fractional-order capacitors (FOCs) to replace the inductors and capacitors in a traditional integer-order LCL (IOLCL) filter, respectively. The principle and frequency characteristics of an FOLCL filter are systematically studied, and five important properties are derived and demonstrated in-depth. One of the most important achievements is that we discover the necessary and sufficient condition for the existence of resonance for an FOLCL filter, that is, the sum of the order of the FOIs and the FOC is equal to 2, which provides a theoretical basis for avoiding the resonance of an FOLCL filter effectively in design. The correctness of the theoretical derivation and analysis are verified by digital simulation. Compared with an IOLCL filter, an FOLCL filter presents more flexible and diverse operating characteristics and has a broader application prospect.

Keywords— fractional-order capacitor; fractional-order inductor; fractional-order LCL Filter; frequency characteristics; resonance

1. Introduction

The development of fractional calculus and the research on the implementation methods of fractional-order elements have opened up new approaches for the design, modelling, and analysis of circuits [1–11]. Fractional-order circuit has become an emerging area of interdisciplinary research and has drawn much interest [5].

Fractional-order inductors (FOIs) and fractional-order capacitor (FOCs) [12, 13] are the basic building elements of fractional-order circuits. The mathematical models of an FOI and an FOC can be expressed as [6–8]

$$\begin{cases} L \frac{d^\alpha i_L}{dt^\alpha} = u_L, \\ C \frac{d^\beta u_C}{dt^\beta} = i_C, \end{cases} \quad (1)$$

where, L is the inductance of an FOI, and C is the capacitance of an FOI. α and β are the order of an FOI and an FOC, respectively. α and β satisfy $0 < \alpha, \beta < 2$. Specially, when $\alpha = 1$ and $\beta = 1$, an FOI and an FOC are a traditional inductor and a capacitor,

respectively. u_L is the inductor voltage, i_L is the inductor current, u_C is the capacitor voltage, and i_C is the capacitor current.

From equation (1) we know that in addition to the traditional parameter (such as L or C), the fractional-order element introduces an additional parameter, namely order, which makes the element and system more complex and more difficult to analyze on the one hand, but on the other hand, it increases the flexibility and degree of freedom and brings in immense versatility towards design and application. For more information about the operating characteristics of FOIs and FOCs, please refer to [3, 6–8, 12, 13].

In recent years, many realizations and fabrication methods of FOIs and FOCs have been reported [14–30], and the orders of some capacitors such as super-capacitors were verified may be much faraway from 1 by experiments [27]. At the same time, FOIs and FOCs have gradually been applied to various traditional integer-order circuits, and counterpart fractional-order circuits are constructed, such as fractional-order chaotic circuits [3], fractional-order sinusoidal oscillators [4], fractional-order RLC/RC/RL circuits [6, 7, 10, 11], fractional-order power electronic converters [8, 9, 31–36], etc.

The traditional integer-order LCL (IOLCL) filter, which consists of two inductors L_1 and L_2 , and one capacitor C , is an important circuit widely applied in power electronic converters, such as be used to a grid-connected inverter [37–39]. The IOLCL filter plays a key role in filtering high-order harmonics in the AC side voltage and improving the quality of the current injected into the grid [37–39]. Compared with L filters, IOLCL filters can achieve a better high-frequency harmonic attenuation effect. However, the IOLCL filter has an inherent resonant peak, which will destroy the stability of the system when the system is not designed and controlled properly [37–39].

An FOC is introduced to replace the integer-order capacitor in an IOLCL filter to form an $LC^\alpha L$ filter in [40]. Theoretical analysis and numerical simulations show that the $LC^\alpha L$ filter provides wider bandwidth to mitigate

higher-order resonant frequencies than its integer-order counterpart. The circuit and mathematical model of a fractional-order LCL (FOLCL) filter are preliminarily given in [41]. However, the detailed theoretical derivation and analysis of the frequency characteristics are missing. Besides, we find the resonant frequency expression

$$\omega = \sqrt{\alpha + \beta} \sqrt{[(L_1 + L_2)/(L_1 L_2 C)]}$$

given in the paper is incorrect, and the correct expression of resonant frequency will be derived later in this paper.

Based on the Caputo fractional calculus [1], this paper deal with the circuit constructing, mathematical modeling, and frequency characteristic analysis of an FOLCL filter, aiming to extend the LCL filter from integer-order domain to fractional-order domain, and to systematically reveal the principle and frequency characteristics of an FOLCL filter. The rest of this paper is organized as follows: the circuit, mathematical models of an FOLCL filter are given in Section 2. In Section 3, the frequency characteristics of an FOLCL filter are studied in-depth, and five important properties of an FOLCL filter are systematically presented. In Section 4, digital simulations are performed to verify the correctness of the theoretical derivation and analysis. Finally, conclusions are drawn in Section 5.

2. The circuit and mathematical models of an FOLCL filter

By introducing the FOIs and the FOC to replace the inductors and the capacitor in an IOLCL filter, respectively, the main circuit of an FOLCL filter can be obtained as shown in Fig. 1(a). Where, the FOLCL filter is used for the AC side filtering of a grid-connected inverter. L_1 denotes the inductance of the input side FOI in units of $\text{H}/\text{sec}^{(1-\alpha_1)}$, its order is α_1 ; L_2 denotes the inductance of the output side FOI in units of $\text{H}/\text{sec}^{(1-\alpha_2)}$, its order is α_2 ; C denotes the capacitance of the FOC in units of $\text{F}/\text{sec}^{(1-\beta)}$, its order is β . All the value range of α_1 , α_2 , and β are (0, 2). Obviously, an FOLCL filter is the general form of an LCL filter, and an IOLCL filter is just the special case of an FOLCL filter when $\alpha_1 = \alpha_2 = \beta = 1$. Besides, u_{dc} is the DC voltage provided by the renewable energy generation or energy storage unit; u_i , u_g , and u_c are the input voltage, the output voltage, and the FOC's terminal voltage of an FOLCL filter, respectively. i_1 , i_2 , and i_c are the input current, the output current, and the FOC's current of an FOLCL filter, respectively. When an FOLCL filter is used for the AC

side filtering of a grid-connected inverter, u_i , u_g , and i_2 are also known as the AC side voltage of the inverter, the power grid voltage, and the current injected into the grid, respectively.

According to the Kirchhoff voltage law and the Kirchhoff current law, the time domain mathematical model of an FOLCL filter can be expressed as

$$\begin{cases} L_1 \frac{d^{\alpha_1} i_1}{dt^{\alpha_1}} = u_i - u_c, \\ C \frac{d^{\beta} u_c}{dt^{\beta}} = i_1 - i_2, \\ L_2 \frac{d^{\alpha_2} i_2}{dt^{\alpha_2}} = u_c - u_g. \end{cases} \quad (2)$$

By performing the Laplace transform on equation (2), the s -domain expression of an FOLCL filter can be obtained as

$$\begin{cases} L_1 s^{\alpha_1} i_1(s) = u_i(s) - u_c(s), \\ C s^{\beta} u_c(s) = i_1(s) - i_2(s), \\ L_2 s^{\alpha_2} i_2(s) = u_c(s) - u_g(s). \end{cases} \quad (3)$$

According to equation (3), we obtain the model structure of an FOLCL filter, as shown in Fig. 1(b).

Fig. 1. The main circuit topology and the model structure of an FOLCL filter. (a) The main circuit topology of an FOLCL filter. (b) the model structure of an FOLCL filter.

According to equation (3) and Fig. 1(b), the output current expression of an FOLCL filter can be obtained as

$$\begin{aligned} i_2(s) = & \frac{1}{L_1 L_2 C s^{\alpha_1 + \alpha_2 + \beta} + L_1 s^{\alpha_1} + L_2 s^{\alpha_2}} u_i(s) \\ & - \frac{L_1 C s^{\alpha_1 + \beta} + 1}{L_1 L_2 C s^{\alpha_1 + \alpha_2 + \beta} + L_1 s^{\alpha_1} + L_2 s^{\alpha_2}} u_g(s). \end{aligned} \quad (4)$$

In order to simplify the analysis, let $\alpha_1 = \alpha_2 = \alpha$, then equation (4) can be simplified as

$$\begin{aligned} i_2(s) = & \frac{1}{L_1 L_2 C s^{2\alpha + \beta} + (L_1 + L_2) s^{\alpha}} u_i(s) \\ & - \frac{L_1 C s^{\alpha + \beta} + 1}{L_1 L_2 C s^{2\alpha + \beta} + (L_1 + L_2) s^{\alpha}} u_g(s). \end{aligned} \quad (5)$$

Generally, $u_g(s)$ is regarded as the disturbance of the

system, so the transfer function from the input voltage $u_1(s)$ to the output current $i_2(s)$ can be obtained, that is, the transfer function of an FOLCL filter is

$$G_{gi}(s) = \frac{i_2(s)}{u_1(s)} = \frac{1}{L_1 L_2 C s^\alpha} \frac{1}{s^{\alpha+\beta} + A}. \quad (6)$$

where

$$A = (L_1 + L_2) / (L_1 L_2 C), \quad \alpha + \beta \in (0, 4). \quad (7)$$

From equation (6), we know that an FOLCL filter is a fractional-order system with order $2\alpha + \beta$, where $\alpha, \beta \in (0, 2)$. Up to now, there is no report on the

frequency characteristics of this type of fractional-order system.

According to equation (6), the frequency domain model of an FOLCL filter can be expressed as

$$G_{gi}(j\omega) = \frac{1}{L_1 L_2 C (j\omega)^\alpha} \frac{1}{(j\omega)^{\alpha+\beta} + A}. \quad (8)$$

Substituting the expression

$$(j\omega)^\alpha = \omega^\alpha e^{j\alpha\pi/2} = \omega^\alpha \cos \frac{\alpha\pi}{2} + j\omega^\alpha \sin \frac{\alpha\pi}{2}$$

into equation (8), the frequency domain expression of an FOLCL filter can be further expressed as

$$G_{gi}(j\omega) = \frac{1}{L_1 L_2 C \omega^\alpha \left(\cos \frac{\alpha\pi}{2} + j \sin \frac{\alpha\pi}{2} \right)} \frac{1}{\omega^{\alpha+\beta} \cos \frac{(\alpha+\beta)\pi}{2} + A + j \omega^{\alpha+\beta} \sin \frac{(\alpha+\beta)\pi}{2}}. \quad (9)$$

According to equation (9), the expressions of the magnitude-frequency characteristic and the phase-frequency characteristic of an FOLCL filter can be obtained as

$$|G_{gi}(j\omega)| = \frac{1/(L_1 L_2 C \omega^\alpha)}{\sqrt{\left(\omega^{\alpha+\beta} + A \cos \frac{(\alpha+\beta)\pi}{2} \right)^2 + A^2 \sin^2 \frac{(\alpha+\beta)\pi}{2}}}. \quad (10)$$

$$\angle G_{gi}(j\omega) = -\arctan \left(\tan \frac{\pi\alpha}{2} \right) - \arctan \frac{\omega^{\alpha+\beta} \sin \frac{(\alpha+\beta)\pi}{2}}{\omega^{\alpha+\beta} \cos \frac{(\alpha+\beta)\pi}{2} + A}. \quad (11)$$

3. The frequency characteristics and Analysis of an FOLCL filter

3.1 The resonance characteristic of an FOLCL filter

In equation (10), when $\alpha + \beta \in (0, 1] \cup [3, 4)$, $\cos[(\alpha + \beta)\pi/2] \geq 0$, so the denominator of $|G_{gi}(j\omega)|$ increases with the angular frequency increase from 0, namely $|G_{gi}(j\omega)|$ decreases with the angular frequency increase. Obviously, there is no resonance in the magnitude-frequency characteristic of an FOLCL filter in this case.

When $\alpha + \beta \in (1, 3)$, $\cos[(\alpha + \beta)\pi/2] < 0$, let's define an intermediate variable

$$\omega_r = \left[-A \cos(\alpha + \beta)\pi/2 \right]^{\frac{1}{\alpha+\beta}}, \quad \alpha + \beta \in (1, 3), \quad (12)$$

namely $\omega_r^{\alpha+\beta} = -A \cos[(\alpha + \beta)\pi/2]$.

Substituting the expression of ω_r into equation (10), we obtain

$$|G_{gi}(j\omega)| = \frac{1/(L_1 L_2 C \omega^\alpha)}{\sqrt{\left(\omega^{\alpha+\beta} - \omega_r^{\alpha+\beta} \right)^2 + A^2 \sin^2 \frac{(\alpha+\beta)\pi}{2}}}, \quad \alpha + \beta \in (1, 3) \quad (13)$$

When the angular frequency satisfies

$$\omega = \omega_r = \left[-A \cos \frac{(\alpha + \beta)\pi}{2} \right]^{\frac{1}{\alpha+\beta}}, \quad \alpha + \beta \in (1, 3), \quad (14)$$

$|G_{gi}(j\omega)|$ can be expressed as

$$|G_{gi}(j\omega_r)| = \frac{1/(L_1 L_2 C \omega_r^\alpha)}{A |\sin[(\alpha + \beta)\pi/2]|}, \quad \alpha + \beta \in (1, 3). \quad (15)$$

According to equation (15) we know that when $\alpha + \beta = 2$, $A \sin[(\alpha + \beta)\pi/2] = 0$, that is $|G_{gi}(j\omega_r)| = \infty$.

This means that the magnitude-frequency characteristic of an FOLCL filter has a resonance peak when $\omega = \omega_r$. On the other hand, if the magnitude-frequency characteristic of an FOLCL filter has a resonance peak when $\omega = \omega_r$, that is $|G_{gi}(j\omega_r)| = \infty$, then there must be $\alpha + \beta = 2$. For the convenient of distinguishing, we denote the ω_r when $\alpha + \beta = 2$ as ω_{rp} , which represents the resonant frequency.

According to the above analysis, we discover that an FOLCL filter has the following resonance property:

Property 1: When $\alpha_1 = \alpha_2 = \alpha$, the necessary and sufficient condition for an FOLCL filter to have a resonance peak is

$$\alpha + \beta = 2. \quad (16)$$

Substituting $\alpha + \beta = 2$ into equation (12), the expression of the resonant frequency ω_p can be obtained as follows:

$$\omega_p = \sqrt{A} = \sqrt{(L_1 + L_2)/(L_1 L_2 C)}. \quad (17)$$

According to equation (17), we further obtain the following resonant frequency property of an FOLCL filter:

Property 2: When an FOLCL filter has a resonant peak, that is the condition of $\alpha + \beta = 2$ holds, the expression of the resonant frequency is $\omega_p = \sqrt{(L_1 + L_2)/(L_1 L_2 C)}$. The resonant frequency ω_p is only determined by the values of L_1 , L_2 , and C , and is independent of the order α of the FOIs and the order β of the FOC.

Property 1 reveals the condition for the existence of the resonance of an FOLCL filter essentially and provides a direct criterion for judging whether an FOLCL filter has a resonant peak. For example, since both α and β are 1, that is $\alpha + \beta = 2$, thus an IOLCL filter has a resonant peak definitely. Furthermore, property 1 provides a theoretical basis for avoiding the resonance of an FOLCL filter effectively in design. When an FOLCL filter has a resonance peak, it is easy to cause system instability [37–39]. For an IOLCL filter, an additional passive or active damper is usually introduced to eliminate resonance and improve the stability against variations [37, 38]. However, the former will lead to additional loss and weaken the attenuation ability of high-frequency harmonics, and the latter will increase the complexity of controller design. Whereas, for an FOLCL filter, we can make $\alpha + \beta \neq 2$ by properly choosing the order of the FOIs and the FOC, so as to effectively avoid the resonance peak fundamentally and improve the stability of the system without an additional damper.

3.2 The corner frequency and log magnitude-frequency characteristic of an FOLCL filter

To analyze the corner frequency of an FOLCL filter, according to the value range of $\alpha + \beta$, the following two intermediate variables of angular frequency are defined.

$$\omega_{t1} = \left| A \cos \frac{(\alpha + \beta)\pi}{2} \right|^{\frac{1}{\alpha + \beta}}, \alpha + \beta \in (0, 0.5] \cup [1.5, 2.5] \cup [3.5, 4). \quad (18)$$

$$\omega_{t2} = \left| A \sin \frac{(\alpha + \beta)\pi}{2} \right|^{\frac{1}{\alpha + \beta}}, \alpha + \beta \in (0.5, 1.5) \cup (2.5, 3.5). \quad (19)$$

Fig. 2 shows the waveforms of $|\cos(\alpha + \beta)\pi/2|^{\frac{1}{\alpha + \beta}}$ in equation (18) and $|\sin(\alpha + \beta)\pi/2|^{\frac{1}{\alpha + \beta}}$ in equation (19). From Fig. 2 we know that when $\alpha + \beta \in (0, 0.5] \cup [1.5, 2.5] \cup [3.5, 4)$ there is $\omega_{t1} > \omega_{t2}$; when $\alpha + \beta \in (0.5, 1.5) \cup (2.5, 3.5)$, there is $\omega_{t1} < \omega_{t2}$.

Fig. 2. The waveforms of $|\cos(\alpha + \beta)\pi/2|^{\frac{1}{\alpha + \beta}}$ and $|\sin(\alpha + \beta)\pi/2|^{\frac{1}{\alpha + \beta}}$ versus to $\alpha + \beta$.

Substituting equations (18) and (19) into equation (10), the magnitude-frequency characteristic of an FOLCL filter can be expressed as

$$|G_{gi}(j\omega)| = \frac{1/(L_1 L_2 C \omega^\alpha)}{\sqrt{(\omega^{\alpha + \beta} + \sigma \omega_{t1}^{\alpha + \beta})^2 + \omega_{t2}^{2(\alpha + \beta)}}}, \quad (20)$$

where

$$\sigma = \begin{cases} 1, & \alpha + \beta \in (0, 1] \cup [3, 4), \\ -1, & \alpha + \beta \in (1, 3). \end{cases} \quad (21)$$

(1) The analysis when $\alpha + \beta \in (0, 0.5] \cup [1.5, 2.5] \cup [3.5, 4)$

According to the value range of ω , equation (20) can be further expressed as

$$|G_{gi}(j\omega)| = \begin{cases} \frac{1/(L_1 L_2 C \omega_{t1}^{\alpha + \beta} \omega^\alpha)}{\sqrt{[(\omega/\omega_{t1})^{\alpha + \beta} + \sigma]^2 + (\omega_{t2}/\omega_{t1})^{2(\alpha + \beta)}}}, & \omega \ll \omega_{t1}, \\ \frac{1/[L_1 L_2 C \omega^{2\alpha + \beta}]}{\sqrt{[1 + \sigma(\omega_{t1}/\omega)^{\alpha + \beta}]^2 + (\omega_{t2}/\omega)^{2(\alpha + \beta)}}}, & \omega \gg \omega_{t1}. \end{cases} \quad (22)$$

For the first formula of equation (22), when $\omega \ll \omega_{t1}$, $(\omega/\omega_{t1})^{\alpha + \beta} \approx 0$, considering that $\sigma^2 = 1$ always holds, so it can be simplified as

$$|G_{gi}(j\omega)| \approx \frac{1/(L_1 L_2 C \omega^\alpha)}{\sqrt{\omega_{t1}^{2(\alpha + \beta)} + \omega_{t2}^{2(\alpha + \beta)}}} = \frac{1}{L_1 L_2 C A \omega^\alpha}, \omega \ll \omega_{t1}. \quad (23)$$

According to equation (23), the log magnitude-frequency characteristic and its asymptotic slope of an FOLCL filter when $\omega \ll \omega_{t1}$ can be obtained as

$$L(\omega) \approx -20 \lg(L_1 L_2 C A) - 20\alpha \lg \omega, \omega \ll \omega_{t1}. \quad (24)$$

$$\frac{dL(\omega)}{d\lg \omega} \approx -20\alpha \text{ dB/dec}, \omega \ll \omega_{t1}. \quad (25)$$

For the second formula of equation (22), when $\omega \gg \omega_{t1}$, $(\omega_{t1}/\omega)^{\alpha+\beta} \approx 0$. Besides, considering that there is $\omega_{t1} > \omega_{t2}$ when $\alpha+\beta \in (0,0.5] \cup [1.5,2.5] \cup [3.5,4)$, then $(\omega_{t2}/\omega)^{\alpha+\beta} \approx 0$. So it can be simplified as

$$|G_{gi}(j\omega)| \approx \frac{1}{L_1 L_2 C \omega^{2\alpha+\beta}}, \omega \gg \omega_{t1}. \quad (26)$$

According to equation (26), the log magnitude-frequency characteristic and its asymptotic slope of an FOLCL filter when $\omega \gg \omega_{t1}$ can be obtained as

$$L(\omega) = -20\lg(L_1 L_2 C) - 20(2\alpha + \beta)\lg \omega, \omega \gg \omega_{t1}. \quad (27)$$

$$\frac{dL(\omega)}{d\lg \omega} \approx -20(2\alpha + \beta) \text{ dB/dec}, \omega \gg \omega_{t1}. \quad (28)$$

(2) The analysis when $\alpha + \beta \in (0.5,1.5) \cup (2.5,3.5)$

When $\alpha + \beta \in (0.5,1.5) \cup (2.5,3.5)$, according to the value range of angular frequency, equation (20) can be further expressed as

$$|G_{gi}(j\omega)| = \begin{cases} \frac{1/(L_1 L_2 C \omega^\alpha \omega_{t2}^{\alpha+\beta})}{\sqrt{[(\omega/\omega_{t2})^{\alpha+\beta} + \sigma(\omega_{t1}/\omega_{t2})^{\alpha+\beta}]^2 + 1}}, & \omega \ll \omega_{t2}, \\ \frac{1/[L_1 L_2 C \omega^{2\alpha+\beta}]}{\sqrt{[1 + \sigma(\omega_{t1}/\omega)^{\alpha+\beta}]^2 + (\omega_{t2}/\omega)^{2(\alpha+\beta)}}}, & \omega \gg \omega_{t2}. \end{cases} \quad (29)$$

For the first formula of equation (29), when $\omega \ll \omega_{t2}$, $(\omega/\omega_{t2})^{\alpha+\beta} \approx 0$, considering that $\sigma^2 = 1$ always holds, so it can be simplified as

$$|G_{gi}(j\omega)| \approx \frac{1/(L_1 L_2 C \omega^\alpha)}{\sqrt{\omega_{t1}^{2(\alpha+\beta)} + \omega_{t2}^{2(\alpha+\beta)}}} = \frac{1}{L_1 L_2 C A \omega^\alpha}, \omega \ll \omega_{t2}. \quad (30)$$

According to equation (30), we obtain

$$L(\omega) \approx -20\lg(L_1 L_2 C A) - 20\alpha \lg \omega, \omega \ll \omega_{t2}, \quad (31)$$

$$\frac{dL(\omega)}{d\lg \omega} \approx -20\alpha \text{ dB/dec}, \omega \ll \omega_{t2}, \quad (32)$$

For the second formula of equation (29), when $\omega \gg \omega_{t2}$, $(\omega_{t2}/\omega)^{\alpha+\beta} \approx 0$. Besides, considering that there is $\omega_{t2} > \omega_{t1}$ when $\alpha + \beta \in (0.5,1.5) \cup (2.5,3.5)$, then $(\omega_{t1}/\omega)^{\alpha+\beta} \approx 0$. So it can be simplified as

$$|G_{gi}(j\omega)| \approx \frac{1}{L_1 L_2 C \omega^{2\alpha+\beta}}, \omega \gg \omega_{t2}. \quad (33)$$

According to (33), we obtain

$$L(\omega) \approx -20\lg(L_1 L_2 C) - 20(2\alpha + \beta)\lg \omega, \omega \gg \omega_{t2}, \quad (34)$$

$$\frac{dL(\omega)}{d\lg \omega} \approx -20(2\alpha + \beta) \text{ dB/dec}, \omega \gg \omega_{t2}. \quad (35)$$

(3) A summary of the corner frequency and log magnitude-frequency characteristic

Based on the above theoretical derivation, we conclude that the corner frequency of an FOLCL filter when $\alpha + \beta \in (0,0.5] \cup [1.5,2.5] \cup [3.5,4)$ is ω_{t1} , and when $\alpha + \beta \in (0.5,1.5) \cup (2.5,3.5)$ is ω_{t2} . ω_{t1} and ω_{t2} can be unified to the following corner frequency.

$$\omega_t = \begin{cases} \omega_{t1} = \left| A \cos \frac{(\alpha + \beta)\pi}{2} \right|^{\frac{1}{\alpha + \beta}}, & \alpha + \beta \in (0,0.5] \cup [1.5,2.5] \cup [3.5,4), \\ \omega_{t2} = \left| A \sin \frac{(\alpha + \beta)\pi}{2} \right|^{\frac{1}{\alpha + \beta}}, & \alpha + \beta \in (0.5,1.5) \cup (2.5,3.5). \end{cases} \quad (36)$$

Besides, for the convenience of discussion, the frequency range when $\omega \ll \omega_t$ and when $\omega \gg \omega_t$ are defined as the low-frequency band and high-frequency band, respectively.

Comparing equations (30)-(35) with equations (23)-(28) one by one, we can see that when $\alpha + \beta \in (0, 0.5] \cup [1.5,2.5] \cup [3.5,4)$ and $\alpha + \beta \in (0.5,1.5) \cup (2.5,3.5)$, the expressions of the magnitude-frequency characteristic of an FOLCL filter in the two cases are the same, and the main difference lies in their corner frequency expressions.

In conclusion, the log magnitude-frequency characteristics of an FOLCL filter can be further summarized as follows.

Property 3: When $\omega \ll \omega_t$, the asymptotic slope of the log magnitude-frequency characteristic in the low-frequency band of an FOLCL filter is $-20\alpha \text{ dB/dec}$; when $\omega \gg \omega_t$, the asymptotic slope of the log magnitude-frequency characteristic in the high-frequency band is $-20(2\alpha + \beta) \text{ dB/dec}$.

The property 3 is very important for analyzing the filtering performance of an FOLCL filter. From property 3, it can be seen that the asymptote slope of the log magnitude-frequency characteristic of an FOLCL filter in the low-frequency band is only determined by the order α of the FOIs, and is independent of the order β of the FOC; while the asymptote slope in the high-frequency band is affected by both α and β . Considering that

$\alpha, \beta \in (0, 2)$, we know that the asymptote slope range of the log magnitude-frequency characteristic of an FOLCL filter are (0dB/dec, -40dB/dec) and (0dB/dec, -120dB/dec) in the low-frequency band and the high-frequency band, respectively.

According to equation (36), we find that the corner frequency of an FOLCL filter is not only affected by the values of L_1 , L_2 and C , but also related to the order α of the FOIs and the order β of the FOC. Besides, when $\alpha + \beta = 2$, there is $\omega_p = \omega_{t1}$, namely the resonant frequency is equal to the corner frequency. In particular, when $\alpha = \beta = 1$, according to equations (25), (28), and (36), we obtain the corner frequency of an IOLCL filter is $\omega_t = \sqrt{A} = \sqrt{(L_1 + L_2)/(L_1 L_2 C)}$, and the asymptote slope in the low-frequency band and the high-frequency band are -20dB/dec and -60dB/dec, respectively.

3.3 The phase-frequency characteristic of an FOLCL filter

It is known to all that for a complex number $z = x + jy$, ($z \neq 0$), the value range of $-\pi < \arg z \leq \pi$ is generally called the principal argument of z , and there is

$$\arg z = \begin{cases} \arctan(y/x), & z \text{ is in the first and fourth quadrants,} \\ \pi + \arctan(y/x), & z \text{ is in the second quadrant,} \\ -\pi + \arctan(y/x), & z \text{ is in the third quadrant,} \end{cases} \quad (37)$$

where, $-\pi/2 < \arctan(y/x) < \pi/2$.

From equation (37), the first term of equation (11) can be further expressed as

$$\arctan[\tan(\pi\alpha/2)] = \begin{cases} \pi\alpha/2, & \alpha \in (0, 1], \\ (\pi\alpha/2 - \pi) + \pi = \pi\alpha/2, & \alpha \in (1, 2), \end{cases} \quad (38)$$

which means $\arctan[\tan(\pi\alpha/2)] = \pi\alpha/2$ always holds when $\alpha \in (0, 2)$.

To facilitate the derivation, let the center frequency

$$\omega_o = A^{1/(\alpha+\beta)} = \omega_o^{\alpha+\beta} \sqrt{L_1 + L_2 / (L_1 L_2 C)}, \quad (39)$$

then, the equation (11) can be further simplified as

$$\angle G_{gi}(j\omega) = -\frac{\pi\alpha}{2} - \arctan \frac{\sin \frac{(\alpha+\beta)\pi}{2}}{\cos \frac{(\alpha+\beta)\pi}{2} + (\omega_o/\omega)^{\alpha+\beta}}. \quad (40)$$

When $\omega \ll \omega_o$, considering that the characteristics of the sin and cos functions, the following relationship always holds.

$$\sin \frac{(\alpha+\beta)\pi}{2} \ll \cos \frac{(\alpha+\beta)\pi}{2} + (\omega_o/\omega)^{\alpha+\beta},$$

then, there is

$$-\arctan \frac{\sin \frac{(\alpha+\beta)\pi}{2}}{\cos \frac{(\alpha+\beta)\pi}{2} + (\omega_o/\omega)^{\alpha+\beta}} \approx 0, \quad \omega \ll \omega_o.$$

Therefore, the phase-frequency characteristic of an FOLCL filter when $\omega \ll \omega_o$ is

$$\angle G_{gi}(j\omega) \approx -\pi\alpha/2, \quad \omega \ll \omega_o. \quad (41)$$

When $\omega \gg \omega_o$, $\omega_o/\omega \approx 0$, so equation (40) can be further simplified as

$$\angle G_{gi}(j\omega) \approx -\frac{\pi\alpha}{2} - \arctan \left[\tan \frac{(\alpha+\beta)\pi}{2} \right], \quad \omega \gg \omega_o. \quad (42)$$

According to equation (37), the second term in equation (42) can be further expressed as

$$\arctan \left[\tan \frac{(\alpha+\beta)\pi}{2} \right] = \begin{cases} \frac{(\alpha+\beta)\pi}{2}, & \alpha + \beta \in (0, 2], \\ \frac{(\alpha+\beta)\pi}{2} - 2\pi, & \alpha + \beta \in (2, 4). \end{cases} \quad (43)$$

By synthesizing equations (42) and (43), the phase-frequency characteristic of an FOLCL filter when $\omega \gg \omega_o$ can be obtained as

$$\angle G_{gi}(j\omega) \approx \begin{cases} -\pi(\alpha + \beta/2), & \alpha + \beta \in (0, 2], \omega \gg \omega_o, \\ -\pi(\alpha + \beta/2) + 2\pi, & \alpha + \beta \in (2, 4), \omega \gg \omega_o. \end{cases} \quad (44)$$

Based on equations (41) and (44), the phase-frequency characteristics of an FOLCL filter can be summarized as follows.

Property 4: When $\omega \ll \omega_o$, the phase of an FOLCL filter is $-\pi\alpha/2$; when $\omega \gg \omega_o$, if $\alpha + \beta \in (0, 2]$, the phase is $-(\alpha + \beta/2)\pi$; if $\alpha + \beta \in (2, 4)$, the phase is $-(\alpha + \beta/2)\pi + 2\pi$.

According to property 4, we can see that the phase of an FOLCL filter when $\omega \ll \omega_o$ is only determined by the order α of the FOIs, and independent of the order β of the FOC; while the phase when $\omega \gg \omega_o$ is affected by both α and β .

3.4 The phase crossover frequency and gain margin of an FOLCL filter

It is well known that the frequency at which the open-loop phase-frequency characteristic intersects the horizontal line $-\pi$ is usually referred to as the phase crossover frequency, and denoted as ω_g . For an FOLCL filter,

according to equation (40) we know that the phase crossover frequency ω_g should satisfy the following equation.

$$\angle G_{gi}(j\omega_g) = -\frac{\pi\alpha}{2} - \arctan \frac{\sin \frac{(\alpha+\beta)\pi}{2}}{\cos \frac{(\alpha+\beta)\pi}{2} + (\omega_o/\omega_g)^{\alpha+\beta}} = -\pi. \quad (45)$$

By solving equation (45), we obtain

$$\omega_g^{\alpha+\beta} = \frac{A \sin[(1-\alpha/2)\pi]}{-\sin[(\alpha+\beta/2)\pi]}. \quad (46)$$

Because $0 < \alpha < 2$, there is $0 < (1-\alpha/2)\pi < \pi$, so the numerator of equation (46) satisfies $A \sin[(1-\alpha/2)\pi] > 0$. Considering that the value range of α and β , if and only if $\alpha + \beta/2 \in (1, 2)$, the denominator of equation (46) satisfies $-\sin[(\alpha+\beta/2)\pi] > 0$, then the right side of equation (46) is greater than 0, that is, there is a real number solution for equation (46) and a phase crossover frequency. To sum up, the phase crossover frequency characteristic of an FOLCL filter can be summarized as follows.

Property 5: If and only if $\alpha + \beta/2 \in (1, 2)$, the phase-frequency characteristic of an FOLCL filter has a phase crossover frequency ω_g , and the expression of ω_g is

$$\omega_g = \sqrt[\alpha+\beta]{\frac{A \sin[(1-\alpha/2)\pi]}{-\sin[(\alpha+\beta/2)\pi]}}. \quad (47)$$

In particular, when $\alpha = \beta = 1$, there is $\omega_g = \sqrt{A} = \sqrt{(L_1 + L_2)/(L_1 L_2 C)} = \omega_p$, that is, the phase crossover frequency is equal to the resonant frequency.

When $\alpha + \beta/2 \in (1, 2)$, the gain margin of an FOLCL filter and its logarithmic expression can be obtained as

$$K_g = \frac{1}{|G_{gi}(j\omega_g)|} = L_1 L_2 C \omega_g^\alpha \sqrt{\omega_g^{2(\alpha+\beta)} + 2A\omega_g^{\alpha+\beta} \cos \frac{(\alpha+\beta)\pi}{2} + A^2}. \quad (48)$$

$$20K_g = -20\lg|G_{gi}(j\omega_g)| = -20\lg \frac{1/L_1 L_2 C \omega_g^\alpha}{\sqrt{\omega_g^{2(\alpha+\beta)} + 2A\omega_g^{\alpha+\beta} \cos \frac{(\alpha+\beta)\pi}{2} + A^2}}. \quad (49)$$

3.5 The gain crossover frequency and phase margin of an FOLCL filter

As is known to all, the frequency ω_c satisfies

$|G_{gi}(j\omega_c)| = 1$ or $20\lg|G_{gi}(j\omega_c)| = 0$ is usually referred to as the phase crossover frequency. According to equation (10), we know that the gain crossover frequency ω_c of an FOLCL filter should satisfy the following equation.

$$\frac{1/(L_1 L_2 C \omega_c^\alpha)}{\sqrt{\omega_c^{2(\alpha+\beta)} + 2A\omega_c^{\alpha+\beta} \cos \frac{(\alpha+\beta)\pi}{2} + A^2 \sin^2 \frac{(\alpha+\beta)\pi}{2}}} = 1. \quad (50)$$

By sorting out equation (50), we obtain

$$\omega_c^{2(2\alpha+\beta)} + 2A\omega_c^{3\alpha+\beta} \cos \frac{(\alpha+\beta)\pi}{2} + A^2 \omega_c^{2\alpha} - \frac{A^2}{(L_1 + L_2)^2} = 0. \quad (51)$$

The above formula is a very complicated nonlinear equation, it is very difficult to find the analytical solution of the gain crossover frequency ω_c of an FOLCL filter. In order to obtain the trend graph of the change of ω_c with the order of the FOIs and the order of the FOC, we successively used the fsolve function, the particle swarm optimization algorithm [42], and the differential evolution (DE) algorithm [43] to solve equation (49) based on MATLAB software and find the DE algorithm has the best solution effect and relatively stable results.

Fig. 3 shows the change diagram of the digital solution of ω_c obtained by the DE algorithm versus $\alpha - \beta$ plane when $\alpha, \beta \in (0.8, 1.2)$. Where, The main circuit parameters of an FOLCL filter as: $L_1 = 600 \mu\text{H}/\text{sec}^{(1-\alpha)}$, $L_2 = 150 \mu\text{H}/\text{sec}^{(1-\alpha)}$, $C = 10 \mu\text{F}/\text{sec}^{(1-\beta)}$.

Fig. 3. The change diagram of the digital solution of ω_c obtained by the DE algorithm versus $\alpha - \beta$ plane.

From Fig. 3, we can see that when the combination of α and β is near $\alpha + \beta = 2$, there is a phenomenon that the value of ω_c suddenly increases. The reason is that an FOLCL filter has a resonance peak when $\alpha + \beta = 2$, which makes the log magnitude-frequency characteristic curve cross the 0dB line many times. This phenomenon will be further verified in the following digital simulations.

Besides, according to property 3 and Fig. 3, we know:

(1) Except for the case of $\alpha + \beta = 2$, as the order α of the FOIs increases, the asymptotic slope of the log magnitude -frequency characteristic curve in the low-frequency band increases correspondingly, and the gain crossover frequency decreases.

(2) Considering that the order β of the FOC does not affect the low-frequency asymptotic slope of the log magnitude -frequency characteristic curve of an FOLCL filter, except for the case of $\alpha + \beta = 2$, changing the order β theoretically will not affect the gain crossover frequency.

The formula for calculating the phase margin of an FOLCL filter is the same as that of an IOLCL filter, namely

$$\gamma = 180^\circ + \angle G_{\text{gi}}(j\omega_c). \quad (52)$$

4. Simulation results and analysis

In order to verify the correctness of the theoretical derivation, analyze the frequency characteristics of an FOLCL filter, and further explore the influence of the orders of the FOIs and the FOC on the frequency characteristics of an FOLCL filter, this paper establishes the digital model of an FOLCL filter and perform

simulations based on MATLAB software. The main circuit parameters of an FOLCL filter as the same as in Fig. 3 are employed, and the orders of the FOIs and the FOC will be given later.

4.1 The frequency characteristic simulation curves of an FOLCL filter

(1) The frequency characteristic curves when α is constant and β changes

Simulation condition I: let the order α of the FOIs be 0.8, 1.0 and 1.2 respectively. The order β of the FOC in each case is increased from 0.6 to 1.4 at intervals of 0.2.

The Bode diagram of an FOLCL filter based on Simulation condition I as shown in Fig. 4 and the corresponding frequency characteristic indicators are summarized in Table I. Where *Nan* means that there is no indicator, *inf* denotes that the indicator is infinite.

(2) The frequency characteristic curves when β is constant and α changes

Simulation condition II: let the order β of the FOC be 0.8, 1.0 and 1.2 respectively. The order α of the FOIs in each case is increased from 0.6 to 1.4 at intervals of 0.2.

The Bode diagram of an FOLCL filter as shown in Fig. 5 and the corresponding frequency characteristic indicators are summarized in Table II.

Fig. 4. The frequency characteristic curves of an FOLCL Filter when α is constant and β changes. (a) The frequency characteristics when $\alpha = 0.8$. (b) The frequency characteristics when $\alpha = 1.0$. (c) The frequency characteristics when $\alpha = 1.2$.

Fig. 5. The frequency characteristic curves of an FOLCL Filter when β is constant and α changes. (a) The frequency characteristics when $\beta = 0.8$. (b) The frequency characteristics when $\beta = 1.0$. (c) The frequency characteristics when $\beta = 1.2$.

Table I

The frequency characteristic indicators of an FOLCL Filter when α is constant and β changes.

Serial number	α	β	ω_g (rad/s)	$20\lg K_g$ (dB)	ω_c (rad/s)	γ (°)	ω_t (rad/s)
1	0.8	0.6	5257083	53.39	8059	107.98	2024102
2	0.8	0.8	508310	28.80	8075	107.93	329599
3	0.8	1.0	98864	9.06	8186	107.76	87883
4	0.8	1.2	28867	-79.03	33194	-72.00	28867
5	0.8	1.4	NaN	Inf	13749	241.70	11092
6	1.0	0.6	429574	47.38	1334	90.00	329598
7	1.0	0.8	92922	27.10	1334	89.99	87883
8	1.0	1.0	28867	-75.58	29510	90.00	28867
9	1.0	1.2	NaN	Inf	1345	90.16	11092
10	1.0	1.4	NaN	Inf	1379	91.43	4772
11	1.2	0.6	87883	45.96	402	71.99	87882
12	1.2	0.8	28867	-57.73	28950	-108.0	28867
13	1.2	1.0	NaN	Inf	402	72.01	11092
14	1.2	1.2	NaN	Inf	402	72.07	4772
15	1.2	1.4	NaN	Inf	403	72.33	2487

Table II

The frequency characteristic indicators of an FOLCL Filter when β is constant and α changes.

Serial number	α	β	ω_g (rad/s)	$20\lg K_g$ (dB)	ω_c (rad/s)	γ (°)	ω_t (rad/s)
1	0.6	0.8	19437846	50.51	16538	124.85	2024630
2	0.8	0.8	508310	28.80	8075	107.93	329599
3	1.0	0.8	92922	27.10	1334	89.99	87883
4	1.2	0.8	28867	-57.73	28950	-108.0	28867
5	1.4	0.8	NaN	Inf	171	54.00	11092
6	0.6	1.0	686670	13.13	368962	56.77	329598
7	0.8	1.0	98864	9.06	8186	107.76	87882
8	1.0	1.0	28867	-75.58	29510	-90.00	28867
9	1.2	1.0	NaN	Inf	402	72.01	11092
10	1.4	1.0	NaN	Inf	171	54.01	4772
11	0.6	1.2	107916	-7.69	133810	-19.98	87883
12	0.8	1.2	28867	-93.42	33192	-72.00	28867
13	1.0	1.2	NaN	Inf	1345	90.16	11092
14	1.2	1.2	NaN	Inf	402	72.07	4771
15	1.4	1.2	NaN	Inf	170.7	54.04	2487

4.2 The analysis of the frequency characteristics of an FOLCL Filter

From Fig. 4, Fig. 5, Table I, and Table II, we know:

(1) When $\alpha + \beta = 2$, such as the combinations of α and β are (0.8,1.2), (1.0,1.0) and (1.2,0.8), the magnitude-frequency characteristic curves of these FOLCL filters cross the 0dB line many times and have a resonance peak at the resonant frequency; besides, the phase-frequency characteristic curves jump -180° at the resonant frequency, for example, when the combination of α and β is (0.8,1.2), the phase-frequency characteristic curve from -72° jump to -252° . Furthermore,

it can be seen that the resonant frequency ω_p is only determined by L_1 , L_2 and C , and is independent of α and β . In this case, the theoretically calculated resonant frequency is

$$\omega_p = \sqrt{(L_1 + L_2)/(L_1 L_2 C)} \approx 28867.5 \text{ rad/s},$$

and the measured resonant frequency from the curves is 28867 rad/s. Obviously, the two frequencies are basically equal, and the slight error is due to the fact that the curves as shown in the Bode diagram is connected by discrete points. In conclusion, the correctness of property 1 and property 2 have been verified.

(2) In each sub-graphs of Fig. 4, when the order α of the FOIs is constant and the order β of the FOC changes, the low-frequency bands of the five log magnitude-frequency characteristic curves overlap and the high-frequency bands diverge, which indicates that the order β of the FOC does not affect the log magnitude-frequency characteristic of an FOLCL filter in the low-frequency band, but it will affect the characteristic in the high-frequency band. The asymptotic slope of the log magnitude-frequency characteristic curve of an FOLCL filter in the low-frequency band is $-20\alpha \text{ dB/dec}$, and this slope is only related to α . For example, when α is 0.8, 1.0, and 1.2, the theoretically low-frequency slopes calculated based on property 3 are -16 dB/dec , -20 dB/dec , and -24 dB/dec , and the measured values from the curves are -15.96 dB/dec , -19.9 dB/dec , and -23.87 dB/dec , respectively, which shows that the theoretical calculated values are consistent with the actual measured values.

Besides, the asymptotic slope of the log magnitude-frequency characteristic curve in the high-frequency band is $-20(2\alpha + \beta) \text{ dB/dec}$, and this slope is affected by both α and β . For example, when the combinations of α and β are (0.8,1.4), (1.0,1.4), and (1.2,1.4), the theoretically high-frequency slopes are -60 dB/dec , -68 dB/dec , and -76 dB/dec , and the measured values from the curves are -60.07 dB/dec , -68.01 dB/dec , and -76.00 dB/dec , respectively, which demonstrates that the theoretical calculated values are consistent with the actual measured values, too.

In each sub-graphs of Fig. 5, when β is constant while α takes different values, the slopes of the low and high-frequency bands of the five log magnitude-frequency characteristic curves are different, which further shows

that the asymptote slope of the log magnitude-frequency characteristic of an FOLCL filter in the low-frequency band is only determined by α , while the asymptote slope in the high-frequency band is affected by both α and β .

Furthermore, when $\alpha + \beta \in (0, 0.5] \cup [1.5, 2.5] \cup [3.5, 4)$, the corner frequency ω_c is calculated according to the first formula of (36), for example, when $\alpha = \beta = 1.0$, the theoretically calculated and the measured corner frequency ω_c is 28868 rad/s and 28867 rad/s, respectively, which are basically equal. When $\alpha + \beta \in (0.5, 1.5) \cup (2.5, 3.5)$, the corner frequency should be calculated according to the second formula of (36), for example, when the combinations of α and β are (0.8, 0.6) and (1.2, 1.4), the theoretically calculated ω_c are 2024272 rad/s and 2487 rad/s, respectively; and the corner frequencies obtained from the curve are 2024102 rad/s and 2487 rad/s, respectively. Which shows that the theoretical calculated values are consistent with the actual measured values.

To sum up, the correctness of property 3 has been verified.

(3) In each sub-graphs of Fig. 4, when α is constant and β changes, the five phase-frequency characteristic curves overlap when $\omega \ll \omega_0$ and diverge when $\omega \gg \omega_0$, which indicates that β only affects the phase-frequency characteristic of an FOLCL filter in the high-frequency band, but not in the low-frequency band. The low-frequency phase changes according to the law of $-\pi\alpha/2$. For example, when α is 0.8, 1.0, and 1.2, both the theoretically calculated and the measured low-frequency phases are -72° , -90° , and -108° , respectively, and we can see the low-frequency phase is only determined by α .

Besides, when $\alpha + \beta \in (0, 2]$, the high-frequency phase changes according to the law of $-(\alpha + \beta/2)\pi$. For example, when the combinations of α and β are (0.8, 0.6), (1.0, 0.6), and (1.2, 0.6), both the theoretically calculated and the measured high-frequency phases are equal and are -198° , -234° , and -270° , respectively. When $\alpha + \beta \in (2, 4)$, the high-frequency phase changes according to the law of $-(\alpha + \beta/2)\pi + 2\pi$. For example, when the combinations of α and β are (0.8, 1.4), (1.0, 1.4), and (1.2, 1.4), both the theoretically calculated and the measured high-frequency phases are equal and

are 90° , 54° , and 18° , respectively.

On the whole, the correctness of property 4 has been verified.

(4) The measured value of the phase crossover frequencies ω_g shown in Table I and Table II are equal to the theoretical value calculated by (47). For example, when the combinations of α and β are (0.8, 0.8), (0.8, 1.0), (1.0, 1.0), and (0.6, 1.2), both the calculated and the measured phase crossover frequencies are equal and are 508310 rad/s, 98864 rad/s, 28867 rad/s, and 107916 rad/s, respectively. In conclusion, the correctness of property 5 has been verified.

(5) From Table I and Fig. 4, we know that when α is constant and β changes, the gain crossover frequency ω_c is almost constant except for $\alpha + \beta = 2$. For example, when the combinations of α and β are (1.0, 0.8) and (1.0, 1.2), the gain crossover frequencies are basically equal and are 1334 rad/s and 1345 rad/s, respectively. Besides, when $\alpha + \beta$ approaches 2, there is a resonance in the magnitude-frequency characteristic of an FOLCL filter, so the gain crossover frequency will increase suddenly. Furthermore, from Table II and Fig. 5, we know that when β is constant and α changes, with the decrease of α , the gain crossover frequency increases gradually except for $\alpha + \beta = 2$. On the whole, these simulation results are consistent with the phenomenon shown in Fig. 4.

In general, the correctness of the theoretical derivation and analysis of the frequency characteristics of an FOLCL filter is verified by digital simulation, and the influences of the order α of the FOIs and the order β of the FOC on the frequency characteristics of an FOLCL filter has been highlighted.

5. Conclusion

On the basis of constructing the main circuit of a more general LCL filter namely an FOLCL filter and establishing its mathematical models, this paper pioneered the theoretical analysis method for the frequency characteristics of an FOLCL filter, and systematically analyzed and summarized the frequency characteristics of an FOLCL filter for the first time. Five important properties of an FOLCL filter are derived and demonstrated in-depth, which are the most important theoretical achievements in this paper. The main differences of frequency characteristic between an IOLCL filter and an FOLCL filter are summarized in Table III.

Table III The comparison between an IOLCL filter and an FOLCL filter.

Properties	IOLCL filters	FOLCL filters	Notes on FOLCL filters
Variables	Three Variables (L_1, L_2, C)	Five Variables ($L_1, L_2, C, \alpha, \beta$)	α is the order of the FOIs, and β is the order of the FOC.
Range of α and β	$\alpha = \beta = 1$	$\alpha, \beta \in (0, 2)$	An IOLCL filter is the special case of an FOLCL filter when $\alpha = \beta = 1$.
The transfer function, $G_{gi}(s)$	$\frac{1}{L_1 L_2 C s^2 + A}$	$\frac{1}{L_1 L_2 C s^\alpha} \frac{1}{s^{\alpha+\beta} + A}$	
Resonance peak	Exists a resonance peak	Exists a resonance peak when $\alpha + \beta = 2$	The necessary and sufficient condition for the existence of a resonance peak is $\alpha + \beta = 2$.
Resonant frequency, ω_p	$\sqrt{(L_1 + L_2)/(L_1 L_2 C)}$	$\sqrt{(L_1 + L_2)/(L_1 L_2 C)}$	ω_p is determined by the values of L_1 , L_2 , and C , and is independent of α and β .
Corner frequency, ω_t	$\sqrt{(L_1 + L_2)/(L_1 L_2 C)}$	$\begin{cases} \omega_{t1} = \alpha+\beta \sqrt{A \cos[(\alpha + \beta)\pi/2]} \\ \alpha + \beta \in (0, 0.5] \cup [1.5, 2.5] \cup [3.5, 4) \\ \omega_{t2} = \alpha+\beta \sqrt{A \sin[(\alpha + \beta)\pi/2]} \\ \alpha + \beta \in (0.5, 1.5) \cup (2.5, 3.5) \end{cases}$	According to the different value range of $\alpha + \beta$, there are two calculation formulas of ω_t . ω_t is affected by both α and β .
Slope of the log magnitude-frequency characteristic, $dL(\omega)/d\lg \omega$	$\begin{cases} -20\text{dB/dec}, \omega \ll \omega_t \\ -60\text{dB/dec}, \omega \gg \omega_t \end{cases}$	$\begin{cases} -20\alpha\text{dB/dec}, \omega \ll \omega_t \\ -20(2\alpha + \beta)\text{dB/dec}, \omega \gg \omega_t \end{cases}$	The slope is only determined by α when $\omega \ll \omega_t$, while is affected by both α and β when $\omega \gg \omega_t$. The range of slope is (0dB/dec, -40dB/dec) and (0dB/dec, -120dB/dec) when $\omega \ll \omega_t$ and $\omega \gg \omega_t$, respectively.
Center frequency, ω_o	$\sqrt{(L_1 + L_2)/(L_1 L_2 C)}$	$\alpha+\beta \sqrt{(L_1 + L_2)/(L_1 L_2 C)}$	ω_o is affected by both α and β .
Phase-frequency characteristic, $\angle G_{gi}(j\omega)$	$\begin{cases} -\pi/2, \omega < \omega_o \\ -3\pi/2, \omega > \omega_o \end{cases}$	$\begin{cases} -\pi\alpha/2, \omega < \omega_o \\ -\pi(\alpha + \beta/2), \alpha + \beta \in (0, 2], \omega > \omega_o \\ -\pi(\alpha + \beta/2) + 2\pi, \alpha + \beta \in (2, 4), \omega > \omega_o \end{cases}$	$\angle G_{gi}(j\omega)$ is only determined by α when $\omega \ll \omega_o$, while is affected by both α and β when $\omega \gg \omega_o$. The high-frequency phase curve changes from $-\pi\alpha/2$ to the more lagging direction when $\alpha + \beta \in (0, 2]$, and to the opposite direction when $\alpha + \beta \in (2, 4)$.
Phase crossover frequency, ω_g	$\sqrt{(L_1 + L_2)/(L_1 L_2 C)}$	$\alpha+\beta \sqrt{\frac{A \sin[(1-\alpha/2)\pi]}{-\sin[(\alpha + \beta/2)\pi]}}$	If and only if $\alpha + \beta/2 \in (1, 2)$, the phase-frequency characteristic of an FOLCL filter has a ω_g .

Compared with an IOLCL filter, since the extra parameters of α and β are included, although an FOLCL filter is more complicated to analyze, it has many novel behaviors and a wider operating range, which brings great flexibility and advantages to the application. The research in this paper provides a theoretical basis for the design of an FOLCL filter. By appropriately selecting the values of α and β , the FOLCL filter which can better coordinate filtering performance and system stability can be obtained. In the future, an FOLCL filter is expected to apply to replace the integer-order LCL filter in traditional integer-order power electronic systems (such as inverters and rectifiers), and to build fractional-order power electronic systems with better performance.

Besides, the frequency characteristic analysis method of an FOLCL filter presented in this paper fills the gap in the research of the frequency characteristic of a general fractional system with $2\alpha + \beta$ -order (where $\alpha, \beta \in (0, 2)$), and has good theoretical significance.

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Junhua Xu was born in Hunan, China. He received his M.S. degree in 2011 and Ph.D. degree in 2021 from the College of Electrical Engineering at Guangxi University, China, both in Electrical Engineering. He joined the College of Electrical Engineering at Guangxi University in 2012. Now he is a master supervisor. His research interests include power system stability and control; fractional-order system modeling, analysis, and control. Corresponding author, E-mail: minghuxjh@126.com.

Xiaocong Li is a professor, doctoral supervisor with College of Electrical Engineering, Guangxi University, Nanning, China. And as a guest professor, he is employed by Nanning University, Nanning, China. His research interests are power system stability and control, power system nonlinear control. Corresponding author, E-mail: lhtlht@gxu.edu.cn.

Xueli Luo is a M.S. student at College of Electrical Engineering of Guangxi University. Her research interest is fractional-order system modeling, analysis, and control .

Liliang Hou is a M.S. student at College of Electrical Engineering of Guangxi University. His research interest is fractional-order system modeling, analysis, and control.

Jianbo Qin received the B.S. degree in automation from Guilin University of Electronic Technology in 2006 and the M.S. degree in control theory and control Engineering from Guangdong University of Technology in 2009, China. He is currently pursuing the Ph.D. in the College of Electrical Engineering, Guangxi University, China. His research interests include power system stability and control.

Xuanren Meng received his B.S. degree in 2017 and M.S. degree in 2020 from the College of Electrical Engineering at Guangxi University, China, both in Electrical Engineering. He joined the Electric Power Research Institute of Guangxi Power Grid Corporation in 2020. His research interest is power system non-linear control; power system relay protection; fractional-order system modeling, analysis, and control.