The way to list all infinite real numbers and to construct a bijection between natural numbers and real numbers

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Research Article

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Abstract

Georg Cantor defined countable and uncountable sets for infinite sets. The set of natural numbers is defined as a countable set, and the set of real numbers is proved to be uncountable by Cantor's diagonal argument. Most scholars accept that it is impossible to construct a bijection between the set of natural numbers and the set of real numbers. However, the way to construct a bijection between the set of natural numbers and the set of real numbers is proposed in this paper. The set of real numbers can be proved to be countable by Cantor's definition. Cantor's diagonal argument is challenged because it also can prove the set of natural numbers to be uncountable. The process of argumentation provides us new perspectives to consider about the size of infinite sets.

Introduction

"Infinite" is an unclear concept, and many scholars try to describe or define it. Most mathematicians believe that "infinite for natural numbers" and "infinite for real numbers" are different. Mathematicians can easily list all infinite natural numbers by order: 1, 2, 3, 4, 5…. However, people cannot give any rule to list all infinite real numbers by order. In set theory, the sets with infinite members are concerned and debated. Georg Cantor defined countable and uncountable sets for infinite sets. For example, the set of natural numbers (N) is countable, and the set of real numbers (R) is uncountable. The main concepts of Cantor's definition for countable sets are:

Concept 1: If all of the infinite members in a set can be listed by any rule, then the infinite set is countable. Otherwise, the infinite set is uncountable.

Concept 2: According to the definition given above, N is a countable set.

Concept 3: For any infinite set X, X is countable if and only if there is a bijection between X and N.

The concepts are approved and applied by most scholars up to now. Under the concepts and definition, Georg Cantor believed and suggested that it is impossible to construct a bijection between N and R. Furthermore, R is proved to be uncountable by Cantor's diagonal argument [1, 2]. The proof can be briefly described as follows:

StepA1: Assuming that R is countable.

StepA2: Under the assumption, the members in R can be listed by order. Any part of the members in R can be listed by order. Real numbers between 0 and 1 can be listed by order.

StepA3: Each real number can be represented by infinite decimal. For example:

0.1 = 0.100000000……

0.25 = 0.250000000……

0.597 = 0.597000000……

StepA4: Each real number between 0 and 1 can be represented by infinite decimal and can be listed. Mark them as s1, s2, s3, ………, sn, ……

s1 = 0.**1**00000000……

s2 = 0.3**3**3333333……

s3 = 0.59**7**570255……

s4 = 0.627**8**98900……

s5 = 0.2555**5**5555……

s6 = 0.77777**7**777……

s7 = 0.101010**1**01……

s8 = 0.9766625**5**5……

s9 = 0.01010101**0**……

⋮

⋮

⋮

StepA5: When all real numbers between 0 and 1 are listed. We can construct a number S and let S differs from sn in its nth digit (Notice bold digits marked in StepA4):

1st digit of S cannot be 1

2nd digit of S cannot be 3

3rd digit of S cannot be 7

4th digit of S cannot be 8

5th digit of S cannot be 5

⋮

⋮

StepA6: S is a real number. S is not any one real number listed above, since their nth digits differ.

StepA7: All real numbers between 0 and 1 are listed, so S should be listed (Step A5). However, S is not any one real number in the list (Step A6). There is a contradiction under the assumption at Step A1.

StepA8: The assumption at Step A1 is wrong, so R is uncountable.

However, N will be proved to be uncountable using Cantor's diagonal argument, and R will be proved to be countable by Cantor's definition in this paper. The results of argumentation will subversively change the concept of "infinite" in set theory, and the process of argumentation will provide us new perspectives to consider about the size of infinite sets.

Argument 1: The set of natural numbers could be uncountable

N can be proved to be uncountable by Cantor's diagonal argument:

StepB1: We know that nature number could be represented as different formats:

1 = 01 = 001 = 0001 = ……01

2 = 02 = 002 = 0002 = ……02

3 = 03 = 002 = 0003 = ……03

:

:

StepB2: Rewrite all natural numbers by opposite left-right direction. For example:

12345 could be rewrote as 54321

However, 54321 = ……0000054321

So it could be rewritten as 1234500000……

StepB3: After rewriting, all natural numbers can be listed:

N1 = 100000000000000……

N2 = 200000000000000……

N3 = 300000000000000……

:

:

N12345 = 543210000000000……

:

:

StepB4: A rewritten natural number S can be constructed as:

S differs from Nn in its nth digit

StepB5: By the construction, S differs from each Nn, since their nth digits differ. According to the logic of Cantor’s diagonal argument, the N has been proved to be uncountable.

I suggest that Cantor’s diagonal argument cannot to prove an infinite set is countable or not.

Argument 2: The set of real numbers is countable

Most scholars accept that it is impossible to construct a bijection between N and R. Also, people cannot give any rule to list all of the infinite R members.

However, I give a rule to list all of the positive R members:

Order positive real numbers

1 → ……00000001.00000000……

2 → ……00000002.00000000……

3 → ……00000003.00000000……

:

:

9 → ……00000009.00000000……

10 → ……00000000.10000000……

11 → ……00000001.10000000……

12 → ……00000002.10000000……

:

:

99 → ……00000009.90000000……

100 → ……00000010.00000000……

101 → ……00000011.00000000……

102 → ……00000012.00000000……

:

:

999 → ……00000099.90000000……

1000 → ……00000000.01000000……

1001 → ……00000001.01000000……

1002 → ……00000002.01000000……

:

:

123456788 → ……00013578.86420000……

123456789 → ……00013579.86420000……

123456790 → ……00013570.96420000……

:

:

To simplify the description, the above rule just lists all of the positive R members. It is easy to expand the rule to lists all of the R members and unnecessary to go into details here. Based on the above rule, we can construct a bijection between N and R by following steps:

StepC1 ~ StepC3: Rewrite all natural numbers as the same method described at StepB1 ~ StepB3

StepC4: Rewrite all real numbers as the sequence:

real number rewritten real numbers

1st digit on the left of decimal point → 1st digit

1st digit on the right of decimal point → 2nd digit

2nd digit on the left of decimal point → 3rd digit

2nd digit on the right of decimal point → 4th digit

:

:

For example:

…….00000001.00000000…… → 1000000000000000……

…….00000002.00000000…… → 2000000000000000……

…….00000003.00000000…… → 3000000000000000……

…….00097531.24680000…… → 1234567890000000……

Then, we get a bijection between positive real numbers and natural numbers. Consider of positive number, negative number and zero, we get a bijection between the set integer numbers (Z) and R.

StepC5: According to the countable set theory, there is a bijection between N and Z. So there is a bijection between N and R. According to aforesaid Concept 3, R is countable.

Moreover, it is easy to see that there is a bijection between N and the set of complex numbers (C) by similar demonstration process. Each complex number could be written as x + yi, and both x and y are real numbers. We could rewrite complex number by following rules:

x’s 1st digit on the left of decimal point → 1st digit

y’s 1st digit on the left of decimal point → 2nd digit

x’s 1st digit on the right of decimal point → 3rd digit

y’s 1st digit on the right of decimal point → 4th digit

x’s 2nd digit on the left of decimal point → 5th digit

y’s 2nd digit on the left of decimal point → 6th digit

x’s 2nd digit on the right of decimal point → 7th digit

y’s 2nd digit on the right of decimal point → 8th digit

:

:

For example:

…051.370… + …062.480…i → 123456780000……

Then, we can finally get a bijection between N and C.

Discussion

There is a new perspective of the contradiction in the Cantor’s diagonal argument. If we examine the Cantor’s diagonal argument carefully, we can find it is not “diagonal” exactly. We simply consider the arrangements of 2 numbers. There will be 4 arrangements:

1 1

1 2

2 1

2 2

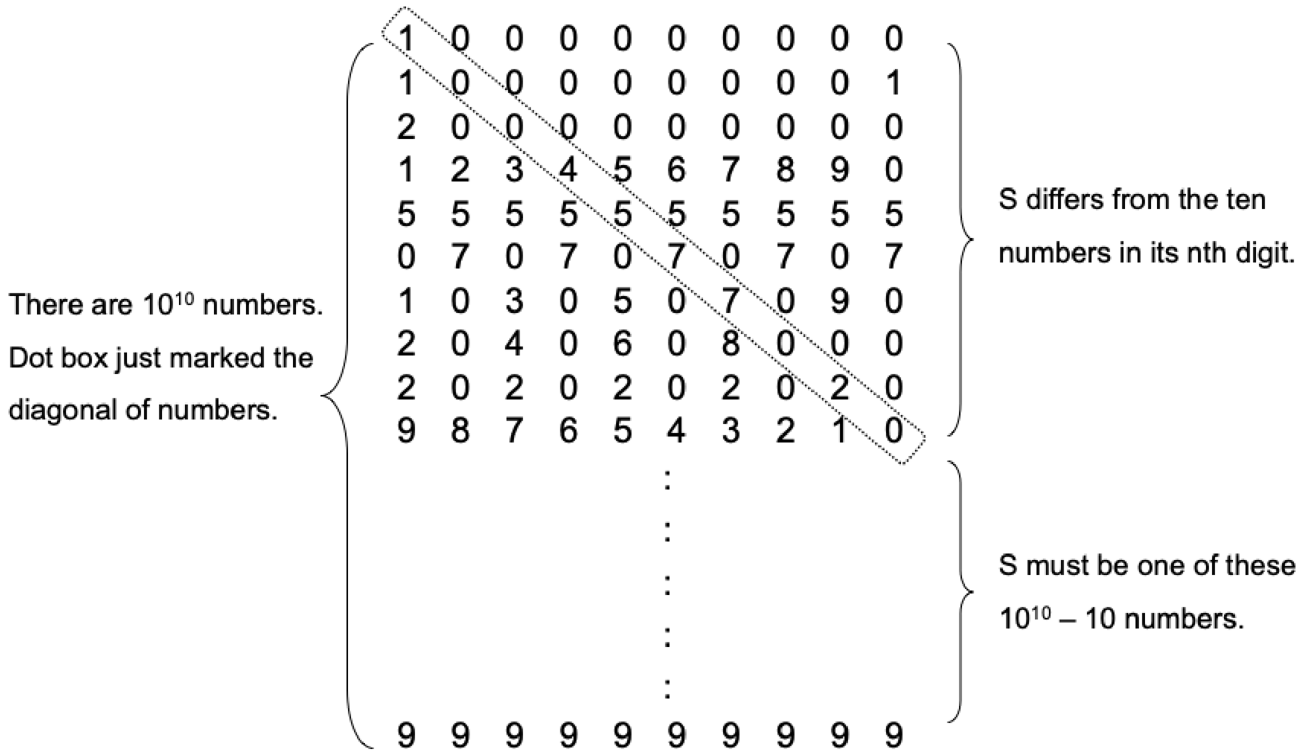
Cantor’s diagonal argument regard it as 2 arrangements, so it looks like “diagonal”.

$$\begin{matrix}1 & 1\\ 1 & 2\end{matrix}$$

We can construct S (as StepA6) and let its 1st digit is not 1 and its 2nd digit is not 2. S is not any one listed above, since their nth digits differ. However, S will be one of the full 4 arrangements:

$$\begin{matrix} & & 1 & 1\\ & & 1 & 2\\ S & \rightarrow & 2 & 1\\ & & 2 & 2\end{matrix}$$

If we can list all real numbers (such as the rule given above), then we cannot construct S and let S differs from sn in its nth digit (StepA4 ~ A5). For example:



The contradiction at StepA7 makes Georg Cantor reject the assumption at StepA1, but it actually make us reject the StepA4 ~ A5. The construction of S is after that all infinite numbers are listed. There should be a rational feature of infinite numbers: infinite numbers cannot be listed of all, or they are not infinite. In my opinion, the number S constructed at StepA5 proves that StepA4 is impossible to complete, but not proves that the assumption at StepA1 is wrong.

I propose that we always can find appropriate rewrite rules to get a bijection between any two infinite sets under current concepts of set theory. In these demonstration processes, we find that it is not necessary to define countable or uncountable for infinite sets. It is nonsense to compare sizes of infinite sets because their members are all infinite. The term, size, is defined to describe finite numbers. Related question developed from the concepts of countable or uncountable, such as continuum hypothesis, will be solved by the new concept without countable or uncountable.

Declarations

Conflicts of interest/Competing interests: The author declares no conflict of interest.

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