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Nonlocal Euler-Bernoulli beam theories with material nonlinearity and their application to single-walled carbon nanotubes

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Abstract: The small-scale effect and the material nonlinearity significantly impact the mechanical properties of nanobeams. However, the combined effects of two factors have not attracted the attention of researchers. In the present paper, we proposed two new nonlocal theories to model mechanical properties of slender nanobeams for centroid locus stretching or inextensional effect respectively. Two new theories consider both the material nonlinearity and the small-scale effect induced by the nonlocal effect. The new models are used to analyze the static bending and the forced vibration for single-walled carbon nanotubes (SWCNTs). The results indicate that the stiffness softening effect induced by the material nonlinearity has more prominent impact than the nonlocal effect on SWCNT's mechanical properties. Therefore, neglecting the material nonlinearity may cause qualitative mistakes.

1. Introduction

Although Nanobeams have immense potential applications in nanoelectromechanical systems, how to model the beams' mechanical properties is still an open question [1-4]. One of the most important properties is the nanobeam's elastic properties that are central to determine the beam's overall mechanical behaviors [5-6]. The small-scale effect and the nonlinearity are two critical factors that significantly affect the structure's mechanical properties at the nanoscale. The small-scale effect reveals that the mechanical properties and the geometric size are strongly related if the structure's geometric size down to the nanometer scale [7-11]. The nonlinearity includes the geometric and material nonlinearity [12-16]. There are a lot of researches of the nanobeam's small-scale effect [17-21]. However, the researchers have rarely paid attention to the material nonlinearity of nanobeams [22]. Moreover, the nonlinearity in the thermal-electro-mechanical coupling also is an important field [23-24]. Researchers usually modify the classical continuum mechanics to capture the small-scale effect through three different paths: the nonlocal stress gradient model

$\left[1 - (e_0 a)^2 \nabla^2\right] \sigma_{ij} = \bar{\sigma}_{ij}$ [7], the strain gradient model $\bar{\sigma}_{ij} = \left[1 - (e_0 a)^2 \nabla^2\right] \varepsilon_{ij}$ [9, 25-26],

and the surface stress model [27]. Here $\bar{\sigma}_{ij}$ and ε_{ij} are the local stress and strain; σ_{ij} is the

nonlocal stress; e_0 is a small-scale parameter; and a is the material characteristic length that is the Carbon-Carbon bonding length for SWCNTs. The stress or strain gradient model is widely used to study carbon nanotubes (CNTs) and graphene [28-30]. Nevertheless, determining the scale parameter of CNTs and graphene, e_0 , is still a controversial open question [5, 31] because the

quantum mechanics of mechanical properties has not been thoroughly proposed [32]. The material nonlinearity has restricted nonlinear elastic parameters' calculations through molecular dynamics (MD) and density functional theory (DFT). For example, the nonlinear elasticity of CNTs [16], graphene [14], and silicon nanowires [13] have attracted widespread attentions. Since the material nonlinearity may complicate mechanical models of CNTs, it has been neglected in most studies. From the existing researches, the stain's cubic terms of strain in the potential energy (corresponding to the quadratic terms in the stress-strain relationship) of graphene and SWCNTs significantly affect their mechanical properties [22, 33]. The accurate understanding of the nanobeam's mechanical properties is the basis of applications. Therefore, it is necessary to consider both the small-scale effect and the material nonlinearity comprehensively. Under the Euler-Bernoulli assumption of displacements, the present paper will propose two theories, which include both the nonlocal effect and the material nonlinearity, to characterize accurately the nanobeam's mechanical properties for different boundary conditions.

2. Methods

We restrict our attention to slender beams in the present research. Hence the Euler-Bernoulli hypothesis is employed [34]: the cross-sections which are perpendicular to the centroid locus before deformation remain plane and perpendicular to the deformed locus and suffer no strain in their planes. Under this hypothesis, we may only consider the beam's longitudinal (x -direction) strain component ε_{xx} , as shown in Fig. 1. Existing atomic calculations show that the potential energies of graphene and silicon materials contain at least the strain's cubic nonlinear terms [12-13]. Geometrically, an SWCNT can be viewed as a graphene sheet that has been rolled into a tube, so the stress-strain relationship of the big diameter SWCNTs may be consistent with graphene. This was confirmed by MD simulations [16]. For simplicity, we assume that the beam's strain is finite but small, so only cubic terms of the potential energy are kept, so the local longitudinal stress can be expressed as $\bar{\sigma}_{xx} = \bar{\sigma}_{xx}^0 + E\varepsilon_{xx} + D\varepsilon_{xx}^2$ [12, 15-16, 22]. Here, $\bar{\sigma}_{xx}^0$ is the initial prestress, E and D are the second-order and third-order elastic coefficients, respectively. Following the nonlocal differential constitutive relationship, has $\left[1 - (e_0 a)^2 \nabla^2\right] \sigma_{xx} = \bar{\sigma}_{xx}$. Then the nanobeam's nonlocal constitutive relationship with the material nonlinearity is written as

$$\left[1 - \mu^2 \nabla^2\right] \sigma_{xx} = \bar{\sigma}_{xx}^0 + E\varepsilon_{xx} + D\varepsilon_{xx}^2, \quad (1)$$

here $\mu = e_0 a$. In order to establish the corresponding nanobeam model, it is necessary to give the beam's strain-displacement relations according to different boundary constraints [34-36]. In this paper, we consider two conditions: (1) the two ends of beams are immovable along the x direction, such as a hinged-hinged beam, hence the effect of axial elongation needs to consider; (2) beams are inextensional, such as a simply supported beam. The present paper will establish motion equations for two conditions, respectively.

Fig. 1. Schematic configuration of a SWCNT, (a) Front elevation, (b) Plan, (c) Equivalent cross-section

2.1 Model with axial stretching effect

If the two ends of a beam cannot move along the x -axis, such as a hinged-hinged beam or clamped-clamped beam, the bending deformations may induce the axial extension, and the axial strain is [34]

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - y \frac{\partial^2 w}{\partial x^2}. \quad (2)$$

Here, u and w are axial displacements of the beam in the x and y directions respectively, as shown in Fig. 1. Substituting Eq. (2) into the expression of local stress, $\bar{\sigma}_{xx} = \bar{\sigma}_{xx}^0 + E\varepsilon_{xx} + D\varepsilon_{xx}^2$, has the local axial force and bending moment as,

$$\bar{N} = \int_A \bar{\sigma}_{xx} dA = -N_0 + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + DA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right]^2, \quad (3)$$

$$\bar{M} = \int_A y \bar{\sigma}_{xx} dA = -EI \frac{\partial^2 w}{\partial x^2} - 2ID \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2}. \quad (4)$$

Here, A and I are the beam's cross-sectional area and the moment of inertia [34]; $N_0 = \int_A \bar{\sigma}_{xx}^0 dA$ is the initial axial load at the ends, as shown in Fig. 1. Neglecting the quartic terms

of w and noticing $\partial u / \partial x = O(\partial w / \partial x)^2$ for slender beams [34-35], the force reduces to

$$\bar{N} = -N_0 + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2, \quad (5)$$

The equations of motion with the stretching effect can write as [34]

$$\frac{\partial N}{\partial x} = m \frac{\partial^2 u}{\partial t^2}, \quad (6)$$

$$\frac{\partial^2 M}{\partial x^2} + N \frac{\partial^2 w}{\partial x^2} = m \frac{\partial^2 w}{\partial t^2} + \bar{F}(x, t). \quad (7)$$

Here m is the mass of the beam per unit span, N and M are the nonlocal axial force and the nonlocal moment. Taking into account Eq. (4) and Eq. (5), the nonlocal constitutive Eq. (1) transforms into

$$M - \mu^2 \frac{\partial^2 M}{\partial x^2} = -EI \frac{\partial^2 w}{\partial x^2} - 2ID \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2}, \quad (8)$$

$$N - \mu^2 \frac{\partial^2 N}{\partial x^2} = -N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right]. \quad (9)$$

Substituting the motion equations, Eq. (6) and Eq. (7), into the constitutive equations, Eq. (8) and Eq. (9), gets

$$M - \mu^2 \left(m \frac{\partial^2 w}{\partial t^2} - N \frac{\partial^2 w}{\partial x^2} + \bar{F} \right) = -EI \frac{\partial^2 w}{\partial x^2} - 2ID \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2}, \quad (10)$$

$$N - \mu^2 \frac{\partial}{\partial x} \left(m \frac{\partial^2 u}{\partial t^2} \right) = -N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right]. \quad (11)$$

Differentiates Eq (10) twice with respect to x , then substitutes it into the Eq. (7), gets

$$\begin{aligned} & -EI \frac{\partial^4 w}{\partial x^4} + N \frac{\partial^2 w}{\partial x^2} - 2ID \frac{\partial^2}{\partial x^2} \left\{ \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2} \right\} \\ & + \mu^2 \frac{\partial^2}{\partial x^2} \left(m \frac{\partial^2 w}{\partial t^2} - N \frac{\partial^2 w}{\partial x^2} + \bar{F} \right) = m \frac{\partial^2 w}{\partial t^2} + \bar{F}(x, t). \end{aligned} \quad (12)$$

From Eq. (11), gets

$$N = \mu^2 \frac{\partial}{\partial x} \left(m \frac{\partial^2 u}{\partial t^2} \right) - N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right]. \quad (13)$$

Substituting Eq. (13) into Eq. (12) and ignoring the inertia term $\partial^2 u / \partial t^2$, gets the lateral motion equation as following.

$$\begin{aligned} & -EI \frac{\partial^4 w}{\partial x^4} + \left\{ -N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\} \frac{\partial^2 w}{\partial x^2} \\ & - 2ID \frac{\partial^2}{\partial x^2} \left\{ \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2} \right\} - \mu^2 \frac{\partial^2}{\partial x^2} \left\{ \left[-N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right. \right. \\ & \left. \left. + EA \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right] \frac{\partial^2 w}{\partial x^2} \right\} = m \left(\frac{\partial^2 w}{\partial t^2} - \mu^2 \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) + \bar{F} - \mu^2 \frac{\partial^2 \bar{F}}{\partial x^2}. \end{aligned} \quad (14)$$

Differentiates Eq. (13) with respect to x , then substitutes it into Eq. (6), gets

$$ID \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + EA \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] = m \frac{\partial^2 u}{\partial t^2} - \mu^2 m \frac{\partial^4 u}{\partial t^2 \partial x^2}. \quad (15)$$

Eq. (14) and Eq. (15) are the nanobeam's plane motion equations with the nonlocal nonlinear constitutive and the stretching effect. Their boundary conditions of displacements are identical with the classical beam theory. If bending motion is considered only, the inertia terms of Eq. (15) can be ignored [35-36], then the equation simplifies as

$$\frac{\partial^2 u}{\partial x^2} = - \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \lambda \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right], \quad (16)$$

here $\lambda = DI / EA$. Integrating Eq. (16) with respect to x , gets

$$\begin{aligned} \frac{\partial u}{\partial x} &= - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - \lambda \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + C_1(t), \\ u &= - \frac{1}{2} \int_0^x \left[\left(\frac{\partial w}{\partial s} \right)^2 + 2\lambda \left(\frac{\partial^2 w}{\partial s^2} \right)^2 \right] ds + C_1(t)x + C_2(t). \end{aligned} \quad (17)$$

Where C_1 and C_2 are functions of time t , which can be determined by imposing boundary conditions on w . For a beam with two unmovable ends, gets [35-36]

$$C_1(t) = \frac{1}{2l} \int_0^l \left[\left(\frac{\partial w}{\partial x} \right)^2 + 2\lambda \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx, \quad C_2 = 0. \quad (18)$$

Substituting Eq. (17) into Eq. (14) and omitting the biquadratic terms of w , gets

$$\begin{aligned} m \left(\frac{\partial^2 w}{\partial t^2} - \mu^2 \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) + C \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} + N_0 \left(\frac{\partial^2 w}{\partial x^2} - \mu^2 \frac{\partial^4 w}{\partial x^4} \right) - 2\lambda ID \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} \right)^3 \\ - \frac{EA}{2l} \left[\frac{\partial^2 w}{\partial x^2} - (2\lambda + \mu^2) \frac{\partial^4 w}{\partial x^4} \right] \int_0^l \left[\left(\frac{\partial w}{\partial x} \right)^2 + 2\lambda \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx = \bar{F} + \mu^2 \frac{\partial^2 \bar{F}}{\partial x^2}. \end{aligned} \quad (19)$$

In Eq. (19), we add a linear damping term $C \partial w / \partial t$. For a hinged-hinged beam, the boundary conditions are [34-36]

$$w(0, t) = w(l, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(l, t) = 0. \quad (20)$$

2.2 The model with the centroid locus inextensional effect

For a nanobeam with an end being fixed or hinged, the other end being free or sliding, the beam are inextensional, and the strain can be written as [34-36]

$$\varepsilon_{xx} = -y \left[\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 w}{\partial x^2} \right], \quad (21)$$

Substituting Eq. (21) into the local nonlinear constitutive equation, $\bar{\sigma}_{xx} = \bar{\sigma}_{xx}^0 + E \varepsilon_{xx} + D \varepsilon_{xx}^2$,

and noting $\int_A y^3 dA = 0$ for the symmetrical cross-section, the local axial force and the moment are

$$\bar{N} = \int_A \bar{\sigma}_{xx} dA = -N_0 - ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2, \quad (22)$$

$$\bar{M} = \int_A y \bar{\sigma}_{xx} dA = -EI \left[\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial^2 w}{\partial x^2} \right) \right]. \quad (23)$$

Analogous to the virtual work principle of the classical beam theory [34], the beam's motion equations with the nonlocal nonlinear constitutive are,

$$\frac{\partial N}{\partial x} = m \frac{\partial^2 u}{\partial t^2}, \quad (24)$$

$$\begin{aligned} & \frac{\partial^2 M}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left[\frac{\partial M}{\partial x} \left(\frac{\partial w}{\partial x} \right)^2 \right] + N \frac{\partial^2 w}{\partial x^2} \\ &= m \frac{\partial^2 w}{\partial t^2} + \frac{1}{2} m \frac{\partial}{\partial x} \left\{ \frac{\partial w}{\partial x} \int_l \left[\frac{\partial^2}{\partial t^2} \int_l \left(\frac{\partial w}{\partial x} \right)^2 ds \right] ds \right\} + \bar{F}(x, t). \end{aligned} \quad (25)$$

For simplicity, ignores the M 's nonlinear term in Eq. (25), and gets

$$\frac{\partial^2 M}{\partial x^2} \approx m \frac{\partial^2 w}{\partial t^2} + \frac{1}{2} m \frac{\partial}{\partial x} \left\{ \frac{\partial w}{\partial x} \int_l \left[\frac{\partial^2}{\partial t^2} \int_l \left(\frac{\partial w}{\partial x} \right)^2 ds \right] ds \right\} - N \frac{\partial^2 w}{\partial x^2} + \bar{F}(x, t). \quad (26)$$

Neglecting $m \partial^2 u / \partial t^2$, Eq. (24) can be simplified to $\partial N / \partial x = 0$, so gets

$$\frac{\partial^2 N}{\partial x^2} = 0 \quad (27)$$

Substituting Eq. (21) into Eq. (1), the nonlocal constitutive relationship is transformed into

$$M - \mu^2 \frac{\partial^2 M}{\partial x^2} = -EI \left[\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial^2 w}{\partial x^2} \right) \right], \quad (28)$$

$$N - \mu^2 \frac{\partial^2 N}{\partial x^2} = -N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2. \quad (29)$$

Substituting Eq. (26) and Eq. (27) into Eq. (28) and Eq. (29), gets

$$\begin{aligned} M &= -EI \left[\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial^2 w}{\partial x^2} \right) \right] - \mu^2 N \frac{\partial^2 w}{\partial x^2} + \mu^2 \bar{F}(x, t) \\ &+ \mu^2 \left\{ m \frac{\partial^2 w}{\partial t^2} + \frac{m}{2} \frac{\partial}{\partial x} \left\{ \frac{\partial w}{\partial x} \int_l \left[\frac{\partial^2}{\partial t^2} \int_l \left(\frac{\partial w}{\partial x} \right)^2 ds \right] ds \right\} \right\}, \end{aligned} \quad (30)$$

$$N = -N_0 + ID \left(\frac{\partial^2 w}{\partial x^2} \right)^2. \quad (31)$$

Differentiates Eq. (30) once and twice with respect to x , then substitutes the outcomes and Eq. (31) into Eq. (25), gets

$$\begin{aligned} & -EI \frac{\partial^4 w}{\partial x^4} - N_0 \left(\frac{\partial^2 w}{\partial x^2} - \mu^2 \frac{\partial^4 w}{\partial x^4} \right) + \frac{\mu^2 N_0}{2} \frac{\partial}{\partial x} \left[\left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^3 w}{\partial x^3} \right] - EI \frac{\partial}{\partial x} \left[\frac{\partial w}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right. \\ & \left. + \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^3 w}{\partial x^3} \right] + ID \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^3 - \mu^2 \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} \right)^3 \right] = m \frac{\partial^2 w}{\partial t^2} - \mu^2 m \frac{\partial^4 w}{\partial t^2 \partial x^4} \\ & - \frac{\mu^2 m}{2} \frac{\partial}{\partial x} \left[\frac{\partial^3 w}{\partial t^2 \partial x} \left(\frac{\partial w}{\partial x} \right)^2 \right] + \left(\frac{m}{2} \frac{\partial}{\partial x} - \frac{\mu^2 m}{2} \frac{\partial^3}{\partial x^3} \right) \left\{ \frac{\partial w}{\partial x} \int_l^s \left[\frac{\partial^2}{\partial t^2} \int_l^s \left(\frac{\partial w}{\partial x} \right)^2 ds \right] ds \right\} \\ & + \bar{F} - \mu^2 \left\{ \frac{\partial^2 \bar{F}}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left[\frac{\partial \bar{F}}{\partial x} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\}. \end{aligned} \quad (32)$$

Here the nonlinear terms only keep up to cubic terms in Eq. (32). For slender beams, the nonlinear inertia terms can be omitted, and Eq. (32) can be simplified to

$$\begin{aligned} & m \left(\frac{\partial^2 w}{\partial t^2} - \mu^2 \frac{\partial^4 w}{\partial t^2 \partial x^4} \right) + C \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} + N_0 \left(\frac{\partial^2 w}{\partial x^2} - \mu^2 \frac{\partial^4 w}{\partial x^4} \right) \\ & - \frac{\mu^2 N_0}{2} \frac{\partial}{\partial x} \left[\left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^3 w}{\partial x^3} \right] + EI \frac{\partial}{\partial x} \left[\frac{\partial w}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^3 w}{\partial x^3} \right] \\ & - ID \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^3 - \mu^2 \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} \right)^3 \right] = \bar{F} - \mu^2 \left\{ \frac{\partial^2 \bar{F}}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left[\frac{\partial \bar{F}}{\partial x} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\} \end{aligned} \quad (33)$$

Displacement's boundary conditions of Eq. (33) are identical with the classical beam theory. For example, the conditions of simply supported beams are

$$w(0, t) = w(l, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(l, t) = 0. \quad (34)$$

Below we will analyze Eq. (19) and Eq. (33). The above two models can divide into four categories: the classical nonlinear model (CNM) for $D = 0$ and $\mu = 0$, the nonlinear constitutive model (NCM) for $D \neq 0$ and $\mu = 0$, the nonlocal nonlinear model (NNM) for $D = 0$ and $\mu \neq 0$, and the nonlocal nonlinear constitutive model (NNCM) for $D \neq 0$ and $\mu \neq 0$.

2.3 Solutions of models

At this point, it is convenient to introduce dimensionless variables in Eq. (19) and Eq.(33). Lets $\bar{x} = x/l$, $\bar{w} = w/l$, $\bar{x} = x/l$, $\bar{t} = t/\omega_0$, and $\omega_0^2 = \pi^4 EI/l^4 m$. Assumes the loads on the

beams are uniform, so $\partial^2 \bar{F} / \partial x^2 = 0$. Eq. (19) for the stretching effect can rewrite as

$$\begin{aligned} & \frac{\partial^2}{\partial \bar{t}^2} \left(\bar{w} - \frac{\mu^2}{l^2} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right) + \frac{C}{m\omega_0} \frac{\partial \bar{w}}{\partial \bar{t}} + \frac{1}{\pi^4} \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \frac{N_0}{m\omega_0^2 l^2} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \frac{\mu^2}{l^2} \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \right) - \frac{2\lambda ID}{m\omega_0^2 l^6} \frac{\partial^2}{\partial \bar{x}^2} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^3 \\ & - \frac{EA}{2m\omega_0^2 l^2} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \frac{2\lambda + \mu^2}{l^2} \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \right) \int_0^1 \left[\left(\frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 + \frac{2\lambda}{l^2} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^2 \right] d\bar{s} = \frac{\bar{F}}{m\omega_0^2 l}. \end{aligned} \quad (35)$$

Eq. (33) for inextensible beams can rewrite as

$$\begin{aligned} & \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - \frac{\mu^2}{l^2} \frac{\partial^4 \bar{w}}{\partial \bar{t}^2 \partial \bar{x}^2} + \frac{C}{m\omega_0} \frac{\partial \bar{w}}{\partial \bar{t}} + \frac{1}{\pi^4} \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \frac{N_0}{m\omega_0^2 l^2} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \frac{\mu^2}{l^2} \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \right) \\ & - \frac{\mu^2 N_0}{2m\omega_0^2 l^4} \frac{\partial}{\partial \bar{x}} \left[\left(\frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 \frac{\partial^3 \bar{w}}{\partial \bar{x}^3} \right] + \frac{EI}{2m\omega_0^2 l^4} \frac{\partial}{\partial \bar{x}} \left[\frac{\partial \bar{w}}{\partial \bar{x}} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^2 + \left(\frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 \frac{\partial^3 \bar{w}}{\partial \bar{x}^3} \right] \\ & - \frac{DI}{m\omega_0^2 l^4} \left[\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^3 - \frac{\mu^2}{l^2} \frac{\partial^2}{\partial \bar{x}^2} \left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^3 \right] = \frac{\bar{F}}{m\omega_0^2 l}. \end{aligned} \quad (36)$$

For simplicity, we take a hinged-hinged beam as an example to discuss the stretching effect and take a simply supported beam to discuss the inextensible beams. So their normalized boundary conditions are identical:

$$\bar{w}(0, t) = \bar{w}(1, t) = \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}(0, t) = \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}(1, t) = 0. \quad (37)$$

It is difficult to accurately solve nonlinear Eq. (35) and Eq. (36). Hence we apply the Galerkin method and the multi-scale method to solve the equations approximately [34-36]. The two methods are widely used to solve nonlinear equations of macroscopic or microscopic structures [18, 20-22, 37-39]. Under boundary conditions Eqs. (37), the approximate solutions of Eq. (35) and Eq. (36) can be written as

$$\bar{w} = \sum_{j=1}^{\infty} \bar{\eta}_j(t) \sin j\pi x. \quad (38)$$

Substitutes Eq. (38) into Eq. (35) and Eq. (36), respectively, only takes the first term of Eq. (38) for simplicity, and lets $\bar{\eta}_1 = \eta$. And then $\sin(\pi x)$ is multiplied by both sides of the equations, and integrates in the interval $[0, 1]$ (Galerkin truncation [34, 36]), gets

$$m_i \ddot{\eta} + c_i \dot{\eta} + k_i \eta + d_i \eta^3 = F, \quad i = 1, 2. \quad (39)$$

Here $i = 1$ indicates the hinged-hinged beam, and $i = 2$ indicates the simply supported beam. The parameters in Eqs. (39) are

$$\begin{aligned}
m_1 = m_2 &= 1 + \frac{\pi^2 \mu^2}{l^2}, \quad c_1 = c_2 = \frac{C}{m\omega_0}, \quad k_1 = k_2 = 1 - \frac{N_0 \pi^2}{m\omega_0^2 l^2} \left(1 + \frac{\pi^2 \mu^2}{l^2} \right), \\
d_1 &= \frac{\pi^4 EA}{4m\omega_0^2 l^2} \left(1 + \frac{2\pi^2 \lambda}{l^2} \right)^2 - \frac{3\pi^8 \lambda ID}{2m\omega_0^2 l^6} + \frac{\pi^4 EA \mu^2}{4m\omega_0^2 l^4} \left(1 + \frac{2\pi^2 \lambda}{l^2} \right), \\
d_2 &= -\frac{3\pi^6 \mu^2 N_0}{8m\omega_0^2 l^4} + \frac{\pi^6 EI}{6m\omega_0^2 l^4} + \frac{3\pi^6 DI}{4m\omega_0^2 l^4} \left(1 + \frac{\pi^2 \mu^2}{l^2} \right), \quad F = \frac{4\bar{F}}{\pi m \omega_0^2 l}.
\end{aligned} \tag{40}$$

It is clear from Eq. (40) that the linear coefficients of the two models are identical, but the nonlinear coefficients are different. Moreover, the nonlocal effect and the material nonlinearity couple in nonlinear terms. The formulae of d_1 demonstrates that the influence of the material nonlinearity will decrease accompanying the increase of beam's length, as an SWCNT example shown in Fig. 2. Nevertheless, if the nonlocal effect is neglected, the material nonlinearity's influence on d_2 is invariant accompanying the length's change due to $\omega_0^2 l^4 = \pi^4 EI / m$, as an SWCNT example shown in Fig. 9.

For an SWCNT not subjected to the end force, $N_0 = 0$, the nonlinear terms induced by the finite deformations and the nonlocal effect produces a hardening effect in two models due to $\mu^2 > 0$; and the nonlinear terms induced by the material nonlinearity has a softening effect in two models due to $D < 0$. It is interesting, for hinged-hinged SWCNTs, that the hardening effect will be smaller than the softening effect for the small length tubes, as shown in Fig.2. The following discussions will show that this competitive relationship significantly affects the mechanical behavior of SWCNTs.

Neglecting the inertia and damping terms in Eqs. (39), the equations to determine the static bending deformations of the beam's middle points are obtained as

$$k_i \eta + d_i \eta^3 = F, \quad i = 1, 2. \tag{41}$$

We will research the mechanical behaviors of hinged-hinged beams and simply supported beams according to Eq. (39) and Eq. (41). In order to solve the beams' vibrations, we rewrite Eq. (39) as

$$\ddot{\eta} + \bar{c}_i \dot{\eta} + \bar{\omega}_i^2 \eta + \bar{d}_i \eta^3 = \bar{f}. \tag{42}$$

Where $\bar{c} = c / m$, $\omega^2 = k_1 / m$, $\bar{d}_i = d_i / m_i$, $\bar{f} = F / m_i$. The multiple scale method [35], which widely use to solve weak nonlinear differential equations for macro-structures [37-39], will be applied to solve Eq. (42). Lets $\tilde{c}_i = \varepsilon^2 c_i$, $\bar{f} = \varepsilon^3 f \cos(\Omega t)$, gives

$$\ddot{\eta} + \omega_i^2 \eta + \varepsilon^2 \tilde{c}_i \dot{\eta} + d_i \eta^3 = \varepsilon^3 f \cos(\Omega t), i = 1, 2.. \tag{43}$$

Supposes

$$\eta(t; \varepsilon) = \varepsilon \eta_1(T_0, T_2) + \varepsilon^3 \eta_3(T_0, T_2), \quad (44)$$

and substitutes it into Eq. (43), then equates the coefficients of ε and ε^3 on both sides, has

$$\varepsilon: \quad \frac{\partial^2 \eta_1}{\partial T_0^2} + \omega_i^2 \eta_1 = 0, \quad (45)$$

$$\varepsilon^3: \quad D_0^2 \eta_3 + \omega_i^2 \eta_3 = -2D_0 D_2 \eta_1 - 2c_i D_0 \eta_1 - d_i \eta_1^3 + \frac{1}{2} f \exp(i\Omega T_0). \quad (46)$$

The solution of Eq. (45) is

$$\eta_1 = A(T_2) \exp(i\omega_i T_0) + cc, \quad (47)$$

here, cc means the complex conjugate. Substituting Eq. (47) into Eq. (46), gets

$$D_0^2 \eta_3 + \omega_i^2 \eta_3 = \frac{1}{2} f \exp(i\Omega T_0) - [i2\omega_i (A' + c_i A) + 3d_i A^2 \bar{A}] \exp(i\omega_i T_0) + NST, \quad (48)$$

here NST denotes non-secular terms [36]. When the load's frequency Ω approaches the nanobeam's modal frequency ω , the beam will appear a relatively large amplitude response. Under

this condition, lets $\Omega = \omega_i + \varepsilon^2 \sigma$, so the solvable condition of Eq. (48) is written as:

$$-i2\omega_i (A' + c_i A) - 3d_i A^2 \bar{A} + \frac{1}{2} f \exp(i\sigma T_2) = 0, \quad i = 1, 2. \quad (49)$$

Letting $A = \frac{1}{2} \alpha \exp(i\beta)$, and substituting it into Eq. (49), then separating the real part and the imaginary part, gets

$$\begin{aligned} \alpha' &= -c_i \alpha + \frac{f}{2\omega_i} \sin \gamma, \\ \alpha \gamma' &= \sigma \alpha - \frac{3}{8\omega_i} d_i \alpha^3 + \frac{f}{2\omega_i} \cos \gamma, \end{aligned} \quad (50)$$

here $\gamma = \sigma T_2 - \beta$. Steady-state motions occur when $\alpha' = \gamma' = 0$ [36], which corresponds to the singular points of Eqs. (50). The steady-state solutions can be obtained from following algebraic equations [36]

$$\left[c^2 + \left(\sigma - \frac{3d_i}{8\omega_i^2} \alpha^2 \right)^2 \right] \alpha^2 = \frac{f^2}{4\omega_i^2}, \quad i = 1, 2. \quad (51)$$

The stability of the steady-state solutions can be judged by investigating the nature of the singular points of Eqs. (50). Lets $a = a_0 + a_1$ and $\gamma = \gamma_0 + \gamma_1$, and substitutes them into Eqs. (50). Then

expands for small a_0 and γ_0 and keeps linear terms in a_1 and γ_1 , gets

$$\begin{aligned}\alpha_1' &= -c_i \alpha_1 + \frac{\gamma_1 f \cos \gamma_0}{2\omega_i}, \\ \gamma_1' &= -\left(\frac{f_i \cos \gamma_{01}}{2\omega_i \alpha_0^2} + \frac{3d_i \alpha_0}{4} \right) \alpha_1 - \frac{\gamma_1 f \sin \gamma_0}{2\omega_i \alpha_0}.\end{aligned}\tag{52}$$

Here, that α_0 and γ_0 are the singular points of Eqs. (50) is used in Eqs. (52). The stability of the steady-state motions depends on the coefficient matrix's eigenvalues on the right-hand sides of Eq. (52). If the real parts of eigenvalues are greater than zero, the solutions are unstable [36]. Hence the steady-state motions are unstable when

$$c_i^2 + \left(\frac{3d_i \alpha_0^2}{8\omega_i} - \sigma \right) \left(\frac{9d_i \alpha_0^2}{8\omega_i} - \sigma \right) < 0.\tag{53}$$

Eq. (53) indicates that the nonlinear term affects the stability of the steady-state solution. So the nonlocal effect and the material nonlinearity also affect the solutions' stability.

3 Results and discussions

This section use a (15,15) SWCNT to demonstrate the differences between the four models (CNM, NCM, NNM, and NNCM). At this point, the SWCNT's diameter is $d = 2.034\text{nm}$, and the thickness is taken as $h = 0.34\text{nm}$, the other physical and geometric parameters are as follows[15, 22] : $E = 1\text{TPa}$, $D = -2\text{TPa}$, $m = 4.86 \times 10^{-15}\text{kgm}^{-1}$, $I = 1.115\text{nm}^4$, $A = 2.171\text{nm}^2$, $\lambda = -1.027 \times 10^{-18}\text{nm}^2$. As mentioned in the introduction, determining the nonlocal parameter e_0 of SWCNTs is an open problem, and the results from atomic calculations have also shown confusion [17, 31]. However, if the vibration frequency in CNTs is in the terahertz range, a conservative estimate of the scale coefficient is $e_0 a < 2\text{nm}$ [5]. In the present research, we take the SWCNT's scale coefficient $\mu = e_0 a = 1\text{nm}$. It is bigger than most values in the existing SWCNT's researches [5]. Using a big nonlocal coefficient helps compare the differences between the material nonlinearity and the nonlocal effect.

3.1 Hinged-hinged SWCNTs

It can be found from Eqs. (40) that the nonlinear parameters are related to the beam length, so we first focus the length's effect on d_1 , as shown in Fig. 2. The figure and Eqs. (40) show that d_1 in the CNM and the NNM is positive for a (15,15) SWCNT. This indicates that the geometric nonlinear terms induced by finite deformations and the nonlocal effect produce a hardening effect for SWCNTs. However, if the material nonlinearity is considered in the models (NCM and NNCM) the nonlinear parameter, d_1 , will change from positive to negative with the length's decrease, as

shown in Fig. 2 and Eqs. (40). This means that the nonlinear terms in Eqs. (39) change from a soft spring to a hard spring. The soft spring has significant influence on the nonlinear mechanical properties of structures [35-39].

Here we take $l = 8\text{nm}$ and $l = 6\text{nm}$ as examples to study the differences between four models. Since the nonlinear coefficients in the four models are all greater than zero for $l = 8\text{nm}$, the SWCNT is a hard spring system ($d_1 > 0$) at this condition. However, for $l = 6\text{nm}$, the models with material nonlinearity (NCM and NNCM) are soft spring systems ($d_1 < 0$), while other models

(CNM and NNM) are still hard spring systems ($d_1 > 0$). Static load-deformation curves can be obtained from Eq. (1) as shown in Fig. 3 and Fig. 4. The two figures show that the softening effect remarkably increases the deformation amplitudes of the static bending. Further, the material nonlinearity may significantly impacts on the SWCNT's vibrations, as shown in from Fig. 5 to Fig. 7 obtained from Eq. (51). For example, the four models' load-response curves at $\sigma = 10$ are significantly different for $l = 8\text{nm}$ and $l = 6\text{nm}$, as shown in Fig. 5 and Fig. (6). Since the softening effects will appear in the small beam's length, the influence of the material nonlinearity will increase with the decrease of the SWCNT's length. Ignoring the material nonlinearity may lead to evident errors for short SWCNTs, as shown in Fig 7 and Fig. 8. The material nonlinearity produces the softening effect in the NCM and NNCM at $l = 6\text{nm}$, and the softening effect makes their frequency-response curves deviate to the left. On the contrary, the CNM and the NNM are hard springs, so the frequency-response curves is skewed to the right, as shown in Fig. 7. We calculated Eq. (42) numerically through the Runge-Kutta method for $(l, \sigma, f) = (6, 5, 5)$. The results show that ignoring the material nonlinearity may seriously underestimate the vibration amplitudes, as shown in Fig. 8. The numerical calculations also show that perturbation solutions are precise.

Fig. 2. Nonlinear coefficients d_1 as functions of hinged-hinged beam lengths l .

Fig. 3. Static deformations as functions of loads for hinged-hinged beams at $l = 8\text{nm}$.

Fig. 4. Static deformations as functions of loads for hinged-hinged beams at $l = 6\text{nm}$.

Fig. 5. Amplitudes of response as functions of the amplitudes of loads for hinged-hinged beams at $(l, \sigma) = (8, 10)$.

Fig. 6. Amplitudes of response as functions of the amplitudes of loads for hinged-hinged beams at $(l, \sigma) = (6, 10)$.

Fig. 7. Frequency-response curves for hinged-hinged beams at $(l, f) = (6, 5)$

Fig. 8. Time series of hinged-hinged beams for $(l, \sigma, f) = (6, 5, 5)$ at an initial value $(x_0, \dot{x}_0) = (0.2, 0)$

3.2 Simply supported SWCNTs

Here we demonstrate the tube length's effect on the nonlinear coefficients d_2 at first, as shown in Fig. 9. It is found from Fig. 9 and Eqs. (40) that the material nonlinearity makes $d_2 < 0$ for the NCM and NNCM. This indicates that both the nonlocal effect and the material nonlinearity have a stiffness softening effect on SWCNTs. The softening effect makes the static bending deformations of the NNM and the NNCM significantly larger than these of the CNM and the NNM, as shown in Fig. 10 obtained from Eq. (41). It is more important that the material nonlinearity significantly affects the dynamic behaviors of SWCNTs. For example, the response amplitudes of the NCM and the NNCM have jumps accompanying excitation amplitude's change, while the CNM and NNM do not produce amplitude's jumps for $(l, \sigma) = (6, -5)$, as shown in Fig. 11 obtained from Eq. (51). Similar to the hinged-hinged beams, the frequency-response curves of the NCM and the NNCM appear maximum vibration amplitudes for $\Omega < \omega_2$. In contrast, the maximum vibration amplitudes of the CNM and the NNM appear for $\Omega > \omega_2$, as shown in Fig. 12 obtained from Eq. (51). The above results indicate that neglecting the material nonlinearity may produce qualitatively incorrect results, and that the material nonlinearity may more significantly impact the SWCNT's mechanical properties than the nonlocal effect. Here we implement numerical calculations for Eq. (42) through the Runge-Kutta method to check the perturbation solutions Eq. (51) for the simply supported SWCNT. The numerical simulations confirm the accuracy of the analytical solutions, as shown in Fig. 13.

Fig. 9. Nonlinear coefficients d_2 as functions of simply supported beam lengths l .

Fig. 10. Static deformations as functions of loads for simply supported beams at $l = 8\text{nm}$.

Fig. 11. Amplitudes of response as functions of the amplitudes of loads for simply supported beams at $(l, \sigma) = (6, -5)$.

Fig. 12. Frequency-response curves for simply supported beams at $(l, f) = (6, 2)$

Fig. 13. Time series of hinged-hinged beams for $(l, \sigma, f) = (6, -4, 2)$ at an initial value

$$(x_0, \dot{x}_0) = (0.1, 0)$$

4. Conclusions

The present paper combines the nonlocal effect and the material nonlinearity to suggest two new Euler-Bernoulli models for nanobeams: the integral-partial differential equation model for the axial stretching effect, and the partial differential equation model for the axial inextensional effect. The two models consider both the small-scale effect and the material nonlinearity of nanobeams. A (15,15)SWCNT is used as an example to research the static bending and the forced vibration for the hinged-hinged and simply supported nanobeams. The results show that the material nonlinearity significantly softens SWCNT's stiffness, and the softening effect more significantly impacts on the mechanical properties than the nonlocal effect. Under certain conditions, neglecting the material nonlinearity may induce noticeable mistakes.

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Availability of data and material

Not applicable.

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Authors' contributions

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