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Reconstruction of nonlinear flows from noisy time series

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Nonlinear dynamics is a rapidly developing subject across all disciplines involving spatial or temporal evolution. The reconstruction of the equations of motion for a nonlinear system from observed time series has been a hot topic for a long time. Nevertheless, in practice only partial information is available for many systems which are very likely contaminated with noise. Here, based on the invariance of the evolution equation during time translation, a globally valid local approximation of the trajectory is determined, which could be reliably used for the reconstruction of the vector fields with unknown parameters or functional forms, even with partial information of the state evolution. The global consideration very effectively alleviates noise interference and bestows exceptional robustness to the technique, which asks only for solution of linear equations and thus is very efficient. The new scheme is nicely demonstrated in the Lorenz equation in different conditions, while an FHN neural network model is used to show its strength in high-dimensions.

I. INTRODUCTION

With rapid development of science and technology, large amount of data is generated in different fields. Revealing the information structure and possible governing equations underlying the dynamics has become an important task of system analysis [1–6]. Due to the advancement of nonlinear dynamics, a lot of progress has been made in various fields, including quantitative finance, neural networks, earthquake prediction, environmental pollution control, *etc.*, but much more is needed in both theoretical development and real-world applications [7–13].

In nonlinear dynamics, time series analysis was started quite early. In 1980, Packard [14] et al. proposed a method to reconstruct a strange attractor in the phase space of an unknown nonlinear system using given time series. At nearly the same time, Takens [15] proposed the great time-delay embedding technique which enables the construction of strange attractors by only employing scalar time series with delay coordinates. On the basis of this reconstruction, many methods [16–19] were developed to estimate important dynamically invariant quantities of the system, such as the fractal dimension, Lyapunov exponents and unstable periodic orbits. But these important quantities are not able to supply all the relevant information of the system concisely, a direct reconstruction of the equations of motion from time series seems necessary on many occasions. Polynomials are often used for this type of reconstruction [20], even with one or several such hidden variables [21]. A standard system for parameter inference is proposed [22–26] by repeatedly taking derivatives of the time series when there is only one observable. However, in the presence of noise, the derivation could amplify the noise and lead to the failure of the reconstruction. Lu considered the influence of noise in the polynomial fitting of the

Lorenz and the Chen equation [27], and concluded that with the increase of noise in the observation, the reconstruction error increases rapidly. In [28], it also shows that the polynomial model is not only sensitive to noise, but also sensitive to sampling rate. Apart from the method of least squares impressive alternative techniques may be invoked to fit the equations of motion, such as the multiple shooting method proposed by Baake [29] which demands very little data, and a synchronization method [30, 31] which could gradually learn the values of the parameters online. Although all these methods works quite well in the presence of small noise, they may become unstable with moderate or strong noise.

Recently, reconstruction of the interaction of networked nodes has become a popular challenging problem, the goal of which is to reveal the network topology. Different methods are proposed in the literature to detect network structure based on time-series [32–58]. The granger causality test or information theoretic tools could be conveniently used to infer network topology [49–55], weights of links are not evaluated. To overcome this difficulty, various techniques based on the correlation consideration [32, 39–41, 43–47] are designed but either a small noise limit is required or a prerequisite smoothing of the data should be done. Sometimes, a local driving many help the topology inference [34, 59, 60], which may not be possible in practice or existing in the available data. Non-interference or synchronization tools [35–38].

The recently developed framework based on compressed sensing can reconstruct the exact form of system equations through time series but it must be assumed that the system has only a few related terms defining dynamics, and the control equations are sparse in the high-dimensional nonlinear function space [32, 33]. In most complex networks, the number of actual connections is far less than the number of possible connections, which results in the sparsity of complex networks, and only limited data can be observed in practice [57], so the compressed sensing method is also successfully applied to the research of network topology inference. In compressed sensing theory [61–65], L_1 norm [56] caters to the requirement of sparsity, and error control term ensures the robustness of re-

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construction to noise and missing nodes, which thus enables detection of hidden nodes in complex networks [58].

Much attention has been paid to the influence of noise in time series analysis, but many of the existing algorithms just try to eliminate unwanted perturbations based on heuristic approaches. Here, according to the invariance property of the governing equations, we use one single universal expression to fit all local short segments of typical orbits, which is then utilized to derive the deterministic part of the dynamics and the noise characteristics at the same time. All the available points are used for a single-expression fitting so the technique is quite robust against noise contamination and there is no need to use another program to smooth the data first [66, 67]. Also, even with noise of moderate size, the computation will still be quite accurate since its influence on the orbit is properly taken care of. If the dynamics is expressible with polynomials, the evaluation of all the coefficients is done by solving linear equations, which is very efficient. Su *et al* proposed a local polynomial fitting method [68] for phase space reconstruction of multivariate chaotic time series, but their fitting is only valid locally and hence does not exploit the invariant structure of the global dynamics.

The main idea of our approach is detailed in Part II and applied to the famous Lorenz system [69] possibly driven by white noise in Part III under different conditions: with or without a known functional form of the equation of motion, with time series of all or part of the state variables. In Part IV, the new scheme is applied to an FHN neural network model [70] to procure the coupling constants with observed data in a quite high-dimensional phase space. In the final part, results are summarized and possible future research directions are pointed out.

II. MODEL BUILDING

A. Dynamical systems driven by white noise

A general n -dimensional nonlinear dynamical system driven by white noise can be written as:

$$\dot{x}(t) = f(x(t)) + \varepsilon(t) \quad (1)$$

where $x(t) = [x_1(t), x_2(t), x_3(t), \dots, x_n(t)]$ is the state variable, f describes the governing equations, and $\varepsilon(t) = [\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \dots, \varepsilon_n(t)]$ is an extra term that originates from the microscopic or the insignificant unknown part of the dynamics and usually is fast-changing, being simplified as white noise with the following characteristics:

$$\langle \varepsilon_i(t) \rangle = 0, \quad (2a)$$

$$\langle \varepsilon_i(t) \varepsilon_j(t') \rangle = 2D \delta_{ij} \delta(t - t'). \quad (2b)$$

where $\langle \cdot \rangle$ denotes the ensemble average and the delta function is a Kronecker one with respect to the discrete indices i, j but a Dirac one to the continuous time variable $t - t'$, where D represents the noise intensity of ε_i .

For nonlinear dynamical systems without noise, we will use the fourth-order Runge-Kutta algorithm to do the integration.

However, when considering a system driven by white noise, a second-order stochastic integration scheme is employed in this paper. please refer to Appendix A for a detailed explanation for this computation

B. Global polynomial fitting of the local dynamics

For reconstruction of nonlinear systems based on time series, considering the invariance of the governing laws, we employ globally valid analytic expressions for both local trajectories and evolution equations. Thus, the orbit of the system in phase space is divided into thousands of tiny segments and each short segment is fitted with the same expression. Specifically, we start from an initial point on each segment and fit a few steps forward and backward by tweaking the parameter values in the assumed analytic expressions based on a combined error estimation over all the segments. The advantage of this idea is that if data points in the time series are seriously contaminated by noise, the scheme will suppress the error through fitting the whole system and averaging out the irregular noisy fluctuation. Therefore, when the scale of the time series to be processed is very large, the advantage of this method becomes obvious.

The following expression is used for the local solution of Eq. (1)

$$x_i(t_0 + t) = x_i(t_0) + g_i(x_1(t_0), x_2(t_0), \dots, x_n(t_0))t + h_i(x_1(t_0), x_2(t_0), \dots, x_n(t_0))t^2, \text{ for } i = 1, 2, \dots, n. \quad (3)$$

where $|t| \ll 1$ is small. Thus, Eq. (3) only describes a short-time evolution, and in many cases a polynomial expression could be employed for g_i and h_i .

$$g_i(x_1(t_0), x_2(t_0), \dots, x_n(t_0)) = a_i + \sum_{j=1}^n b_{i,j} x_j + \sum_{j,k=1}^n c_{i,j,k} x_j x_k + \sum_{j,k,l=1}^n d_{i,j,k,l} x_j x_k x_l, \quad (4a)$$

$$h_i(x_1(t_0), x_2(t_0), \dots, x_n(t_0)) = a'_i + \sum_{j=1}^n b'_{i,j} x_j + \sum_{j,k=1}^n c'_{i,j,k} x_j x_k + \sum_{j,k,l=1}^n d'_{i,j,k,l} x_j x_k x_l, \quad (4b)$$

where $a_i, a'_i, b_{i,j}, b'_{i,j}, c_{i,j,k}, c'_{i,j,k}, d_{i,j,k,l}, d'_{i,j,k,l}$ are the coefficients in the model to be optimized by fitting all the short pieces of the time series. In a sense, Eq. (3) could be viewed as a Taylor expansion of the solution to the differential equation Eq. (1). If the original vector field f_i is of the polynomial type, Eq. (4a,4b) could be exact. For a general vector field, we hope that Eq. (4a,4b) are good approximations of the corresponding coefficients at least locally. Of course, other types of expressions may possibly be used for g_i and h_i , if extra information is available or could be extracted.

In this paper, we use polynomial approximation for g_i and h_i for simplicity. The system states at different times are used as the initial points $x_i(t_0)$ to fit all the corresponding pieces with Eq. (3). In more detail, if the initial state is $x_i(t_0)$, the forward or backward evolution for a time t (here

we make $t = mt_{unit}$, where $m \in \mathbb{Z}$ and t_{unit} is a chosen elementary time step) is described by Eq. (3). We may pick up $2m + 1$ points along the orbit, namely $x_i(t_0 - mt_{unit}), x_i(t_0 - (m-1)t_{unit}), \dots, x_i(t_0), x_i(t_0 + t_{unit}), x_i(t_0 + 2t_{unit}), \dots, x_i(t_0 + mt_{unit})$ for the fitting. The same expression should be valid for all initial conditions and a reliable description of the system is obtained by minimizing the following cost function

$$P = \sum_{i=1}^n \sum_{t_0=(m+1)t_{unit}}^{(N-m)t_{unit}} \sum_{j=-m}^m (x_i(t_0 + jt_{unit}) - x'_i(t_0 + jt_{unit}))^2, \quad (5)$$

with respect to the unknown coefficients, where $x'_i(t_0 + t)$ is the known time series and $x_i(t_0 + t)$ is computed from Eq. (3). N is the total number of points in the time series and m is the number of local evolution steps forward or backward. In the following computation, $t_{unit} = 0.01$.

C. Separate noise and data

In reality, available data is often polluted with noise for various reasons, which should be considered in the dynamics reconstruction and is assumed to be white here. When noise is weak, locally the noisy orbit may be viewed as a superposition of a deterministic orbit with noisy fluctuations. However, if the noise is moderate or strong, the orbit will deviate from

the deterministic orbit even locally. This influence will be calculated below in a general nonlinear system.

For the dynamical system Eq. (1), we may write down its integral form

$$x_i(t) = x_i(0) + \int_0^t f_i(x_1(s), x_2(s), \dots, x_n(s)) ds + \int_0^t \epsilon_i(s) ds. \quad (6)$$

which has a (deterministically) first-order approximation (stochastic 1/2-order)

$$x_i^{(1)}(t) = x_i(0) + f_i(x_1(0), x_2(0), \dots, x_n(0))t + \int_0^t \epsilon_i(s) ds. \quad (7)$$

and then a (deterministically) second-order approximation (stochastic 3/2-order)

$$x_i^{(2)}(t) = x_i(0) + \int_0^t f_i(x_1^{(1)}(s), x_2^{(1)}(s), \dots, x_n^{(1)}(s)) ds + \int_0^t \epsilon_i(s) ds. \quad (8)$$

A second-order Taylor expansion is carried out on $f_i(x_1^{(1)}(t), x_2^{(1)}(t), \dots, x_n^{(1)}(t))$, resulting in

$$\begin{aligned} & f_i(x_1^{(1)}(t), x_2^{(1)}(t), \dots, x_n^{(1)}(t)) \\ &= f_i(x_1(0), x_2(0), \dots, x_n(0)) + \sum_{j=1}^n (x_j^{(1)}(t) - x_j(0)) \frac{\partial f_i}{\partial x_j} + \frac{1}{2!} \sum_{j,k=1}^n (x_j^{(1)}(t) - x_j(0))(x_k^{(1)}(t) - x_k(0)) \frac{\partial^2 f_i}{\partial x_j \partial x_k}, \end{aligned} \quad (9)$$

where in the second order term, only the lowest order of

the difference $(x_j^{(1)}(t) - x_j(0))$ is needed. Upon Substituting Eq. (9) into Eq. (8), we have

$$\begin{aligned} x_i^{(2)}(t) &= x_i(0) + f_i(x_1(0), x_2(0), \dots, x_n(0))t + \int_0^t ds \sum_{j=1}^n [f_j(x_1(0), x_2(0), \dots, x_n(0))s + \int_0^s \epsilon_j(\tau) d\tau] \frac{\partial f_i}{\partial x_j} \\ &+ \frac{1}{2!} \int_0^t ds \sum_{j,k=1}^n [f_j(x_1(0), x_2(0), \dots, x_n(0))s + \int_0^s \epsilon_j(\tau) d\tau] [f_k(x_1(0), x_2(0), \dots, x_n(0))s + \int_0^s \epsilon_k(\rho) d\rho] \frac{\partial^2 f_i}{\partial x_j \partial x_k} + \int_0^t \epsilon_i(s) ds \\ &= x_i(0) + f_i t + \frac{t^2}{2} \sum_{j=1}^n f_j \frac{\partial f_i}{\partial x_j} + \frac{1}{2!} \sum_{j,k=1}^n \frac{\partial^2 f_i}{\partial x_j \partial x_k} \int_0^t ds \int_0^s \epsilon_j(\tau) d\tau \int_0^s \epsilon_k(\rho) d\rho + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \int_0^t \int_0^s \epsilon_j(\tau) d\tau ds + \int_0^t \epsilon_i(s) ds, \end{aligned} \quad (10)$$

In Eq. (9) and Eq. (10), f_i , $\frac{\partial f_i}{\partial x_j}$ and $\frac{\partial^2 f_i}{\partial x_j \partial x_k}$ are abbreviations of $f_i(x_1(0), x_2(0), \dots, x_n(0))$, $\frac{\partial f_i(x_1(0), x_2(0), \dots, x_n(0))}{\partial x_j}$ and

$\frac{\partial^2 f_i(x_1(0), x_2(0), \dots, x_n(0))}{\partial x_j \partial x_k}$ respectively. By taking an ensemble average of Eq. (10), the deterministic part of the orbit is ob-

tained:

$$\begin{aligned} & \langle x_i^{(2)}(t) \rangle \\ &= x_i(0) + f_i t + \frac{t^2}{2} \sum_{j=1}^n f_j \frac{\partial f_i}{\partial x_j} + \frac{1}{2} D t^2 \sum_{j=1}^n \frac{\partial^2 f_i}{\partial x_j^2}, \quad (11) \end{aligned}$$

where the last term is a second order contribution from the

noise, which serves to push the average off the deterministic orbit and should be taken into account if the noise strength D is not small. The previous local fitting is done actually for this average orbit, which contains noise contribution.

The noise can be estimated by taking a second moment of Eq. (10)

$$\begin{aligned} & \langle (x_i^{(2)}(t) - \langle x_i^{(2)}(t) \rangle)^2 \rangle \\ &= \langle \left[\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \int_0^t \int_0^s \varepsilon_j(\tau) d\tau ds + \int_0^t \varepsilon_i(s) ds + \frac{1}{2!} \sum_{j,k=1}^n \frac{\partial^2 f_i}{\partial x_j \partial x_k} \int_0^t ds \int_0^s \varepsilon_j(\tau) d\tau \int_0^s \varepsilon_k(\rho) d\rho - \frac{1}{2!} D t^2 \sum_{j=1}^n \frac{\partial^2 f_i}{\partial x_j^2} \right]^2 \rangle \\ &= 2Dt + 2D \frac{\partial f_i}{\partial x_i} t^2, \quad (12) \end{aligned}$$

to the lowest orders. From Eq. (12), it is easy to see that if $\partial f_i / \partial x_i$ is large, the second order modification of D should be considered for a better estimation of both the noise and hence the original vector field by Eq. (11).

In practice, once the fitting with Eq. (3) is completed, we regard it as the average orbit given by Eq. (11). Thus, by comparing it with the given observation $x_i^{(2)}(t)$, we may get a rough estimation of the noise strength D , neglecting the second order term in Eq. (12). Then we may estimate the unknown coefficients in the vector field f_i by minimizing the following error function

$$Q = \sum_{i=1}^n \sum_{t_0=(m+1)t_{unit}}^{(N-m)t_{unit}} \sum_{j=-m}^m (x_i(t_0 + jt_{unit}) - \langle x_i^{(2)}(t_0 + jt_{unit}) \rangle)^2, \quad (13)$$

where $x_i(t)$ is given by Eq. (3) and $\langle x_i^{(2)}(t) \rangle$ is from Eq. (11) with the above estimation of D . In Eq. (13), the summation is similar to that in Eq. (5). If the unknown coefficients enter f_i in a linear way, the minimization of Q could easily be done by repeatedly solving linear algebraic equations, starting from a first-order approximation by identifying the coefficient of t in Eq. (3) with f_i (see Eq. (11)).

D. Error estimation

After fitting the data, we examined the root-mean-square (one-step) prediction error MSE [28] and the relative error P_{eer} to properly evaluate the results, which could be explicitly written as follows:

$$MSE = \frac{1}{n(N-2m)(2m+1)} \sum_{i=1}^n \sum_{t_0=(m+1)t_{unit}}^{(N-m)t_{unit}} \sum_{j=-m}^m (x_i(t_0 + jt_{unit}) - x'_i(t_0 + jt_{unit}))^2, \quad (14a)$$

$$P_{eer} = \sum_{i=1}^n \sum_{t_0=(m+1)t_{unit}}^{(N-m)t_{unit}} \sum_{j=-m}^m \frac{(x_i(t_0 + jt_{unit}) - x'_i(t_0 + jt_{unit}))^2}{x'_i(t_0 + jt_{unit})^2}, \quad (14b)$$

where N represents the number of data points along the temporal direction, n is the number of degrees of freedom, m determines the number of points on a local piece of the orbit. The variables $x_i(t)$ and $x'_i(t)$ are defined similarly as in Eq. (5).

In the following, we also calculate the magnitude of the noise with a more precise evaluation of D . Explicitly, Eq. (12) will be used to get a better estimation of D based on the observed data and the polynomial fitting of the average orbit after the vector field f_i is determined. A better evaluation of D incurs better estimation of other unknown parameters in f_i .

This could be repeated again and again for quite accurate determination of all the parameters. The relative error of a parameter is defined as:

$$err_c = \frac{|c - c'|}{c'}, \quad (15)$$

where c is the fitting result and c' is the true value.

III. FITTING LORENZ SYSTEM

A. Lorenz system

The Lorenz system is a typical deterministic dissipative system[69], which reads

$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz,\end{aligned}\quad (16)$$

where the parameters $a > 0, b > 0, r > 0$. When $a = 10, b = \frac{8}{3}, r = 28$, it becomes chaotic and a strange attractor shows up. The Lorenz system has an obvious feature that nearby trajectories separate in an exponential form, which indicates the sensitive dependence of motion on initial conditions, and as a result it is impossible to make long-term prediction, because very small uncertainties will be magnified rapidly with time increasing.

We may arrange the time series data in a matrix with the following form.

$$A = \begin{pmatrix} x_1(t_1) & x_2(t_1) & x_3(t_1) \\ x_1(t_2) & x_2(t_2) & x_3(t_2) \\ \vdots & \vdots & \vdots \\ x_1(t_N) & x_2(t_N) & x_3(t_N) \end{pmatrix} \quad (17)$$

which is extracted from a long trajectory obtained with a fourth-order Runge-Kutta simulation. For example, we may choose the initial point as $[-8, 7, 27]$ and the evolution time step $t_{unit} = 0.01$, so the matrix A with $N = 10000$ basically stores much of the relevant dynamics of a typical trajectory in the phase space.

The fitting equation for the Lorenz system is as follows (see Eq. (4a, 4b)):

$$\begin{aligned}x_i(t + t_0) &= x_i(t_0) + g_i(x_1(t_0), x_2(t_0), x_3(t_0))t \\ &\quad + h_i(x_1(t_0), x_2(t_0), x_3(t_0))t^2.\end{aligned}\quad (18)$$

where $x_i = [x_1, x_2, x_3], i = 1, 2, 3$ denotes the components of the Lorenz system in x, y and z directions for convenience.

With the data stored in the matrix (17), a loss function P may be constructed, which depends on some unknown coefficients as in Eq. (4a) and Eq. (4b). However, based on the principle of least squares, the coefficients $(a_i, a'_i, b_{i,j}, b'_{i,j}, c_{i,j,k}, c'_{i,j,k}, d_{i,j,k,l}, d'_{i,j,k,l})$ of the model could be evaluated by solving linear equations. The error characteristics MSE and P_{eer} defined by Eq. (14a, 14b) with different N, m for the fitting are displayed in Table I.

The expression Eq. (18) is in fact a quite accurate description of the orbit. Upon substituting the point $x(t_0)$ into the model (Eq. (18)) we obtain the point at $t = t_0 + t_{unit}$, which is again taken as the initial point for the next iteration. The above process is repeated L times and we have the prediction of the next L points along the orbit. The x component of the original trajectory and the predicted orbit are depicted in Fig. 1, where the first 5000 points, could be used for the fitting

TABLE I. The error characteristics MSE and P_{eer} given by Eq. (14a, 14b) for different N, m in fitting a trajectory of the Lorenz system (Eq. (16)).

	$m = 1$	$m = 2$	$m = 3$
$N = 100$	$MSE = 4.73e-06$ $P_{eer} = 1.81e-08$	$MSE = 2.7e-03$ $P_{eer} = 1.04e-05$	$MSE = 0.26$ $P_{eer} = 9.91e-04$
$N = 1000$	$MSE = 2.63e-09$ $P_{eer} = 1.01e-11$	$MSE = 2.93e-07$ $P_{eer} = 1.12e-09$	$MSE = 4.55e-06$ $P_{eer} = 1.75e-08$
$N = 5000$	$MSE = 1.92e-09$ $P_{eer} = 7.38e-12$	$MSE = 1.67e-07$ $P_{eer} = 6.43e-10$	$MSE = 2.54e-06$ $P_{eer} = 9.79e-09$

the parameters. The next 400 points could be iteratively computed through the expression Eq. (18), even with a small noise $D = 0.01$. For $L > 5400$, the prediction will be completely invalid, which is due to the extreme sensitivity of the chaotic motion on initial conditions and noise distraction. Even minor disturbances will grow exponentially with time, resulting in uncontrollable errors between the prediction and the real data.

FIG. 1. Prediction: comparison of an exact trajectory (black curve) and its prediction (red curve) with the fitting Eq. (18) for the Lorenz system driven by white noise, where the prediction starts from $L(step) = 5000$, with $m = 1, t_{unit} = 0.01$, and the noise strength $D = 0.01$.

Similarly, with the fitting expression Eq. (18), If we know the form of the dynamics equation and the time series, the parameters in the Lorenz equation could be deduced by minimizing Q in Eq. (13). With a fixed time step $t_{unit} = 0.01$, different N, m produce different yet consistent outputs, which bear disparate relative errors defined in Eq. (15) and shown in Table II. It is easy to see that in all the calculations, the errors err_a, err_b, err_c are quite small and the three equation parameters could be reproduced reasonably well.

From Table I, Table II and Table III, we may draw the following conclusions: when fitting a noise-free time series, the more data points used for the fitting, the more accurate the result and the smaller the error. In addition, it can be seen from the tables that when $N = 1000$, the fitting error of the system is already very small, and further increase of the number of data points has only a minor impact on the error reduction.

TABLE II. The error characteristics err_a, err_b, err_c given by Eq. (15) for different N, m in fitting the parameters in the Lorenz system (Eq. (16)).

	$m = 1$	$m = 2$	$m = 3$
$N = 100$	$err_a = 0.0028$	$err_a = 0.0068$	$err_a = 0.0119$
	$err_b = 1.34e-04$	$err_b = 3.18e-04$	$err_b = 5.12e-04$
	$err_r = 4.94e-04$	$err_r = 0.0012$	$err_r = 0.0021$
$N = 1000$	$err_a = 0.0023$	$err_a = 0.0057$	$err_a = 0.0102$
	$err_b = 3.45e-06$	$err_b = 8.92e-06$	$err_b = 2.27e-05$
	$err_r = 1.51e-04$	$err_r = 3.75e-04$	$err_r = 6.70e-04$
$N = 5000$	$err_a = 0.0022$	$err_a = 0.0055$	$err_a = 0.0099$
	$err_b = 3.21e-06$	$err_b = 7.75e-06$	$err_b = 2.24e-05$
	$err_r = 2.13e-04$	$err_r = 5.31e-04$	$err_r = 9.49e-04$

TABLE III. The error characteristics err_a, err_b, err_c given by Eq. (15) for different t_{unit} in fitting the parameters in the Lorenz system (Eq. (16)), $m = 1$.

	$t_{unit} = 0.01$	$t_{unit} = 0.02$	$t_{unit} = 0.03$
$N = 1000$	$err_a = 0.0023$	$err_a = 0.0088$	$err_a = 0.018$
	$err_b = 3.45e-06$	$err_b = 0.0065$	$err_b = 0.0152$
	$err_r = 1.51e-04$	$err_r = 1.87e-04$	$err_r = 1.1e-04$

In addition, it is also observed that the fitting error increases gradually when m increases and t_{unit} . Two possible reasons could be envisaged for this. For small t_{unit} , the information of the vector field is mainly stored between neighboring points along the orbit. Extra points may not give much additional information on the local field if the noise is small enough and the round-off error is under control. Moreover, with more points included, higher-order polynomials in t should be used to describe the distant points. However, the coefficients of the higher-order terms could be very complicated and the functional form Eq. (4a,4b) may not be valid across different orders. Nevertheless, if the system were contaminated with noise, a few points along the orbit could dampen noise more effectively in principle.

B. The Lorenz system driven by white noise

For a Lorenz system driven by white noise, a matrix A' similar to A can be constructed. In this section, we mainly study the sensitivity of our scheme to noise, so we fix $N = 10000, m = 1, t_{unit} = 0.01$ and only change the size of noise.

The computation is basically similar to that without noise. According to the principle of least squares, based on Eq. (18), we obtain the optimal coefficients, where the influence of noise strength is shown in Table. IV.

TABLE IV. Influence of noise strength on the fitting results for the Lorenz system (Eq. (16)) with $N = 10000, m = 1, t_{unit} = 0.01$.

$D = 0.5$	$D = 1$	$D = 10$
$MSE = 0.0023$	$MSE = 0.0048$	$MSE = 0.0459$
$P_{eer} = 8.7604e-06$	$P_{eer} = 1.8313e-05$	$P_{eer} = 1.6094e-04$

From the table, we see that with the increase of noise, the fitting error seems to increase proportionally to D . However, even with considerably large noise, the fitting error is still within acceptable range. Therefore, the current technique appears robust upon noisy perturbation. Through our formula, even if the noise intensity is strong and the equation of motion is unknown, we can still properly evaluate the noise intensity and extract the original vector field from the noise data. If the analytical form of the vector field (here Eq. (16)) is known, Eq. (11) and Eq. (12) may be used to evaluate the system parameters a, b, r and the noise strength D , using the method of least squares. With $N = 10000, m = 1, t_{unit} = 0.01$ but different noise intensity, the system parameters are computed and displayed in Table. V.

In Table V and VI, D', a', b' and r' respectively represent the computed system parameters. Their errors err_D, err_a, err_r , and err_b are defined in Eq. (15). Even for relatively large noise $D=50$ (the typical trajectory and time series show in Fig. 2), the computed strength D' and a', b' and r' is very close to the true value, and the fitting error grows with the noise in general. One may imagine that the current technique could possibly fail in the presence of very strong noise ($D > 50$) since the perturbation calculation in the previous section may become invalid. As can be seen from Table VI, in the presence of noise, if there are too few data points, it will have a great impact on the results, especially on the estimation of noise intensity.

C. The reconstruction of the unknown vector field

In the previous calculation, we assumed that the form of the system is known (Eq. (16)) and only the unknown parameters a, b, r need to be fixed. However, in actual situations, the functional form of a vector field may not be known and we have to reconstruct it from scratch. In general, a functional basis $\{Z_j\}_{j=1,\dots,d}$ should be taken and the equation of motion is expanded in terms of this basis.

$$\dot{x}_i = \sum_{j=1}^d a_{i,j} Z_j, i = 1, 2, 3 \quad (19)$$

where $a_{i,j}$ are the parameters that need to be determined from the data. For Lorenz-like systems, a polynomial basis $Z = [1, x, y, z, xy, xz, yz, x^2, y^2, z^2]^T$ [47] may be used. With the functional form Eq. (19), the computation scheme above could be employed again and the coefficients $\{a_{i,j}\}$ are conveniently obtained by solving linear equations. In literature, the SINDy's method designed by Brunton [33] utilizes compressed sensing to do similar fitting and has achieved remarkable success by arriving at the often few coefficients present in the analytic expression. Here, to embrace the idea, an L_1 -norm term $\beta \sum_{i,j} |a_{i,j}|$ could simply be added to the error function to improve the fitting, where β is the weight for the sparsity requirement. Table VII compares the fitting results from both methods for the Lorenz equation (Eq. (16)) with $N = 10000, t_{unit} = 0.01, m = 1, \beta = 0.1$. At $D = 1$, both results are good and accurate, even though the current one already seems a little advantageous. However, at $D = 5$, the

FIG. 2. A: The x - z projections of a trajectory of the Lorenz system Eq. (16), B: Time series of x , with $N = 10000, t_{unit} = 0.01, a = 10, b = \frac{8}{3}, r = 28$, and the noise strength $D = 50$.

TABLE V. The recovered noise strength and the parameters of the Lorenz system for different noise with $N = 10000, m = 1, t_{unit} = 0.01$.

$D = 0.5$		$D = 1$		$D = 10$		$D = 50$	
$D' = 0.50$	$err_D = 0.0128$	$D' = 1.0114$	$err_D = 0.0114$	$D' = 10.107$	$err_D = 0.0107$	$D' = 50.525$	$err_D = 0.0105$
$a' = 9.983$	$err_a = 0.0017$	$a' = 9.9771$	$err_a = 0.0029$	$a' = 10.089$	$err_a = 0.0089$	$a' = 9.848$	$err_a = 0.0152$
$r' = 27.9889$	$err_r = 3.98e-04$	$r' = 27.9852$	$err_r = 5.3e-04$	$r' = 27.9552$	$err_r = 0.0016$	$r' = 27.9048$	$err_r = 0.0034$
$b' = 2.6701$	$err_b = 0.0013$	$b' = 2.662$	$err_b = 0.0017$	$b' = 2.6512$	$err_b = 0.0058$	$b' = 2.6941$	$err_b = 0.0103$

TABLE VI. The error characteristics err_a, err_b, err_c given by Eq. (15) for different N, D in fitting the parameters in the Lorenz system (Eq. (16)).

	$D = 1$	$D = 10$	$D = 50$
$N = 100$	$err_D = 0.3025$	$err_D = 0.0651$	$err_D = 0.2913$
	$err_a = 0.0237$	$err_a = 0.0735$	$err_a = 0.1601$
	$err_b = 0.0186$	$err_b = 0.0673$	$err_b = 0.1144$
	$err_r = 0.00055$	$err_r = 0.0163$	$err_r = 0.0378$
$N = 1000$	$err_D = 0.0374$	$err_D = 0.0416$	$err_D = 0.0481$
	$err_a = 0.0087$	$err_a = 0.0288$	$err_a = 0.0391$
	$err_b = 0.005$	$err_b = 0.0176$	$err_b = 0.0103$
	$err_r = 0.0016$	$err_r = 0.0052$	$err_r = 0.0114$

result by our method seems much more accurate. With even larger noise, our method still works, displaying its privilege in dealing with noisy dynamics.

D. Reconstruction of the Lorenz system with partial observations

In many cases, we only observe part of the state variables of a system. For example, in the Lorenz system we may only have access to the data in the x and z directions. In this case, Takens embedding theorem [15] should be invoked to carry out the fitting. Delayed coordinates are used for possible extra dimensions of the system. As the second components are unknown, we are allowed to use only the first and the third column of the time series matrix (A) Eq. (17) as the observed time series data. There is a whole literature for the proper

TABLE VII. The fitted coefficients with general polynomial form for the Lorenz system (Eq. (16)) with $N = 10000, m = 1, t_{unit} = 0.01, D = 1, D = 5$, by the method of SINDy [33] $N = 100000, t_{unit} = 0.001, D = 1, D = 5$

	D=1			D=5			
	$\dot{x}$	$\dot{y}$	$\dot{z}$	$\dot{x}$	$\dot{y}$	$\dot{z}$	
Results of SINDy	x	-9.9614	27.5343	0	-9.67	26.6993	0
	y	9.9796	-0.8038	0	9.7617	-0.3753	0
	z	0	0	-2.6647	0	0	-2.6251
	xy	0	0	1.0003	0	0	0.9917
	xz	0	-0.9900	0	0	-0.9769	0
	x^2	0	0	0	0	0	0
Results of our method	x	-9.9923	27.946	0	-10.0958	28.2978	0
	y	10.0047	-0.9709	0	10.0977	-1.0529	0
	z	0	0	-2.6678	0	0	-2.6426
	xy	0	0	0.9991	0	0	0.9952
	xz	0	-0.9984	0	0	-1.00814	0
	x^2	0	0	0	0	0	0

choice of the embedding dimension and the time delay [5, 71]. Here, we assume that the choice has been made: the embedding dimension is 3 and the time delay is $\tau = 0.06$. We use x, x_τ, z as the state variable for the reconstruction, where x_τ is the value of x delayed by τ . Similar to what has been done before, Eq. (18) gives a local fitting in terms of polynomials in x, x_τ, z , which could be used as the state variables to construct a vector field if there is no extra information about the dynamics. In contrast, if the functional form of the Lorenz system is known (with unknown parameters), we may express the unknown variable y with a combination of x, z and x_τ .

$$y_i(t+t_0) = l(x(t_0), x_\tau(t_0), z(t_0)) + g(x(t_0), x_\tau(t_0), z(t_0))t + h(x(t_0), x_\tau(t_0), z(t_0))t^2 \quad (20)$$

where l, g, h are polynomials of their arguments, being sim-

ilar to Eq. (4a) or Eq. (4b). All the unknown coefficients including the parameters could be determined by substituting Eq. (20) and the local expressions for x, z into Eq. (16) and minimizing the squared error at all the available data points. The parameters of the Lorenz system obtained in this procedure are $a = 10.0240, b = 2.6867, r = 30.7600$ and the relative errors are $err_a = 0.24\%, err_b = 0.75\%, err_r = 9.86\%$. The fitting errors defined by Eq. (14a,14b) are $MSE = 0.6993$ and $P_{eer} = 0.0086$ respectively.

IV. FITTING THE FHN NEURAL NETWORK

A. FHN neural network

The FitzHugh-Nagumo(FHN) system was derived by FitzHugh in a simplification of the Hodgkin-Huxley(HH) model[72]. The FHN equation is a second-order nonlinear system, which can be used to simulate the dynamical behavior of neurons. One particular form of the equation is as follows:

$$\begin{aligned} \dot{v}_i &= v_i - \frac{1}{3}v_i^3 - w_i + c_1 + \sum_{j=1}^N W_{ij}v_j + \Gamma_{1,i}, \\ \dot{w}_i &= \delta(v_i - bw_i + c_2) + \Gamma_{2,i}. \end{aligned} \quad (21)$$

where v_i is the rate of change of the membrane potential, and w_i is a slowly changing recovery variable for the i th neuron. In this model, the dynamical behavior of the system is controlled by the coefficients δ, b, c_1, c_2 and the interaction weight W_{ij} . $\Gamma_{1,i}, \Gamma_{2,i}$ are noise terms, which are usually taken to be white satisfying Eq. (2a-2b). When external noise is added to the system, coherent resonance may occur with proper system parameters [73, 74].

We use an FHN network with $L = 10$ nodes to test our scheme, shown in Fig. 3 as a directed graph. In order to reach a balanced state, the sum of the influences from other nodes on a node should be opposite to the self-influence. Therefore, the interaction between nodes can be expressed as a Laplace matrix W_{ij} . In more detail, when $i \neq j, W_{ij} \in [0, 1)$ being randomly distributed, $i, j = 1, 2, \dots, L$; when $i = j, W_{ii} = -\sum_{i \neq j} W_{ij}$. The parameters in our simulation of the network are chosen to be $\delta = 0.08, b = 0.8, c_1 = 0, c_2 = 0.7$, and the initial state variables are distributed uniformly in the $[0, 1)$ interval. The following stochastic Runge-Kutta scheme [75, 76] is employed to do the integration.

$$x(\Delta t) = x_0 + \frac{1}{2}\Delta t(F_1 + F_2) + (2\Delta t)^{\frac{1}{2}}w_0. \quad (22)$$

where the intermediate variables F_1 and F_2 are computed as follows:

$$\begin{aligned} F_1 &= f(x_0 + (2\Delta t)^{\frac{1}{2}}w_1), \\ F_2 &= f(x_0 + \Delta t F_1 + (2\Delta t)^{\frac{1}{2}}w_2). \end{aligned} \quad (23)$$

and w_0, w_1, w_2 are random numbers satisfying the standard normal distribution. The evolution of the variable v_1 is plotted

in Fig. 4 for the time interval $[0, 10000]$. Over a fluctuating background, several spikes are clearly seen.

FIG. 3. Directed Graph of 10 Nodes for an FHN neuronal network model.

FIG. 4. The change of the variable v_1 vs time of the FHN model (Eq. (21)), where the integration time step $t_{step} = 0.1$ and the noise variance in the Eq. (21) is $D_v = 0.005, D_w = 0.01$.

B. Fitting the FHN model

In the current fitting, the functional form of the vector field is known though the dimension of the phase space is quite high. In order to reduce the amount of calculation, the mutual influence in the v variables between different nodes of the network is approximated by the summation term on the right-hand side of the following equation to the lowest order.

$$\begin{aligned} v_i(t_0 + t) &= v_i(t_0) + \sum_j^L q_{i,j} v_j(t_0) t \\ &\quad + g_{vi}(v_i(t_0), w_i(t_0)) t + h_{vi}(v_i(t_0), w_i(t_0)) t^2, \\ w_i(t_0 + t) &= w_i(t_0) + g_{wi}(v_i(t_0), w_i(t_0)) t \\ &\quad + h_{wi}(v_i(t_0), w_i(t_0)) t^2. \end{aligned} \quad (24)$$

where $i = 1, 2, \dots, L$. To facilitate writing, we use one variable x_i to collectively denote the variables $\{v_i, w_i\}_{i=1, \dots, L}$. The coefficients $g_{vi}, h_{vi}, g_{wi}, h_{wi}$ in Eq. (24) could be written as:

$$g_{vi} = a_i^1 + \sum_{j=1}^2 b_{i,j}^1 x_{ij}(t_0) + \sum_{j,k=1}^2 c_{i,j,k}^1 x_{ij}(t_0) x_{ik}(t_0) + \sum_{j,k,l=1}^2 d_{i,j,k,l}^1 x_{ij}(t_0) x_{ik}(t_0) x_{il}(t_0), \quad (25a)$$

$$h_{vi} = a_i^2 + \sum_{j=1}^2 b_{i,j}^2 x_{ij}(t_0) + \sum_{j,k=1}^2 c_{i,j,k}^2 x_{ij}(t_0) x_{ik}(t_0) + \sum_{j,k,l=1}^2 d_{i,j,k,l}^2 x_{ij}(t_0) x_{ik}(t_0) x_{il}(t_0), \quad (25b)$$

$$g_{wi} = a_i^3 + \sum_{j=1}^2 b_{i,j}^3 x_{ij}(t_0) + \sum_{j,k=1}^2 c_{i,j,k}^3 x_{ij}(t_0) x_{ik}(t_0) + \sum_{j,k,l=1}^2 d_{i,j,k,l}^3 x_{ij}(t_0) x_{ik}(t_0) x_{il}(t_0), \quad (25c)$$

$$h_{wi} = a_i^4 + \sum_{j=1}^2 b_{i,j}^4 x_{ij}(t_0) + \sum_{j,k=1}^2 c_{i,j,k}^4 x_{ij}(t_0) x_{ik}(t_0) + \sum_{j,k,l=1}^2 d_{i,j,k,l}^4 x_{ij}(t_0) x_{ik}(t_0) x_{il}(t_0), \quad (25d)$$

where for convenience we use the notation $x_{i1} = v_i, x_{i2} = w_i$. The superscript in the coefficient distinguishes the coefficients for different variables. With the method of least squares, all these coefficients could be determined from a given long time series.

In this model, $N = 10000$ observation points with time interval $t_{step} = 0.08$ are taken from a typical orbit, according to which local fitting coefficients are computed with Eq. (24) and Eq. (25a,25b,25c,25d). Among them, the calculated coefficients include the elements of matrix W_{ij} . Consequently, the parameters in the FHN model: $\delta, b, c_1, c_2, D_v, D_w$ are then estimated with the method of least squares as well. To refine the computation, we may treat b, c_1, c_2, D_v, D_w as known and minimize an equation similar to Eq. (13) to get a better W_{ij} and δ , which could be fixed in the next turn when computing other parameters. The iteration is carried out until all the parameter values become steady. The fitting results of the coupling coefficients W_{ij} are plotted against the true values in Fig. 5. All the non-diagonal terms of W_{ij} are positive, which are computed with quite high accuracy. The diagonal terms of W_{ij} are all negative and also have a good estimation but not as accurate as the non-diagonal terms, probably due to its size. The estimates of the parameters of the dynamics equation are listed in Table VIII, where the error is calculated according to Eq. (15).

FIG. 5. Comparison of the matrix elements W_{ij} in Eq. (21) with the time step $t_{step} = 0.08, N = 10000$ for the noise strength $D_v = 0.0025, D_w = 0.05$.

TABLE VIII. Estimates of the parameters for the local dynamics of Eq. (21) and the noise strength.

var	original	fitting	error
δ	0.08	0.0766	4.25%
b	0.8	0.8031	0.39%
c_1	0	0.000098	0.0098%
c_2	0.7	0.6907	1.33%
D_v	0.0025	0.0027	8%
D_w	0.05	0.0497	0.6%

V. CONCLUSION

Reliable reconstruction of a nonlinear dynamical system is very important for its proper understanding, prediction or control. In this paper, one same set of invariant polynomials are used to fit all the local pieces of given trajectories of observables, which are then used to deduce the equations of motion. Specifically, the impact of noise on an average trajectory is evaluated to the lowest order, which is seen to be proportional to the noise strength D . When noise is small, this correction may be neglected but it could become significant when noise is moderate or strong. Due to the use of all available points for fitting, the current scheme appears quite robust in the presence of noise, which is demonstrated in the Lorenz system comprehensively. Regardless of whether the functional form of the equation of motion is known or not, the dynamics can be reconstructed with quite high accuracy. A specific procedure is designed to retrieve the parameters in the equation if only part of the state evolution is available. The computation of all unknown coefficients could be carried out by solving linear equations, which bestows high efficiency and is nicely implemented in an FHN neural network model.

In the current paper, we use only polynomials to do all the fitting, which is convenient and capable of handling quite general situations. However, in other circumstances, alternative basis may be used to better represent a system under investigation. It would be very interesting to find a systematic way to design a proper basis of functions in general upon which to do the expansions[2, 5]. Moreover, for very high-dimensional systems, such as turbulent fluids or brain networks[77, 78], we may not be able to write a lot of nonlinear terms in the fitting

expressions. How to choose relevant terms based on the time series itself is worthy of further investigation.

Although the current scheme may handle systems in the presence of weak or moderate noise, it is doomed to fail if the noise is too strong since a perturbation calculation is performed in our derivation of noise effects. A different consideration may be needed to disclose the regular part of the dynamics that is buried under rampant noise. Even when the noise is not so strong but is colored, the current scheme may falter if the correlation time is not short. One way out of this caveat is to model the colored noise with a system only involving white noise[77]. A proper treatment of fluctuations may even help us dump the non-essential degrees of freedom to noise terms and achieve considerable reduction of the complexity of a problem[78].

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA AVAILABILITY STATEMENT

All data, models, generated or used during the study appear in the submitted article, code generated during the study are available from the corresponding author by request.

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