

Closed-form solution for column buckling optimization

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Closed-form solution for optimization of buckling column

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Abstract

An optimization problem for a column, loaded by axial forces, whose direction and value remain constant, is studied in this article. The dimensional analysis introduces the dimensionless mass and rigidity factors, which simplicities the mathematical technique for the optimization problem. With the method of dimensional analysis, the solution of the nonlinear algebraic equations for the Lagrange multiplier is superfluous. The closed-form solutions for Sturm-Liouville and mixed types boundary conditions are derived. The solutions are expressed in terms of the higher transcendental function. The principal results are the closed form solution in terms of the hypergeometric and elliptic functions, the analysis of single- and bimodal regimes, and the exact bounds for the masses of the optimal columns. The proof of isoperimetric inequalities exploits the variational method and the Hölder inequality. The isoperimetric inequalities for Euler's column are rigorously verified. The isoperimetric inequality could be formulated with the sharp inequality sign, because the optimal solution could not be attained for any design.

1. Optimality conditions in problems of stability

The statement of the optimization problem is discussed in Henrot (2006). The Lagrange hypothesis, that the optimal rod possesses the constant cross section, was abandoned in Euler's buckling problem by Tadjbakhsh and Keller (1962). In the cited work, the rigorous mathematical validation of optimality of single-mode solution was given and an analytic solution for two types of boundary conditions: clamping/hinging and clamping/clamping was obtained. The corresponding isoperimetric inequalities were also demonstrated. The optimal shapes of the column were discussed by Barnes (1977).

However, it was shown in Olhoff and Ramussen (1977), that the solution, obtained for a rod clamped at both ends is invalid, since this rod has a different buckling form (bimodal case) and the critical load is less than expected by Tadjbakhsh and Keller (1962). The analytical solution of the bimodal optimization problem for a rod, clamped at both ends, was obtained independently in Seiranyan (1984), Masur (1984). The optimal design problem of columns has logically involved variable cross sections (Elishakoff, 2000).

The problem of maximizing the minimum eigenvalue of a self-adjoint operator is examined in Bratus' and Seyranian (1983). An isoperimetric condition is imposed on the control variable. It was shown in the cited paper, that in a number of cases the optimal solutions are characterized by two or more forms of loss of stability or natural oscillations. In the case of conservative systems described by self-adjoint equations, this signifies multiplicities of eigenvalues, i.e., of critical loads, under which loss of stability frequencies occurs. The necessary conditions for an extremum are obtained in the case when the optimal solution is characterized by a double eigenvalue.

In the paper Bratus' (1991), the problems of eigenvalue optimization for elliptic boundary-value problems were considered. The coefficients of the higher derivatives are determined by the internal characteristics of the medium and play the role of control. The necessary conditions of the first and second order for problems of the first eigenvalue maximization were presented.

In the paper Olhoff and Seyranian (2008), is shown that bimodal solutions exist for columns that rest on a linearly elastic (Winkler) foundation and have clamped-clamped and clamped-simply supported ends. The equilibrium equation for a non-extensible, geometrically nonlinear elastic column is then derived, and the initial post-buckling behavior of a bimodal optimum column near the bifurcation point is studied using a perturbation method.

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In the paper Farshad and Tadjbakhsh (1973), the optimal shape of a column with the most general conservative state of loading is determined. The boundary conditions of a column contain, in general, 16 parameters; it is shown that the boundary values of a column with general conservative loading depend on no more than nine constants. The influence of these parameters on the optimal shape and corresponding buckling load is also obtained.

Egorov and Kondrat'ev (1996) deliberated some estimates of the eigenvalues of self-adjoint boundary-value problems for an ordinary differential equation.

The solution of the optimization problem for a rod, clamped at both ends, to the author's awareness, was not expressed in the form of closed form and with isoperimetric inequalities. The proof of isoperimetric inequality is based on optimality conditions for multiple eigenvalues. The

2. Stability problem of a compressed rod

2.1. Differential equations of stability

Consider the problem of optimizing the stability of a rod. The rod is placed horizontally (along x axis) and is compressed by forces f in x direction applied to its ends. For some critical value of this parameter, the rod buckles. Buckling takes place in the xOz plane. The origin of the coordinate system coincides with middle point of the rod, and the x axis passes through the point of the rod's support. We study the rod's equilibrium in the case of small deflections within the framework of the linear theory of elasticity. Let the deflection of the bent axis of the rod from the line of action of forces compressing the rod and $2L$ its length is:

$$y(x), -L \leq x \leq L$$

For the beginning, it is assumed that the rod has constant rigidity in various cross-sectional planes, and therefore

$$J_{yy} = J_{zz} = J, J_{yz} = 0,$$

where J_{yy} and J_{zz} are the moments of inertia of the transverse cross-sectional area with respect to lines passing through a point on the neutral axis of bending and parallel to the axes y and z , respectively. The bending moment in the cross-section with the bending rigidity EJ is $m = EJy''$. The stability of a compressed elastic rod with certain end conditions is considered:

$$(a^\alpha y'')'' + \Lambda y'' = 0, \Lambda = \frac{\tilde{f}}{k_\alpha E}, EJ = k_\alpha A^\alpha.$$

The function $a = a(x)$ designates the cross-sectional area along the span $-L \leq x \leq L$. The axial force in the column is \tilde{f} . The bending stiffness is $J(x)$. The quantity a^α is proportional to the flexural rigidity, and the exponent α takes the values of 1, 2 and 3. The case $\alpha = 2$ is mostly studied in this manuscript. This case corresponds to a congruent change in the form of the cross-section. For the circular cross-section the constant is

$$k_2 = (4\pi)^{-1}.$$

For the cross-section in the form of an equilateral triangle the value for the constant is

$$k_2 = \sqrt{3}/18.$$

The cases $\alpha = 1$ and $\alpha = 3$ describe the situations in which the form of transverse cross-section undergoes an affine transformation such that one of the geometrical dimensions of the cross-section (the width or height of the cross-section respectively) changes. The values of constants k_α are given in Banichuk (1990).

In terms of the dimensionless coordinate, the stability equation can be written as

This equilibrium equation is of the fourth order. It can be reduced to the second order, if instead of deflection y the bending moment is used as the unknown variable. In terms of the bending moment the equilibrium equation for the bending of a compressed rod may be written

$$m'' + \Lambda a^{-\alpha} m = 0 \text{ for } -L < x < L.$$

The boundary conditions contain the moment and the displacement in the expressions. However, the reduction of the order of the stability equation requires the formulation of boundary in terms of moment only. This problem was solved by Kamke (1938). For derivation of the boundary values in terms of bending moment the twice integration of the equilibrium equation is usually performed. Taking into account the assigned boundary conditions on displacements, the boundary conditions are formulated in terms of the bending moment only.

2.2. Boundary conditions

The following boundary conditions are usually formulated. The rod is clamped at the end $x = x_0$, if this end fulfils the conditions

$$y|_{x=x_0} = 0, \quad y'|_{x=x_0} = 0.$$

The rod is free at the end $x = x_0$, if the moment and bending force Q vanish:

$$m|_{x=x_0} = EJy''|_{x=x_0} = 0, \quad Q|_{x=x_0} = (EJy'')'|_{x=x_0} = 0.$$

The rod is hinged at the end $x = x_0$, if the moment and displacement vanish

$$m|_{x=x_0} = EJy''|_{x=x_0} = 0, \quad y|_{x=x_0} = 0.$$

The following boundary conditions are used:

I. the boundary conditions for the rod which is clamped at $x = -L$ and free at $x = L$ are the following

$$y|_{x=-L} = 0, \quad y'|_{x=-L} = 0, \quad EJy''|_{x=L} = 0, \quad (EJy'')'|_{x=L} = 0.$$

In terms of moments the boundary conditions (I) are

$$m'|_{x=-L} = 0, m|_{x=L} = 0. \tag{1}$$

II. if the rod is clamped $x = -L$ and hinged at $x = L$, the boundary conditions assume the form

$$y|_{x=-L} = 0, \quad y'|_{x=-L} = 0, \quad EJy''|_{x=L} = 0, \quad y|_{x=L} = 0.$$

In terms of moments the boundary conditions (II) are

$$(m' + m)|_{x=-L} = 0, M|_{x=L} = 0. \quad (2)$$

III. If the rod is hinged at $x = L$ and $x = -L$ the boundary conditions are:

$$EJy''|_{x=-L} = 0, y|_{x=-L} = 0, EJy''|_{x=L} = 0, y|_{x=L} = 0.$$

In terms of moments the boundary conditions (III) read as:

$$m|_{x=\pm L} = 0.$$

IV. If the ends $x = -L$ and $x = L$ are clamped, the boundary conditions are

$$y|_{x=L} = 0, y'|_{x=L} = 0, y|_{x=-L} = 0, y'|_{x=-L} = 0.$$

In terms of moments the boundary conditions (IV) are

$$(m' + m)|_{x=-L} = m|_{x=L}, m'|_{x=-L} = m'|_{x=L} \quad (3)$$

In this Chapter, the variational method for establishing the isoperimetric inequality will be applied. We use for brevity for integrals of functions $\varphi(x)$, $-L \leq x \leq L$ the notation:

$$\langle \varphi \rangle = \int_{-L}^L \varphi(\xi) d\xi.$$

The variational principle for the stability of a rod in the dimensionless form can be written as

$$\Lambda[A] = \Lambda[m, A] = \min_{\tilde{m}=\tilde{m}(\xi)} \lambda[\tilde{m}, A]. \quad (4)$$

In Eq. (4), the Rayleigh's quotient for buckling is:

$$\lambda[\tilde{m}, A] = \frac{\langle \tilde{m}'^2 \rangle}{\langle a^{-\alpha} \tilde{m}^2 \rangle}.$$

The admissible buckling moment for the Rayleigh's quotient is $\tilde{m} = \tilde{m}(x)$. The admissible functions $\tilde{m}(x)$ in (4) are all functions, having piecewise continuous first derivatives, satisfying definite boundary conditions. The actual buckling moment $m = m(x)$ delivers the minimal value for the Rayleigh's quotient. Euler's equation for variational problem (4) is (1).

2.3. Dimensionless factors for efficiency evaluation of optimization

Consider hereafter columns having a fixed mass of material with the unit density:

$$\mathfrak{M} = \mathfrak{B} = \text{const}, \mathfrak{B} = \langle a \rangle.$$

The value $\langle a \rangle$ expresses the volume \mathfrak{B} of the material. This volume \mathfrak{B} remains equal for all columns:

$$\langle a \rangle = 1. \quad (5)$$

For the boundary value conditions I to IV there are different values of masses of the optimal rods. The comparison of the results is not straightforward. The influence of the exponent α influences the

estimations for the optimization effects. Another argument for the introduction of the invariant optimization factors is methodical. In the variational calculus is common to get one factor as the optimization objective and the others as the a-priori given constraints. To convert it into an unconstrained problem the method of Lagrange multipliers are commonly used. For the column the optimality expresses with the isoperimetric inequalities. The resulting unconstrained problem with Lagrange multipliers increases number of variables. The new number of unknown variables is the original number of variables plus the original number of constraints. The constraints are usually solved for some of the variables in terms of the others, and the former can be substituted out of the objective function, leaving an unconstrained problem in a smaller number of variables. This method of solution of leads to the nonlinear algebraic equations for Lagrange multipliers. These nonlinear equations in the most cases do not possess the closed analytical solutions and are solvable only numerically.

Instead of dealing with the Lagrange multipliers, we introduce the certain invariant factors. Consider the columns with the same form of cross-sections, fixing the exponent α . For each fixed value of α we introduce two factors:

$$F_A = \frac{\Lambda L^{p_1}}{\sqrt{p_2}}, F_B = \frac{\Lambda L^{p_3}}{B^{p_4}}. \quad (6)$$

These two dimensionless factors are the commensurable physical quantities are of the same kind and can be directly compared to each other, even if they are originally expressed in differing units of measure. For some arbitrary powers p_1, p_2, p_3, p_4 , the factors F_A, F_B alter for any affine transformation of the column. The affine transformation of the column is the product of two elementary transformations, namely homothety and scaling. The homothety of ratio ζ multiplies lengths by ζ . Thus, ζ is the ratio of magnification or dilation factor or scale factor or similitude ratio. The cross-section function $A(\xi)$ scales by another factor ϱ , such that for the affine transformed column the cross-section function will be $\varrho A(\xi)$. Apparently, the eigenvalue Λ alters in course of the affine transformation of the column. If the column will be to twice long but the cross-section function and volume of the column remain constant, the eigenvalue will be 16 times smaller. If the length does not alter, but the volume of the column doubles, the eigenvalue will be four times high.

We use the factors F_A, F_B for the comparisons of different designs. The critical buckling load $\tilde{f} = k_\alpha E \Lambda$ inherits the factor k_α and is proportional to this value. Evidently, that the ratios of the buckling loads for different designs with the same form of the cross-sections do not depend on the constants k_α . For different cross-sections the actual value of k_α have to be used.

With the methods of dimensional analysis, we can immediately determine the distinctive choice of powers

$$p_1 = 2 + \alpha, p_2 = \alpha, p_3 = 3, p_4 = 1,$$

$$F_A = \frac{\Lambda L^{2+\alpha}}{\sqrt{\alpha}}, F_B = \frac{\Lambda L^3}{B}.$$

The factors F_A, F_B do not alter for any affine transformation of the column. Thus, the factors F_A, F_B are invariant to the affine transformation of the column and provide a natural basis for the comparison of different designs.

With the above factors, the estimation of the effect of mass optimization turns out to be trivial. For this purpose, we consider the reference design with the constant cross-section along the span. The invariant factors for the reference design are \tilde{F}_A, \tilde{F}_B . For all exponents α and for the boundary conditions III with both ends hinged both factors are:

$$\tilde{F}_{A(III)} = \tilde{F}_{B(III)} = \pi^2.$$

For all exponents α and for the boundary conditions IV with both ends clamped both factors are also equal:

$$\tilde{F}_{A(IV)} = \tilde{F}_{B(IV)} = 4\pi^2.$$

The greater the factors are, the higher the buckling force for the given length and volume of the column. For example, the buckling force of the reference clamped column is four times the buckling force of the reference column with the hinged ends.

The dual formulations are typical the optimization of buckling column as well. For the dual formulations, the masses of the columns for the fixed lengths and fixed buckling forces are compared. The volumes and masses of the optimal and reference columns relate to each other as the inverse roots of the order α of the factors F_A :

$$\frac{V_{III}}{\bar{V}_{III}} = \alpha \sqrt{\frac{\tilde{F}_{A(III)}}{F_{A(III)}}}, \quad \frac{V_{IV}}{\bar{V}_{IV}} = \alpha \sqrt{\frac{\tilde{F}_{A(IV)}}{F_{A(IV)}}}.$$

Specifically, the column with the higher value of the factor F_A possesses the lower mass. In the same way, the ratio of the averaged stiffness is the inverse ratio of factors F_B :

$$\frac{B_{III}}{\bar{B}_{III}} = \frac{\tilde{F}_{B(III)}}{F_{B(III)}}, \quad \frac{B_{IV}}{\bar{B}_{IV}} = \frac{\tilde{F}_{B(IV)}}{F_{B(IV)}}.$$

We use systematically the method of dimensionless factors for the optimization analysis. The applied method for integration of the optimization criteria delivers different length and volumes of the optimal columns. Instead of the seeking for the columns of the fixed length and volume, we directly compare the columns with the different lengths and cross-sections using the invariant factors.

3. Optimization problem of the Sturm type

3.1. Optimality conditions for the Sturm type boundary conditions

The eigenvalue problems for the ordinary differential equation with the above given boundary conditions are self-adjoint. The conditions (1) and (2) are of the Sturm type; see Hazewinkel, Michiel (2001), Zettl (2005). There exists an infinite set of eigenvalues, all eigenvalues are real and positive and can be arranged as a monotonic sequence, and each eigenvalue is simple:

$$\Lambda_1 < \Lambda_2 < \dots < \Lambda_{2k-1} < \Lambda_{2k} < \dots$$

The column with the thickness distribution $A(\xi)$ which satisfies the necessary optimality condition

$$m^2 = c_L A^{\alpha+1} \tag{7}$$

and the static equation

$$m'' + \Lambda A^{-\alpha} m = 0 \tag{8}$$

with boundary conditions I, II or III could be proved to be optimal for boundary conditions of the Sturm type. Here c is the Lagrange multiplier of the variational calculus problem (8) with the isoperimetric condition (5). The applied dimensional analysis is based on the rescaling of the optimal shape and does not require the determination of the Lagrange multiplier. We set the value as $c_L = 1$.

3.2. Closed form solution of the optimization problem for the Sturm type boundary conditions

3.2.1. Auxiliary solution of generalized Emden-Fowler equation

For the basic solution of the optimization problem, we use the auxiliary ordinary differential equation of the second order:

$$\frac{d^2x}{d\xi^2} = \Lambda \left(\frac{dx}{d\xi}\right)^{1-S} \xi^{P-1}. \quad (9)$$

The equation (9) is the generalization of the Emden-Fowler equation (Berkovic, 1997). The Lane-Emden equation was used to model the thermal behavior of a spherical cloud of gas within the framework of the classical thermodynamics. The Lane-Emden-Fowler type equations are used for description of a number of physical phenomena, originally phase transitions in critical thermodynamic systems of spherical geometry and Ginzburg-Landau theory of phase transitions.

We solve the Emden-Fowler equation (9) using the method of integration factors. The integration factor for Eq. (9) is:

$$\varpi_1 = \left(\frac{dx}{d\xi}\right)^{S-1}. \quad (10)$$

Multiplication of Eq. (9) by ϖ_1 leads to its first integral:

$$\left(\frac{dx}{d\xi}\right)^S - \frac{\Lambda S}{P} \xi^P + C_1 = 0.$$

The first integral delivers the solution of Eq. (9):

$$x(\xi) = \int_{\xi_0}^{\xi} \left(t^P \frac{\Lambda S}{P} - C_1\right)^{1/S} dt. \quad (11)$$

For $\xi_0 = 0, C_1 = \Lambda \cdot S/P$, the integral (11) evaluates in terms of the hypergeometric function ($0 \leq \xi \leq 1$):

$$x(\xi) = \left(-\frac{\Lambda S}{P}\right)^{1/S} {}_2F_2\left(\left[\frac{1}{P}, -\frac{1}{S}\right], \left[\frac{P+1}{P}\right], \xi^P\right) \cdot \xi. \quad (12)$$

The values of the function and its first derivatives on the ends of the interval for $0 \leq \xi \leq 1$ are:

$$x(0) = 0, \quad (13)$$

$$\left.\frac{dx}{d\xi}\right|_{\xi=0} = \left(-\frac{\Lambda S}{P}\right)^{1/S}, \quad (14)$$

$$x(1) = \left(-\frac{\Lambda S}{P}\right)^{1/S} \frac{\Gamma\left(\frac{P+1}{P}\right)\Gamma\left(\frac{S+1}{S}\right)}{\Gamma\left(\frac{PS+P+S}{PS}\right)}, \quad (15)$$

$$\left. \frac{dx}{d\xi} \right|_{\xi=1} = 0. \quad (16)$$

The integral of the function $x(\xi)$ evaluates in closed form as well (Gradshtein, Ryzhik, 2014):

$$\mathfrak{X} = \int_0^1 x(\xi) d\xi = \frac{1}{2} \left(-\frac{\Lambda s}{p} \right)^{1/s} {}_2F_2 \left(\left[\frac{1}{p}, -\frac{1}{s}, \frac{2}{p} \right], \left[\frac{p+2}{p}, \frac{p+1}{p} \right], 1 \right). \quad (17)$$

With the notation

$$p = \frac{1}{P}, q = \frac{1}{Q},$$

the formulas (12) and (17) express as:

$$x(\xi) = \left(-\frac{\Lambda p}{s} \right)^{1/s} {}_2F_2([p, -s], [1+p], \xi^{1/p}) \cdot \xi, \quad (18)$$

$$\mathfrak{X} = \frac{1}{2} \left(-\frac{\Lambda p}{s} \right)^{1/s} {}_2F_2([p, -s, 2p], [1+2p, 1+p], 1).$$

3.2.2. Solution of the optimization problem

In this section, the solution of the optimization problem for the admissible values of the parameter α is studied. The closed-form solution outcomes for an arbitrary permissible value of parameter α . For the application of the above results to the optimization of elastic elements, the dependent and independent variables in the equations (7)-(8) are to be exchanged:

$$\frac{d^2x}{dm^2} = \Lambda \left(\frac{dx}{dm} \right)^3 m^R, \quad R = \frac{1-\alpha}{1+\alpha}. \quad (19)$$

In the new variables, the Eq. (19) turns into the Emden-Fowler equation (9) with the following parameter:

$$S = -2, P = \frac{2}{\alpha+1}, R = P - 1.$$

According to Eq. (11), the general solution of Eq. (19) is:

$$x = -i\sqrt{P} \int_{\nu}^m \frac{dt}{\sqrt{2\Lambda t^P - C}}. \quad (20)$$

The values ν and C are the integration constants. To avoid the solution of the nonlinear transcendental equations for the integration constants we favor to exploit the symmetry thoughts. The sense of the constant ν is the moment in the middle of the column, $\nu = M(x = 0)$. Due to the symmetry the equations with respect to the point $x = 0$, the function $M(x)$ must be an even function of the variable x . This condition requires for the integration constant:

$$C = 2\Lambda\nu^P.$$

With this value, the integral (20) evaluates in the closed form with the hypergeometric function (Gradshtein, Ryzhik, 2014):

$$x(m) = \frac{v^{-\frac{1}{\alpha+1}}}{\sqrt{\Lambda}\sqrt{\alpha+1}} {}_2F_2\left(\left[\frac{1}{2}, \frac{1+\alpha}{2}\right], \left[\frac{3+\alpha}{2}\right], \left(\frac{m}{v}\right)^{\frac{2}{\alpha+1}}\right) \cdot m + c_0, \quad (21)$$

$$c_0 = -\sqrt{\frac{\pi}{\Lambda}} \frac{\Gamma\left(\frac{\alpha+3}{2}\right)}{\sqrt{\alpha+1} \Gamma\left(\frac{\alpha+1}{2}\right)} v^{\frac{\alpha}{\alpha+1}}.$$

The solution (21) could be also expressed in terms of the Jacoby polynomials $\mathbf{P}_a^{(b,c)}[x]$ (Abramowitz et al. 1983):

$$x = \sqrt{\frac{P}{2\Lambda}} \int_v^m \frac{dt}{\sqrt{v^P - t^P}} = c_1 \mathbf{P}_{-1/2}^{\left(\frac{1}{P}, \frac{1}{2}\right)} \left[1 - 2\left(\frac{m}{v}\right)^P\right] + c_0, \quad c_1 = -\frac{m}{v} c_0. \quad (22)$$

Thus, the axial coordinate is the function of the new independent parameter m . According the boundary conditions III, the moment vanishes on the hinged ends. From these conditions the half-length of the rod determines from Eq. (18) as (Gradshtein, Ryzhik, 2014):

$$L_{opt} = \sqrt{\frac{P}{2\Lambda}} \int_0^v \frac{dt}{\sqrt{v^P - t^P}} = -c_0 \equiv \sqrt{\frac{\pi}{\Lambda}} \frac{\Gamma\left(\frac{\alpha+3}{2}\right)}{\sqrt{\alpha+1} \Gamma\left(\frac{\alpha+1}{2}\right)} v^{\frac{\alpha}{\alpha+1}}. \quad (23)$$

The functions of moments m over the axial coordinate x are shown for $\alpha = 2$ on the Figure 1. The moment in the middle of the rod is given for each curve ($v = 2,3,4$). The critical value Λ is equal for all curves. As the result, the half-lengths L_{opt} of the rods are appear to be different. The curves for $\alpha = 3$ area shown on the Figure 3. The higher transcendental functions (22) reduce in the cases $\alpha = 1,2,3$ to the elementary functions (Table 1).

In its turn, the area of cross-section of the optimal column results from Eq. (7) is the function of the new independent variable m :

$$A(m) = m^{\frac{2}{\alpha+1}}.$$

The corresponding distributions of cross-section areas along the span are plotted for $\alpha = 2$ on the Figure 2 and for $\alpha = 3$ on Figure 4.

The half-volume of the column is the integral of the cross-section area of the column (Gradshtein, Ryzhik, 2014):

$$V_{opt} \equiv \sqrt{\frac{P}{2\Lambda}} \int_0^v \frac{A(t)dt}{\sqrt{v^P - t^P}} \equiv \sqrt{\frac{P}{2\Lambda}} \int_0^v \frac{t^{\frac{2}{\alpha+1}} dt}{\sqrt{v^P - t^P}} = \sqrt{\frac{\pi}{\Lambda}} \frac{\sqrt{\alpha+1} \Gamma\left(\alpha + \frac{1}{2}\right)}{2 \Gamma(\alpha+1)} v^{\frac{2\alpha}{\alpha+1}}. \quad (24)$$

We introduce one another notable constant that will be referred as an averaged bending rigidity of the column. The bending rigidity of the column half expresses as an integral of the moment of inertia of the cross-sections along the total length of the column. The moment of inertia of the cross-section reads:

$$J(m) = A^\alpha(m) = m^{\frac{2\alpha}{\alpha+1}}.$$

The averaged bending rigidity of the column expresses as the integral (Gradshtein, Ryzhik, 2014):

$$B_{opt} \equiv \sqrt{\frac{P}{2\Lambda}} \int_0^v \frac{A^\alpha(t) dt}{\sqrt{v^P - t^P}} \equiv \sqrt{\frac{P}{2\Lambda}} \int_0^v \frac{t^{\frac{2\alpha}{\alpha+1}} dt}{\sqrt{v^P - t^P}} = \sqrt{\frac{\pi \sqrt{\alpha+1} \Gamma(\frac{3\alpha+1}{2})}{\Lambda 2 \Gamma(\frac{3\alpha}{2}+1)}} v^{\frac{3\alpha}{\alpha+1}}. \quad (25)$$

Table 1 Closed-form solutions for the optimization problem for rods with hinged ends and their lengths, volumes and averaged rigidity

α	$\Omega = M/v$	L_{opt}	V_{opt}	B_{opt}
1	$x = \sqrt{\frac{2v}{\Lambda} \cdot \frac{\Omega - \sqrt{1 - \Omega} - 1}{1 + \sqrt{1 - \Omega}}}$	$\sqrt{\frac{2v}{\Lambda}}$	$\sqrt{\frac{2v\pi}{\Lambda} \cdot \frac{1}{4}}$	$\sqrt{\frac{2 \cdot 2v^{3/2}}{\Lambda} \cdot \frac{1}{3}}$
2	$x = \sqrt{\frac{3v^3}{16\Lambda} \cdot \left(2 \arcsin\left(\Omega^{\frac{1}{3}}\right) - \pi - 2\Omega^{\frac{1}{3}}\sqrt{1 - \Omega^{\frac{2}{3}}}\right)}$	$\sqrt{\frac{3 \pi v^{2/3}}{\Lambda} \cdot \frac{1}{4}}$	$\sqrt{\frac{3 \cdot 3\pi v^{4/3}}{\Lambda} \cdot \frac{1}{16}}$	$\sqrt{\frac{3 \cdot 5\pi v^2}{\Lambda} \cdot \frac{1}{32}}$
3	$x = \frac{2v^{3/4}}{3\sqrt{\Lambda}} \cdot \frac{\Omega + \sqrt{\Omega} - 2}{\sqrt{1 - \sqrt{\Omega}}}$	$\frac{4v^{3/4}}{3\sqrt{\Lambda}}$	$\frac{5v^{3/2}\pi}{16\sqrt{\Lambda}}$	$\frac{256v^{9/4}}{315\sqrt{\Lambda}}$

We demonstrated above the closed-form solution of the both hinged ends. The other solutions with the Sturm type boundary conditions could be derived from the basic solution using traditional scaling methods.

3.2.3. Isoperimetric inequalities

For the optimal column with the hinged ends the invariant factors are:

$$F_{A.O.} = \frac{4 \cdot 2^{\alpha(2+\alpha)} \pi^{1-\frac{\alpha}{2}} \Gamma^{2+2\alpha}\left(\frac{\alpha+3}{2}\right)}{(\alpha+1)^{1+2\alpha} \Gamma^{2+\alpha}\left(\alpha+\frac{1}{2}\right)}, \quad (26)$$

$$F_{B.O.} = \frac{8\pi}{(\alpha+1)^2} \frac{\Gamma^3\left(\frac{\alpha+3}{2}\right)}{\Gamma^3\left(1+\frac{\alpha}{2}\right)} \frac{\Gamma\left(1+\frac{3\alpha}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3\alpha}{2}\right)}.$$

Table 2 Invariant factors for optimal rods with hinged ends and mass and stiffness ratios

α	$F_{A.O.}$	$F_{B.O.}$	$\frac{V_{III}}{\tilde{V}_{III}}$	$\frac{B_{III}}{\tilde{B}_{III}}$
0	π^2	π^2	$4/e \approx 1.47..$	1
1	$\frac{32}{\pi}$	12	$\frac{\pi^3}{32} \approx 0.968$	$\frac{\pi^2}{12} \approx 0.822$
2	$\frac{4\pi}{3}$	$\frac{6\pi^2}{5}$	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{5}{6} \approx 0.833$
3	$\frac{16777216}{30375\pi^3}$	$\frac{35}{3}$	$\frac{45}{256} \left(\frac{\pi^5}{3}\right)^{1/3} \approx 0.821$	$\frac{3\pi^2}{35} \approx 0.8459$
∞	∞	$\pi\sqrt{12}$	$1/\sqrt{2} \approx 0.707$	$\frac{\sqrt{3}\pi}{6} \approx 0.9068$

The ratios of factors $F_{A.O.}$ and $F_{B.O.}$ to their values of the reference rod with the constant cross-section $\tilde{F}_{A(III)} = \tilde{F}_{B(III)} = \pi^2$ is shown on Figure 5. Together with the ratios of factors $F_{A.O.}/\tilde{F}_{A(III)}$ and $F_{B.O.}/\tilde{F}_{B(III)}$, the limit value $\sqrt{12}\pi^{-1}$ is displayed. The case $\alpha = 0$ corresponds to the rod of the constant cross-section. For this value the factors $F_{A.O.}/\tilde{F}_{A(III)}$ and $F_{B.O.}/\tilde{F}_{B(III)}$ is equal to 1.

With Eq. (26), the expression for the mass and stiffness ratios reads:

$$\frac{V_{III}}{\tilde{V}_{III}} = \pi^{\frac{2+\alpha}{2\alpha}} \cdot (\alpha+1)^{2+\frac{1}{\alpha}} \cdot 2^{-\frac{\alpha^2+2\alpha+2}{\alpha}} \cdot \frac{\Gamma\left(\alpha+\frac{1}{2}\right) \cdot \Gamma^{2/\alpha}\left(1+\frac{\alpha}{2}\right)}{\Gamma^{(2\alpha+2)/\alpha}\left(\frac{\alpha+3}{2}\right)} < 1 \text{ for } \alpha > 1$$

$$\frac{B_{III}}{\bar{B}_{III}} = \frac{\pi(\alpha+1)^2}{8} \frac{\Gamma^3\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{1+3\alpha}{2}\right)}{\Gamma^3\left(\frac{\alpha+3}{2}\right) \Gamma\left(1+\frac{3\alpha}{2}\right)} < 1 \text{ for } \alpha > 0. \quad (27)$$

The mass and stiffness ratios Eq. (27) are displayed on Figure 6.

The optimality expresses in form of the isoperimetric inequalities between the invariant factors:

$$F_{A.O.} \geq F_A, F_{B.O.} \geq F_B.$$

The equality holds if and only if the column is an affine transformation of the column with the optimal shape.

The isoperimetric inequality was stated in the cited article of Tadjbakhsh and Keller:

If the eigenvalues are simple, the optimal column has the largest buckling load among all the columns with the same volume of material:

$$\Lambda[A] \leq \Lambda[A_e] \equiv \Lambda_e, \text{ for } \langle a \rangle = 1. \quad (28)$$

The equality in (28) sign holds only for the column with the cross-area distribution $A = A(\xi)$.

4. Optimality for boundary conditions of mixed type

4.1. Optimality conditions for the mixed type boundary conditions

The above method will be applied for solution of the closed-form solution of the optimization problem for the columns with boundary conditions of the mixed type.

The eigenvalue problem for the ordinary differential equation (7) with boundary conditions IV is self-adjoint. However, the boundary conditions IV are not of the Sturm type and the eigenvalues are not necessarily simple.

Specifically, the conditions (3) are of mixed type. There exists an infinite set of eigenvalues, and all eigenvalues are real. The eigenvalue problem has a double zero eigenvalue with associated fundamental functions. The first fundamental function with the zero eigenvalue is a constant and the second fundamental function is linear function of ξ (see, Seiranyan, 1984 and Karaa, 2003). Only the positive eigenvalues of the buckling problem have physical meaning. The positive eigenvalues can be arranged as two sequences:

$$\Lambda_1 < \Lambda_3 < \dots < \Lambda_{2k-1} < \Lambda_{2k+1} < \dots$$

$$\Lambda_2 < \Lambda_4 < \dots < \Lambda_{2k} < \Lambda_{2k+2} < \dots$$

such that for $k = 1, 2, 3, \dots$:

$$\Lambda_{2k-1} < \Lambda_{2k} < \Lambda_{2k+1}, \Lambda_{2k} < \Lambda_{2k+1} < \Lambda_{2k+2}.$$

Thus, the positive eigenvalues of the boundary value problem are simple or double. Function $A(\xi)$ satisfies isoperimetric condition (5):

$$\langle A \rangle = 1.$$

The derivation of necessary optimality conditions in the case of multiple eigenvalues was performed in papers of Bratus' and Seiranjan (1983), Seiranyan (1984), Bratus' (1991), Banichuk et al (2002). In the cited papers was shown, that there exist the functions $U(\xi)$, $V(\xi)$ and $A(\xi)$ such that:

$$U^2 + V^2 = c_L A^{\alpha+1}, \quad c_L > 0, \quad (29)$$

$$\begin{aligned} \Lambda_1 &= \langle U'^2 \rangle, & U'' + \Lambda_1 A^{-\alpha} U &= 0, \\ \Lambda_2 &= \langle V'^2 \rangle, & V'' + \Lambda_2 A^{-\alpha} V &= 0. \end{aligned}$$

The condition of coincidence for the lowest positive eigenvalues is satisfied:

$$\Lambda_1 = \Lambda_2. \quad (30)$$

The constant c_L in the necessary optimality condition (29) plays the role of Lagrange multiplier for the isoperimetric condition. We can arbitrarily set this constant, using the applied method of dimensional analysis. Thus, the constant c_L will be set to 1. This choice eases of the mathematical formulas

The functions $U(\xi)$, $V(\xi)$ satisfy the boundary conditions (4) and the orthogonality condition

$$\langle A^{-\alpha} UV \rangle = 0, \quad \langle A^{-\alpha} U^2 \rangle = 1, \quad \langle A^{-\alpha} V^2 \rangle = 1. \quad (31)$$

4.2. Isoperimetric inequality for the mixed type boundary conditions

The column with the thickness distribution $A(\xi)$ is optimal in the bimodal sense and isoperimetric inequality is to be stated in the bimodal case.

The functions m_1 and m_2 are the first and the second buckling modes of the optimal column respectively. As mentioned above, the isoperimetric inequality is valid in the single modal case. The analogous isoperimetric inequality for the bimodal case is to be immediately proved.

We consider an arbitrary positive distribution of thickness $a > 0$ and boundary conditions IV. In this case the lowest positive eigenvalues Λ_{*1} , Λ_{*2} and moments m_{*1} , m_{*2} satisfy the equations:

$$\begin{aligned} \Lambda_{*1} &= \langle m_{*1}'^2 \rangle \text{ with} & m_{*1}'' + \Lambda_{*1} a^{-\alpha} m_{*1} &= 0, \\ \Lambda_{*2} &= \langle m_{*2}'^2 \rangle \text{ with} & m_{*2}'' + \Lambda_{*2} a^{-\alpha} m_{*2} &= 0. \end{aligned}$$

As stated above, for the boundary conditions of mixed type the following inequality is valid:

$$\Lambda_{*1} \leq \Lambda_{*2}.$$

Consider two differentiable, admissible functions \tilde{m}_1 and \tilde{m}_2 , that satisfy boundary conditions IV and the orthogonality condition:

$$\langle a^{-\alpha} \tilde{m}_1 \tilde{m}_2 \rangle = 0.$$

As the functions \tilde{m}_1 and \tilde{m}_2 are the admissible functions, the variational principle for the eigenvalue gives that:

$$\Lambda_{*1} \leq \frac{\langle \tilde{m}_1'^2 \rangle}{\langle a^{-\alpha} \tilde{m}_1^2 \rangle}, \quad \Lambda_{*2} \leq \frac{\langle \tilde{m}_2'^2 \rangle}{\langle a^{-\alpha} \tilde{m}_2^2 \rangle}.$$

The addition of these values and division by two delivers:

$$\Lambda_{*1} \equiv \frac{\Lambda_{*1} + \Lambda_{*1}}{2} \leq \frac{\Lambda_{*1} + \Lambda_{*2}}{2} \leq \frac{1}{2} \left[\frac{\langle \tilde{m}_1'^2 \rangle}{\langle a^{-\alpha} \tilde{m}_1^2 \rangle} + \frac{\langle \tilde{m}_2'^2 \rangle}{\langle a^{-\alpha} \tilde{m}_2^2 \rangle} \right]. \quad (32)$$

For the admissible functions, one can choose:

$$\begin{aligned} \tilde{m}_1 &= U \cos \gamma + V \sin \gamma, \\ \tilde{m}_2 &= -U \sin \gamma + V \cos \gamma. \end{aligned} \quad (33)$$

Here α is a root of the equation:

$$2 \sin 2\gamma \cdot \langle A^{-\alpha} UV \rangle = \cos 2\gamma \cdot \langle A^{-\alpha} (U^2 - V^2) \rangle.$$

From (33) follows that:

$$\langle A^{-\alpha} \tilde{m}_1^2 \rangle = \langle A^{-\alpha} \tilde{m}_2^2 \rangle = \frac{1}{2} \langle A^{-\alpha} (U^2 + V^2) \rangle,$$

$$\langle \tilde{m}_1'^2 + \tilde{m}_2'^2 \rangle = \langle U'^2 + V'^2 \rangle.$$

Substitution of $\tilde{m}_1(\xi)$ and $\tilde{m}_2(\xi)$ from (33) into (32) demonstrates, that:

$$\Lambda_{*1} \leq \frac{1}{2} \left[\frac{\langle \tilde{m}_1'^2 \rangle}{\langle a^{-\alpha} \tilde{m}_1^2 \rangle} + \frac{\langle \tilde{m}_2'^2 \rangle}{\langle a^{-\alpha} \tilde{m}_2^2 \rangle} \right] \equiv \frac{\langle U'^2 + V'^2 \rangle}{\langle a^{-\alpha} (U^2 + V^2) \rangle}. \quad (34)$$

In order to state the isoperimetric inequality, the following auxiliary inequality must be established:

$$\frac{\langle U'^2 + V'^2 \rangle}{\langle a^{-\alpha} (U^2 + V^2) \rangle} \leq \frac{\langle U'^2 + V'^2 \rangle}{\langle A^{-\alpha} (U^2 + V^2) \rangle}. \quad (35)$$

The numerators of the fractions to the left and right of the auxiliary inequality (35) are identical. The denominators (35) are, however, different and are to be compared. The inequality for denominators, that has to be proven, reads:

$$\langle A^{-\alpha} (U^2 + V^2) \rangle \leq \langle a^{-\alpha} (U^2 + V^2) \rangle. \quad (36)$$

At this point, the optimality condition (29) is essentially used:

$$U^2 + V^2 = c_L A^{\alpha+1}.$$

Namely, the substitution of the optimality condition (9) into (16) delivers the inequality:

$$\left\langle \frac{A^{\alpha+1}}{A^\alpha} \right\rangle \leq \left\langle \frac{A^{\alpha+1}}{a^\alpha} \right\rangle. \quad (37)$$

The validity of the looked-for inequality (17) follows directly from the Hölder inequality, (Pachpatte, 2005).

Consider two measurable real- or complex-valued functions φ, ψ , that fulfil for $-L \leq \xi \leq L$ the conditions:

$$\varphi \geq 0, \quad \psi \geq 0.$$

Hölder inequality reads for the functions $\varphi(\xi), \psi(\xi)$ as:

$$\langle \varphi \psi \rangle \leq \langle \varphi^\tau \rangle^{1/\tau} \langle \psi^\varsigma \rangle^{1/\varsigma}. \quad (38)$$

The numbers τ and ς in Eq. (38) are said to be Hölder conjugates of each other:

$$\frac{1}{\tau} + \frac{1}{\varsigma} = 1, \quad 1 \leq \tau, \varsigma \leq \infty. \quad (39)$$

For proof of the necessary isoperimetric inequality (37), we substitute into the Hölder inequality (38) the expressions:

$$\begin{aligned} \varphi &= A \cdot a^{-1/\varsigma}, \quad \psi = a^{1/\varsigma}, \\ \varsigma &= \frac{\alpha+1}{\alpha}, \quad \tau = \alpha + 1. \end{aligned} \quad (40)$$

Consequently, from (37) follows the inequality for denominators (36) and finally the desired inequality (35):

$$\Lambda_{*1} \leq \frac{\langle U'^2 + V'^2 \rangle}{\langle a^{-\alpha}(U^2 + V^2) \rangle} \leq \frac{\langle U'^2 + V'^2 \rangle}{\langle A^{-\alpha}(U^2 + V^2) \rangle} \equiv \frac{\Lambda_1 + \Lambda_2}{2}. \quad (41)$$

From (21) follows the isoperimetric inequality

$$\Lambda_{1*} \leq \frac{\Lambda_1 + \Lambda_2}{2} \equiv \Lambda_1 \equiv \Lambda_2. \quad (42)$$

To complete the proof of the isoperimetric inequality (42), it should be shown in the next Section that the column, which satisfies the necessary optimality conditions (29), really exists.

Consequently, it was proved that the column, that obeys the necessary optimality conditions (29), has the largest buckling load among all the columns with the same weight, assuming the bimodal condition is fulfilled for the optimal distribution of thickness $A(\xi)$.

4.3. Closed-form solutions for eigenvalues with mixed-type boundary conditions

4.3.1. Equations of optimization problem with mixed type boundary conditions

The corresponding optimality conditions in the case of multiple eigenvalues were given in (Seiranyan, 1984). The standard method for determination of the optimal shapes in this case leads to the non-linear boundary value problem and reveals no closed form solution for the shape of the column in terms of higher transcendental functions.

For the closed form solution of the optimization problem with both clamped ends, we study two simultaneous equations:

$$m_1'' + \frac{\Lambda m_1}{|\wp|^{\alpha+1}} = 0, \quad m_2'' + \frac{\Lambda m_2}{|\wp|^{\alpha+1}} = 0. \quad (43)$$

The function \wp is the complex function of the real variable x :

$$\wp = m_1 + im_2 = M \cdot \exp(i\theta). \quad (44)$$

The functions $M = |\wp|$ and θ portrays the amplitude and the phase of the complex moment \wp . These functions are the scalar real functions of the real variable x . The cross-sectional area $A = M^P$ follows from M with Eq. (29). The substitution of (44) in (43) leads to the complex differential equation:

$$\aleph = \wp'' + \frac{\Lambda \wp}{|\wp|^{\alpha+1}} = 0.$$

Eq. (43) are the real $\aleph_R = \text{Re } \aleph$, and imaginary parts $\aleph_I = \text{Im } \aleph$ of Eq. (44). With the phase function θ , we get two simultaneous real differential equations:

$$\aleph_R \sin \theta - \aleph_I \cos \theta = -M \cdot \theta'' - 2\theta' \cdot M' = 0, \quad (45)$$

$$\aleph_R \cos \theta + \aleph_I \sin \theta = -M \cdot (\theta')^2 + \Lambda M^R + M'' = 0. \quad (46)$$

The boundary conditions for Eq. (45) are:

$$\theta'(0) = 1, \theta(0) = 0. \quad (47)$$

The boundary conditions (47) ensure, that the function \aleph_R is even and the function \aleph_I is odd. The solution of (45) reads:

$$\vartheta(x) = M^2(0) \int_0^x \frac{dt}{M^2(t)}. \quad (48)$$

The substitution of the function from Eq. (48) in the Eq. (46) leads to the equation for $M(x)$:

$$M'' - M^{-3} + \Lambda M^R = 0. \quad (49)$$

4.3.2. The shape of the optimal column

For closed form solution, the dependent and independent variables in the equation (49) must to be exchanged. As the result, we come to the generalized Emden-Fowler equation for $x = x(M)$:

$$\frac{d^2 x}{dM^2} = \Lambda \left(\frac{dx}{dM} \right)^3 \frac{M^{R+3} - 1}{M^3}. \quad (50)$$

The right side of Eq. (50) is not a homogeneous function of the independent variable M . This circumstance leads to the bulky formulas for the closed form solution.

We proceed now with the analytical solution. For the beginning, one boundary condition of the Eq. (50) is assumed to be $x(1) = 0$. This means, that we temporary fixing the origin of coordinate at the cross-section with the area 1. With this boundary condition two solutions of Eq. (50) read:

$$x(M) = \pm \sqrt{P} \int_1^M \frac{dt}{\sqrt{-2t^{2+P} + PC_1 t^2 - P}}. \quad (51)$$

with the dimensionless constant $P = 1 + R$. Apparently, that $M(x)$ is the even function of x due to the symmetry of the problem with respect to the origin of coordinate system.

The significant task is to determine the integration constant C_1 from the a-priori symmetry conditions. For the beginning we use only one, namely the positive solution of two in Eq. (51). Let the positive function (51) assumes the maximum value in the point $M = \mu$. In other words, $x(\mu) = L$, because the axial coordinate runs from $-L$ to L . The integrand of Eq. (51) must have an integrable singularity in this point. Consequently, the denominator of the integrand must vanish in the point $t = \mu$. From this condition the integration constant results as:

$$C_1 = \frac{2t^{2+P} + P}{P \mu^2}. \quad (52)$$

Notable, is that with the certain value (52) for integration constant, the equation (51) delivers the solution of the optimization problem in integral form. Substantial, that the setting (52) satisfies the

boundary conditions (3). The integrals (51) permit the representation in terms of the higher transcendental functions for α equal to 1,2 and 3.

From now we study the central for applications case $\alpha = 2, P = 2/3$.

Substitution of (52) in (51) leads to the positive solution:

$$x(M) = \frac{\sqrt{6}\mu}{2} \int_1^M \frac{tdt}{\sqrt{t^2 \left(3\mu^{\frac{8}{3}} + 1\right) - \left(3t^{\frac{8}{3}} + 1\right)\mu^2}}. \quad (53)$$

We go further and attempt to express the solution (53) in terms of the higher transcendental functions.

For this purpose, the integration variable will be replaced, $s^2 = t^3$. The integral (53) turns into:

$$x(M) = \frac{3\sqrt{6}\mu}{4} \int_1^{M^{2/3}} \frac{s^2 ds}{\sqrt{F(s)}}, \quad (54)$$

$$F(s) = -3\mu^2 s^4 + \left(3\mu^{\frac{8}{3}} + 1\right) s^3 - \mu^2.$$

The polynomial $F(s)$ is of the fourth order. The roots of the polynomial F are indicated as $E_i, i = 1,2,3,4$. The polynomial F factorizes to the product of the polynomials of the first and the third order:

$$F(s) = (s - \mu^{2/3})G(s) = -3\mu^2 \prod_{i=1}^4 (s - E_i), \quad (55)$$

$$G(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0,$$

$$a_3 = -3\mu^2, a_2 = \mu^{2/3}, a_1 = 1, a_0 = \mu^{4/3}.$$

The first root of the polynomial $F(s)$ is evidently $E_1 = \mu^{2/3}$. Three other roots of the cubic equation $G(s) = 0$

are denoted as E_2, E_3, E_4 . The roots of the cubic equation provides Cardano's formula. One root of is real:

$$E_2 = \varrho^{2/3} = \frac{\sqrt[3]{4\mu\phi^2 + 18\sqrt{2}\mu^3 + 2\sqrt[3]{2\mu + 2\sqrt{\mu\phi}}}}{18\mu^{7/3}\phi^{1/3}} > 0,$$

$$\phi = 243 \mu^{16/3} + 27\mu^{8/3} \sqrt{81 \mu^{16/3} + 14 \mu^{8/3} + 1} + 27\mu^{8/3} + 2.$$

Two other roots are complex conjugated values:

$$E_{3,4} = \check{\alpha} \pm i \check{\beta},$$

$$2\check{\alpha} = -\left(\frac{2\mu^2}{\phi}\right)^{\frac{1}{3}} - \frac{\frac{(2\phi)^{2/3} + 2^{1/3}}{18} - \frac{2\phi^{1/3}}{9}}{\phi^{1/3} \mu^2},$$

$$2\check{\beta} = \left(\frac{2\mu^2}{\phi}\right)^{\frac{1}{3}} - \frac{2^{1/3}\sqrt{3}(2^{1/3}\phi^{2/3} - 2)}{18\phi^{1/3} \mu^2}.$$

The value μ is the maximal bending moment in the column, which attains in the both end points of the rod. The value ϱ is the minimal bending moment in the column, which reaches its value in the middle section. The values E_1 and E_2 are the cross-section areas of the column at these three points. Hence, the improper integral in Eq. (54) converges and reduces to:

$$x(M) = -\frac{3i\sqrt{2}}{4} \int_1^{M^{2/3}} \frac{s^2 ds}{\sqrt{\prod_{i=1}^4 (s - E_i)}}. \quad (56)$$

According to our previous setting, the expression (56) calculates the axial coordinate from the cross-section with the area exactly equal to 1. Now we can correct the formula (56). We calculate the axial coordinate, fixing the coordinate origin in the middle of the column $x = 0$. The axial coordinate starts in the middle of the rod and runs to the end section, such that

$$\varrho \leq M \leq \mu, \quad 0 \leq x \leq L$$

For the right half of the column the solution reads (Gradshtein, Ryzhik, 2014):

$$x(M, P) = -\frac{3i\sqrt{2}}{4} \int_P^M \frac{s^2 ds}{\sqrt{\prod_{i=1}^4 (s - E_i)}} \equiv \frac{3}{4}\sqrt{2} \cdot (f_{(0)} + f_{(1)} + f_{(2)} + f_{(3)}), \quad (57)$$

where the auxiliary functions are:

$$\begin{aligned} f_{(0)} &= \mathbf{f}_{(0)} \cdot (\Phi_0(M) - \Phi_0(P)), & \mathbf{f}_{(0)} &= 1, \\ f_{(1)} &= \mathbf{f}_{(1)} \cdot (\Phi_1(M) - \Phi_1(P)), & \mathbf{f}_{(1)} &= i\sqrt{E_3 - E_1}\sqrt{E_4 - E_2}, \\ f_{(2)} &= \mathbf{f}_{(2)} \cdot (\Phi_2(M) - \Phi_2(P)), & \mathbf{f}_{(2)} &= i \frac{E_2^2 + (E_1 + E_3)E_4 - E_1E_4}{\sqrt{E_3 - E_1}\sqrt{E_4 - E_3}}, \\ f_{(3)} &= \mathbf{f}_{(3)} \cdot (\Phi_3(M) - \Phi_3(P)), & \mathbf{f}_{(3)} &= i \frac{(E_1 + E_2 + E_3 + E_4)(E_2 - E_1)}{\sqrt{E_3 - E_1}\sqrt{E_4 - E_3}}. \end{aligned}$$

$$\varrho^{\frac{2}{3}} = P, \quad M^{\frac{2}{3}} = M,$$

The integral (57) expresses in terms of elliptical functions (Abramowitz et al, 1983):

$$\begin{aligned} \Phi_0(P) &= \frac{\sqrt{E_1 - P} \sqrt{E_3 - P} \sqrt{E_4 - P}}{\sqrt{P - E_2}}, \\ \Phi_1(P) &= \mathbf{E}(\omega_1, \omega_2), & \Phi_2(P) &= \mathbf{F}(\omega_1, \omega_2), \\ \Phi_3(P) &= \mathbf{\Pi}(\omega_1, \omega_3, \omega_2), & \Phi_4(P) &= \mathbf{\Pi}(\omega_1, \omega_4, \omega_2), \\ \omega_1 &= i \frac{\sqrt{E_4 - E_2} \sqrt{E_1 - P}}{\sqrt{E_4 - E_1} \sqrt{P - E_2}}, & \omega_2 &= \frac{\sqrt{E_3 - E_2} \sqrt{E_4 - E_1}}{\sqrt{E_3 - E_1} \sqrt{E_4 - E_2}}, \\ \omega_3 &= \frac{E_4 - E_1}{E_4 - E_2}, & \omega_4 &= \frac{E_4 - E_1}{E_4 - E_2} \cdot \frac{E_2}{E_1}. \end{aligned}$$

The plots of the moments over the axial coordinate are shown on the Figure 7. The closed-form analytical solution (57) is shown with the bold lines for the values of $\mu = 2, 3, 4$.

As already stated, the optimal cross-sectional area follows from the amplitude of the moment M with the necessary optimality condition Eq. (29). The area of the cross-section runs in the interval: $E_2 \leq A \leq E_1$. The cross-section reaches the upper and lower values in the points $x(\mu) = \pm L$ and $x(\rho) = 0$ correspondingly. The second solution of Eq. (51) possesses the negative sign. Accordingly, the multiplication of Eq. (57) by -1 provides the axial coordinate for the left part of the column $-L \leq x(M) \leq 0$. The plots of the areas of the cross-sections over the axis of the rod are shown on the Figure 8. The results from closed-form analytical solution $A = M^{\frac{2}{\alpha+1}}$ are shown with the solid lines for the values of $\mu = 2,3,4$. The second moments of inertia of the cross-section is shown on the Figure 9. Once again, the analytical solution $J = M^{\frac{2\alpha}{\alpha+1}}$ is displayed with the solid lines.

4.4. Length, volume and averaged rigidity

Remarkably, that the half-length, half-volume and the half-averaged stiffness allow the invariant representations for the functionals of the optimization problem in terms of the improper integrals:

$$L_m = -\frac{3i\sqrt{2}}{4} \int_{E_1}^{E_2} \frac{s^2 ds}{\sqrt{\prod_{i=1}^4 (s - E_i)}}, \quad (58)$$

$$V_m = -\frac{3i\sqrt{2}}{4} \int_{E_1}^{E_2} \frac{s^3 ds}{\sqrt{\prod_{i=1}^4 (s - E_i)}}, \quad (59)$$

$$B_m = -\frac{3i\sqrt{2}}{4} \int_{E_1}^{E_2} \frac{s^4 ds}{\sqrt{\prod_{i=1}^4 (s - E_i)}}. \quad (60)$$

To prove the above formulas, we can spot, that the variable y signifies the area of the cross-section. Thus, the integrand in (58) represents the element of axial length. Therefore, if we multiply the element of length by the area and integrate it in the interval $[E_2, E_1]$, we get the half-volume V_m of the rod in Eq. (49). Similarly, the local bending stiffness is the squared area y^2 for $\alpha = 2$. Its integral delivers the averaged stiffness B_m in Eq. (60). These three invariants express in terms of elliptical functions, similar to the expressions (57).

For the evaluation of integral V_m right half of the column the auxiliary function (Gradshteyn, Ryzhik, 2014):

$$V(M, P) = -\frac{9i\sqrt{2}}{16} \int_P^M \frac{s^3 ds}{\sqrt{\prod_{i=1}^4 (s - E_i)}} \equiv \frac{9}{16} \sqrt{2} \cdot (g_{(0)} + g_{(1)} + g_{(2)} + g_{(3)}). \quad (61)$$

The auxiliary functions Eq.(61) are the following:

$$\begin{aligned} g_{(0)} &= \mathbf{g}_{(0)}(M)\Phi_0(M) - \mathbf{g}_{(0)}(P)\Phi_0(P), & 3\mathbf{g}_{(0)}(P) &= 3E_1 + E_2 + 2P + 3E_3 + 3E_4, \\ g_{(1)} &= \mathbf{g}_{(1)} \cdot (\Phi_1(M) - \Phi_1(P)), & \mathbf{g}_{(1)} &= i(E_1 + E_2 + E_3 + E_4)\sqrt{E_3 - E_1}\sqrt{E_4 - E_3}, \\ g_{(2)} &= \mathbf{g}_{(2)} \cdot (\Phi_2(M) - \Phi_2(P)), & \mathbf{g}_{(2)} &= i \frac{3(E_4 - E_2)(E_1 - E_2)^2 + (E_1 - E_2)(E_4 - E_2)(8E_2 + E_3 + 3E_4) - 8E_2^3}{\sqrt{E_3 - E_1}\sqrt{E_4 - E_3}}, \\ g_{(3)} &= \mathbf{g}_{(3)} \cdot (\Phi_3(M) - \Phi_3(P)). & \mathbf{g}_{(3)} &= i \frac{(E_1 - E_2)(3(E_1^2 + E_2^2 + E_3^2 + E_4^2) + 2E_1(E_2 + E_3 + E_4) + 2E_2(E_3 + E_4) + 2E_3E_4)}{3\sqrt{E_3 - E_1}\sqrt{E_4 - E_3}}. \end{aligned}$$

The similar expressions result for the averaged stiffness as well:

$$B(M, P) = -\frac{3i\sqrt{2}}{4} \int_P^M \frac{s^4 ds}{\sqrt{\prod_{i=1}^4 (s-E_i)}}.$$

The formulas are analogous to the recipes, given in Eq. (57) and Eq. (61). The terms rather bulky and will be omitted in this manuscript for brevity.

4.5. Fundamental functions for buckling moments

Once the function $x(M)$ is determined, the phase $\vartheta(M)$ in Eq. (44) follows from (48) after the swapping of independent variables. The improper integral converges and expresses in terms of elliptic functions (Gradshteyn, Ryzhik, 2014). Finally, the phase reads for $\varrho \leq M \leq \mu$ as follows:

$$\vartheta(M) = \frac{9}{8} \int_{\varrho^{\frac{2}{3}}}^{\frac{M^{\frac{2}{3}}}{\varrho^{\frac{2}{3}}}} \frac{ds}{s \sqrt{\prod_{i=1}^4 (s-E_i)}} = \frac{27i\sqrt{2}}{16E_1E_2 \sqrt{E_3-E_1} \sqrt{E_4-E_2}} \Psi(M, P). \quad (62)$$

In Eq. (61) the following notations are used:

$$\Psi(P, M) = (\Phi_2(P) - \Phi_2(N)) \cdot E_1 + (\Phi_4(N) - \Phi_4(P)) \cdot (E_1 - E_2),$$

$$\varrho^{\frac{2}{3}} = P, \quad M^{\frac{2}{3}} = M.$$

The fundamental functions for moments are:

$$m_1 = M \cdot \cos\left(\pi \frac{\vartheta}{\tilde{\vartheta}}\right), \quad m_2 = M \cdot \sin\left(\pi \frac{\vartheta}{\tilde{\vartheta}}\right). \quad (63)$$

The constant $\tilde{\vartheta}$ is used for the normalization of the moments:

$$\tilde{\vartheta} = \lim_{M \rightarrow \mu} \vartheta(M) = \frac{9}{8} \int_{\varrho^{\frac{2}{3}}}^{\frac{\mu^{\frac{2}{3}}}{\varrho^{\frac{2}{3}}}} \frac{ds}{s \sqrt{\prod_{i=1}^4 (s-E_i)}}. \quad (64)$$

The values (62), (63) are the explicit functions of the moment M . The axial coordinate x depends of m , Eq. (57). Consequently, we arrived the parametric form for all necessary functions m_1, m_2, M, ϑ . The plot of eigenvalue solutions for moments m_1, m_2 and for their complex amplitude M are shown on the Figure 10 for $\mu = 2$. The function m_2 is the even function of the axial coordinate x . On Figures 11 and 12 the similar plots of eigenvalue solutions for moments and for their complex amplitude for $\mu = 3$ and $\mu = 4$. All plots are shown for comparison again on the Figure 13.

4.6. Fundamental functions for buckling displacements

The next task is to determine the fundamental functions for the buckling forms. The even and odd functions of the axial coordinate $y_1(x), y_2(x), -L < x < L$ are the solutions of the ordinary differential equations with already determined functions $J(x) = A^{-\alpha}(x), m_1(x), m_2(x)$:

$$\begin{aligned} (J y_1'')'' &= \Lambda m_1'', \\ y_1(L) &= 0, y_1'(L) = 0, y_1'(0) = 0, y_1'''(0) = 0. \\ (J y_2'')'' &= \Lambda m_2'', \end{aligned} \quad (65)$$

$$y_2(L) = 0, y_2'(L) = 0, y_2(0) = 0, y_2''(0) = 0.$$

The eigenfunctions for the displacements result from the solution of the eigenvalue problem (65). The functions for second moment of inertia and for the moments are too complex for the closed form solution. The solution of the boundary value problem is performed with the numerical method with the Program MAPLE 2020 (Maple, 2020). The results are displayed on Figure 14. The dashed lines demonstrate the even eigenfunctions y_2 for the moment amplitudes $\mu = 2, 3$ and 4. The dotted curves show the odd eigenfunctions y_1 .

4.7. Asymptotic solutions

For the estimation of the optimization effects, we determine limit form of the column in the asymptotic case $\mu \rightarrow \infty$. The shape of the optimal column is an hourglass figure with the infinitely small $\varrho \rightarrow 0$. For this purpose, we have to determine at first the asymptotic limits of the roots:

$$\tilde{E}_1 = \mu^{2/3}; \tilde{E}_i = o(\mu), i = 2, 3, 4.$$

For the vanishing roots

$$\tilde{E}_i = 0, i = 2, 3, 4,$$

the improper integral from Eq. (57) reduces to:

$$\tilde{x}(M) = \frac{3\sqrt{2}}{4} \mu^{\frac{2}{3}} \int_0^{(M/\mu)^{\frac{2}{3}}} \frac{\sqrt{s} ds}{\sqrt{1-s}}. \quad (66)$$

for $0 \leq M < \mu$, $0 \leq x < L$.

The integral (66) expresses with the elementary functions:

$$\tilde{x}(M) = \frac{3\sqrt{2}}{8} \mu^{\frac{2}{3}} \arcsin\left(\frac{2M^{\frac{2}{3}}}{\mu^{\frac{2}{3}}} - 1\right) + \frac{3\pi\sqrt{2}}{16} \mu^{\frac{2}{3}} - \frac{3\sqrt{2}M^{\frac{1}{3}}\mu^{\frac{2}{3}}}{4\sqrt{\mu^{\frac{2}{3}} - M^{\frac{2}{3}}}} + \frac{3\sqrt{2}M}{4\sqrt{\mu^{\frac{2}{3}} - M^{\frac{2}{3}}}}. \quad (67)$$

for $0 \leq M \leq \mu$, $0 \leq \tilde{x} \leq L$.

The similar considerations lead to the asymptotical expansions for volume and averaged rigidity (60) and (61). The asymptotic solution for the amplitude moment M , area of cross-section A and second moment J are displayed on Figures 7,8,9. The functions $M = \tilde{x}$ are drawn for the same parameters, as the exact analytical solutions. The colors correspond the colors of the lines for the exact solutions, but the approximate liens are drawn with the dash style.

Finally, we get the estimations of the invariants in the limit case of hourglass figure with the infinitely narrow tackle:

$$\begin{aligned} L_a &= \lim_{m \rightarrow \mu} \tilde{x}(M) = \frac{3\pi\sqrt{2}}{8} \mu^{\frac{2}{3}}, \\ V_a &= \frac{9\pi\sqrt{2}}{32} \mu^{\frac{4}{3}}, \\ B_a &= \frac{15\pi\sqrt{2}}{64} \mu^2. \end{aligned} \quad (68)$$

5. Isoperimetric inequalities

We evaluate the invariant factors $F_{A,c}, F_{B,c}$ for the column with the constant cross-section using the standard formulas of technical mechanics. With the expressions (59)-(61) and (69) we can determine all invariant factors for the optimal columns with the mixed boundary conditions:

$$F_{A.IV.a} = 2\pi^2, F_{B.IV.a} = \frac{9\pi^2}{5}. \quad (69)$$

The comparison of the factors (69) leads to the estimation of the masses for the column with the same lengths and the same critical buckling load.

$$\frac{V_{IV.a}}{\tilde{V}_{IV}} = \left(\frac{\tilde{F}_{A.IV.a}}{F_{A.IV}} \right)^{-1/2} = \frac{1}{\sqrt{2}}, \frac{B_{IV.a}}{\tilde{B}_{IV}} = \left(\frac{\tilde{F}_{B.IV.a}}{F_{B.IV}} \right)^{-1} = \frac{9}{20}. \quad (70)$$

The mass estimation (70) is in good accordance with the numerically evaluated value. The mass of the optimal column V_{IV} in relation to the mass of the reference column \tilde{V}_{IV} from the exact solution is shown on Figure 15 with the red color. This value reduces continuously with the increasing parameter μ . The asymptotical limit of the ratio $\frac{V_{IV.a}}{\tilde{V}_{IV}}$ for the infinitely high values of μ is

$$\lim_{\mu \rightarrow \infty} \frac{V_{IV.a}}{\tilde{V}_{IV}} = \lim_{\mu \rightarrow \infty} \left(\frac{\tilde{F}_{A.IV.a}}{F_{A.IV}} \right)^{-1/2} = \frac{1}{\sqrt{2}},$$

and is equal to the corresponding value from the approximate solution (70). The analogous tendency has the ratio of averaged stiffness of the optimal rod to the averaged stiffness of the constant-section rod.

It is worthwhile to mention, that the mass of the optimal rod reduces continuously with the increasing value of μ . The optimum could not be attained with the finite value of the parameter. The waist of the hourglass ϱ becomes narrow with the increasing values of μ , but the limit value could not be achieved for the finite values of parameter. Another argument for this statement follows from the comparison of the invariant factors. Figure 16 displays the invariant mass parameter $F_{A.IV}$ for the optimal column as the function of the parameter μ . This value increases asymptotically to its limit value $F_{A.IV.a} = 2\pi^2$. The plots on Figure 17 shows the similar behavior of the invariant rigidity parameter $F_{B.IV}$. This parameter increases to its upper limit $F_{B.IV.a} = \frac{9\pi^2}{5}$ with the rising parameter μ and vanishing parameter ϱ . The isoperimetric inequality could be formulated as the sharp inequality. Consequently, the optimal solution could not be attained for any finite value of parameters. The analogous consideration, but using completely different arguments, was deliberated by (Cox, McCarthy, 1998).

6. Conclusions

The method of dimensionless factors is used for the optimization analysis. The applied method for integration of the optimization criteria delivers different length and volumes of the optimal columns. Instead of the seeking for the columns of the fixed length and volume, the columns with the different lengths and cross-sections are compared using the invariant factors. In this manuscript, the isoperimetric inequalities were rigorously justified by means of the Hölder inequality about the mean values. We demonstrated that Euler's column with boundary conditions of mixed type, which is governed by necessary optimality conditions in the bimodal case, possesses the largest buckling load among all columns with the same weight. The optimal column has the shape of hourglass.

Conflict of interests: On behalf of all authors, the corresponding author states that there is no conflict of interest.

Replication of results: No results are presented

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Figures

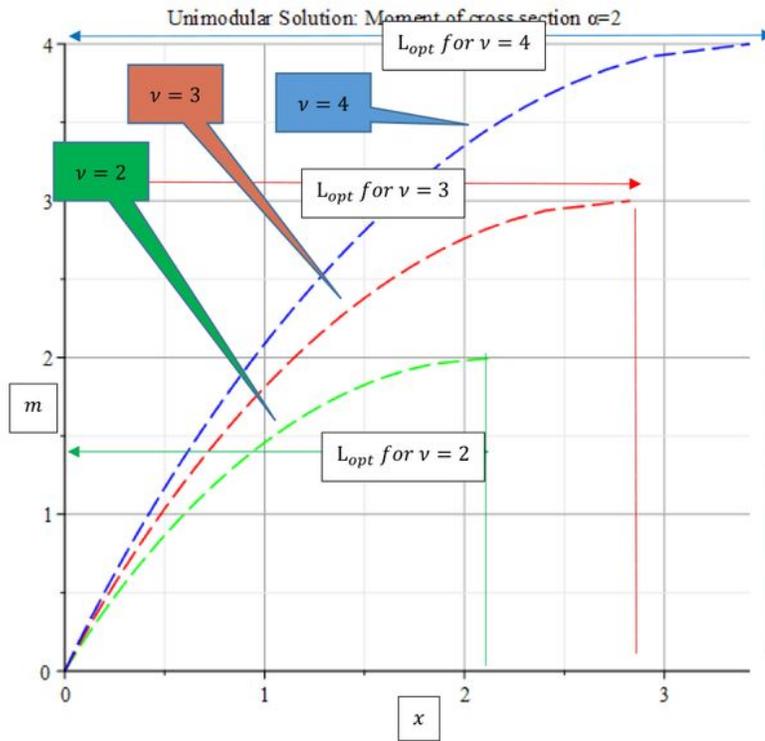


Figure 1 Moments along the half-lengths of the hinged rods for the unimodular solution, Sturm Liouville problem $\alpha = 2$

Figure 1

Please see the figure for the caption.

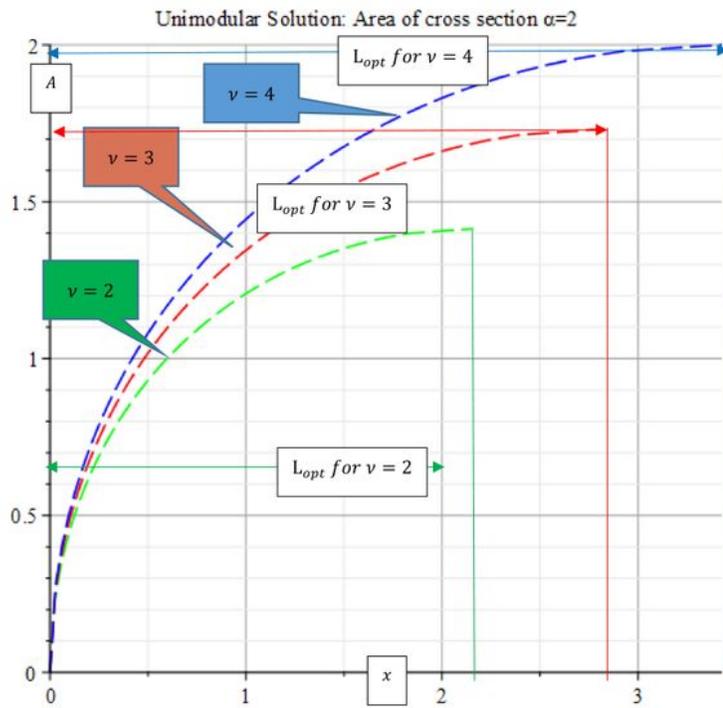


Figure 2 Cross-sectional areas along the half-lengths of the hinged rods for the unimodal solution, $\alpha = 2$ Sturm Liouville problem

Figure 2

Please see the figure for the caption.

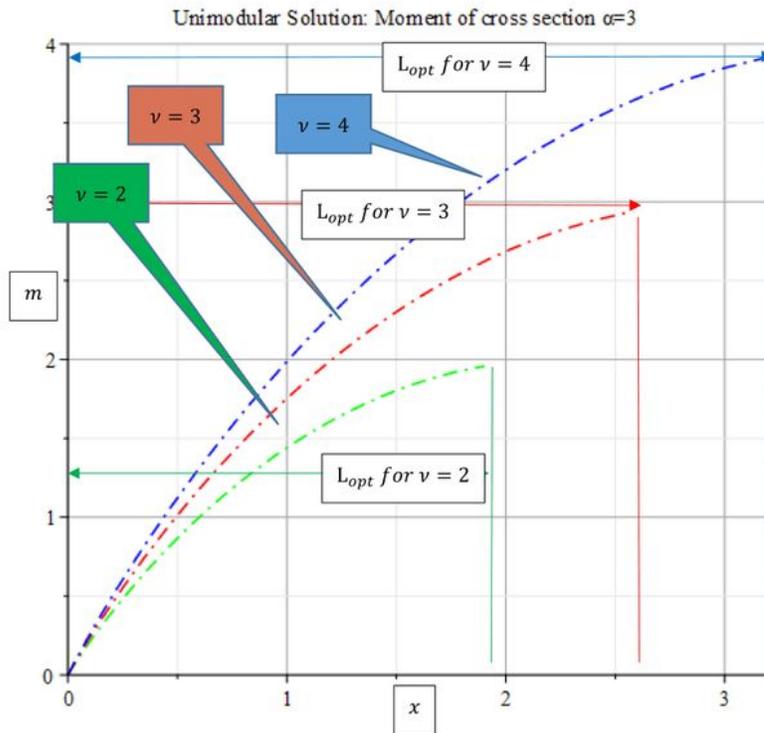


Figure 3 Moments along the half-lengths of the hinged rods for the unimodular solution , $\alpha = 3$, Sturm Liouville problem

Figure 3

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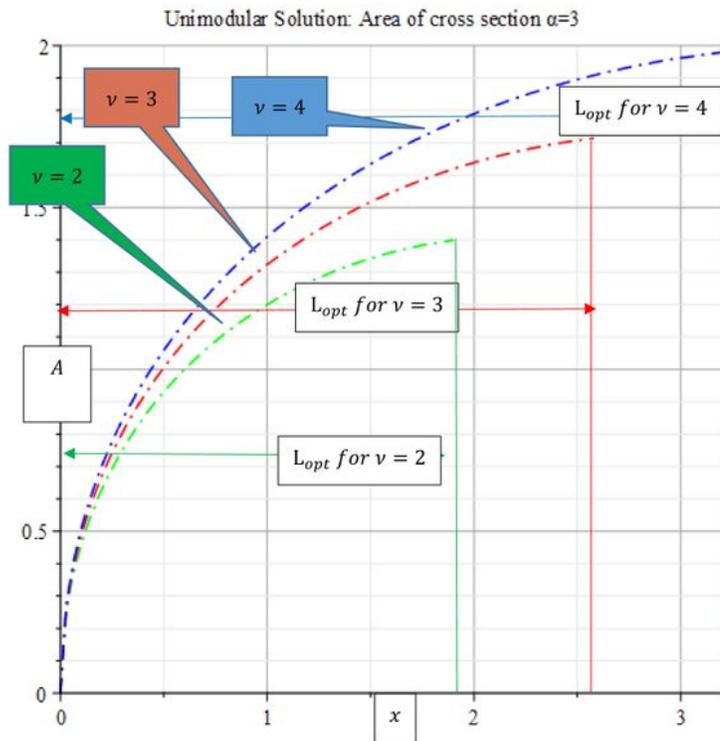


Figure 4 Cross-sectional areas along the half-lengths of the hinged rods for the unimodal solution , $\alpha = 3$, Sturm Liouville problem

Figure 4

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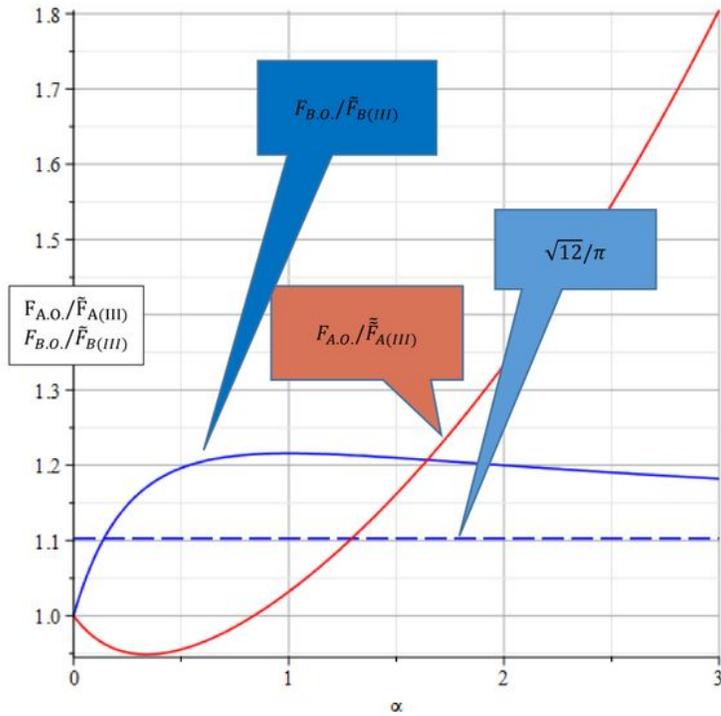


Figure 5 Factors $F_{A,o}/\tilde{F}_{A(II)}$ and $F_{B,o}/\tilde{F}_{B(II)}$ as the functions of exponent α for the Sturm Liouville problem

Figure 5

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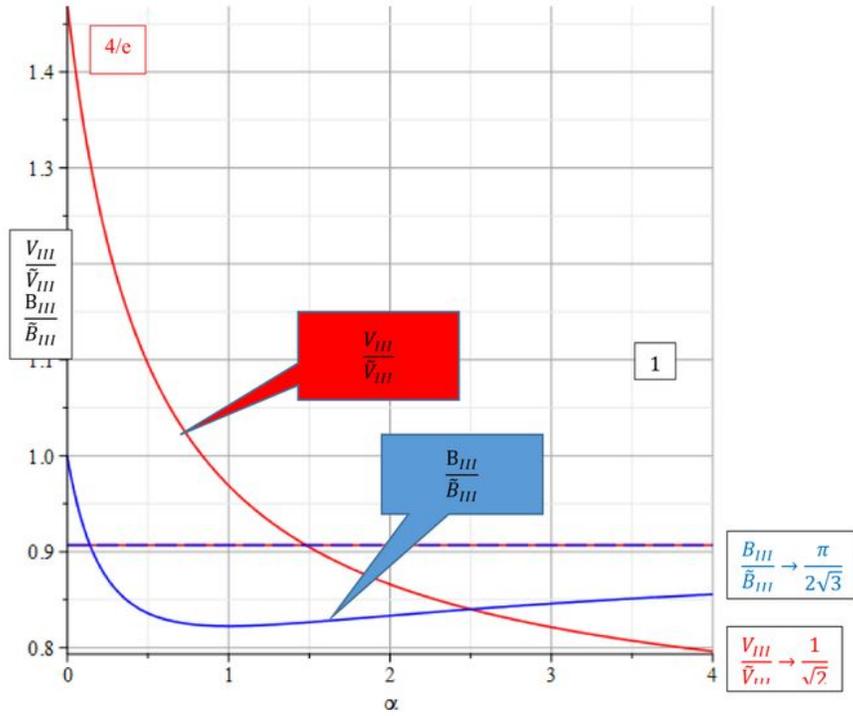


Figure 6 Mass and stiffness ratios $\frac{B_{III}}{\tilde{B}_{III}}$ and $\frac{V_{III}}{\tilde{V}_{III}}$ as the functions of exponent α for the Sturm Liouville problem

Figure 6

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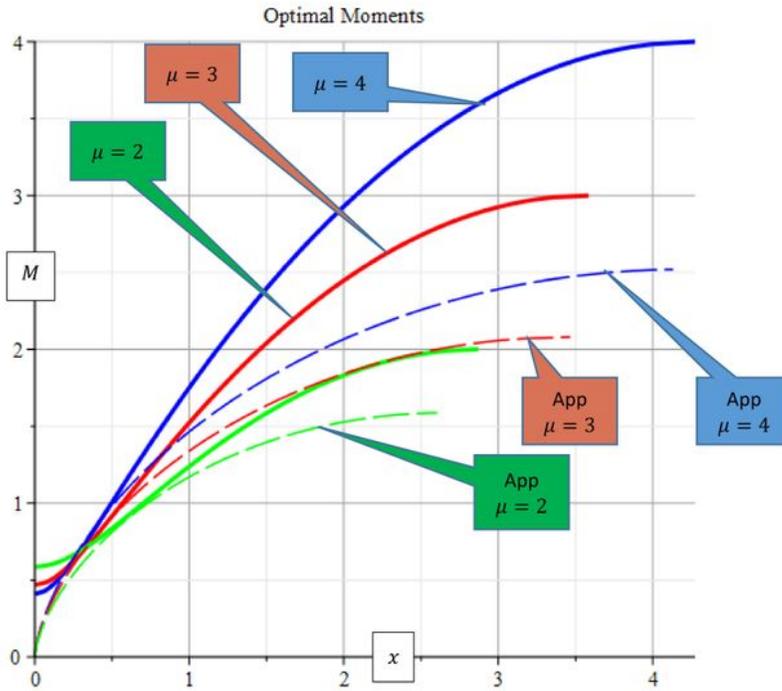


Figure 7 Optimal moments over the span for different values μ . The exact and asymptotical solutions Mixed problem

Figure 7

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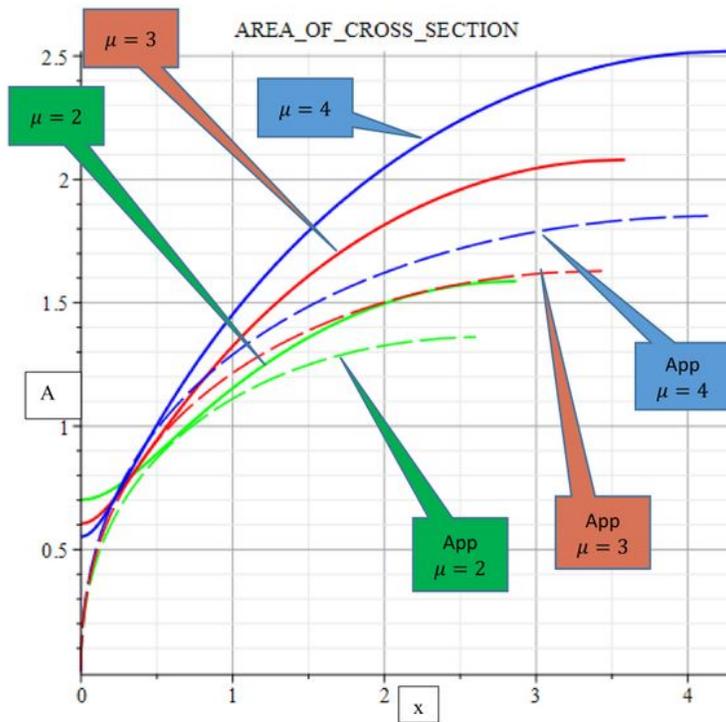


Figure 8 Optimal cross-section areas over the span for different values μ . The exact and asymptotic solutions Mixed problem

Figure 8

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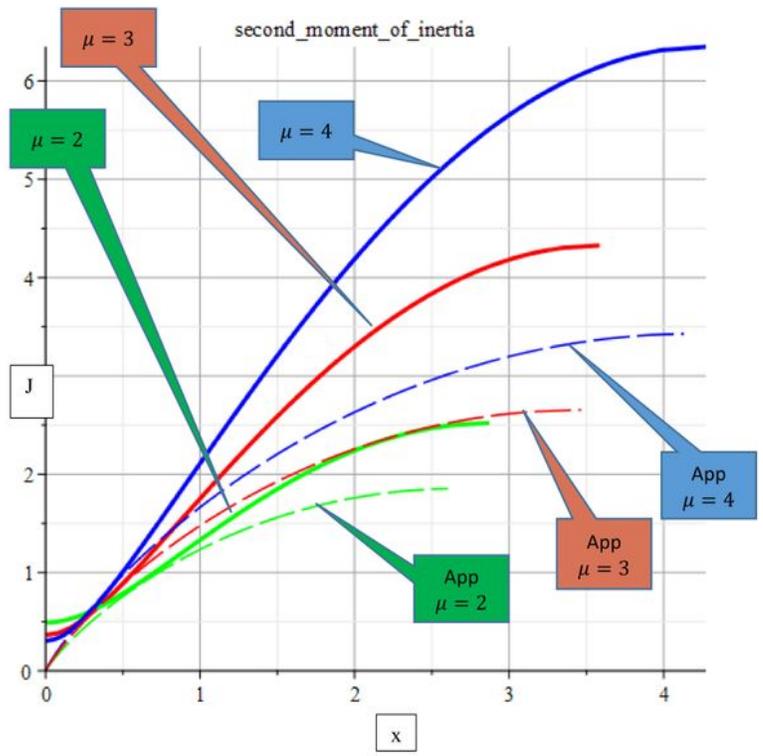


Figure 9 Optimal geometric moment of the second order over the span for different values μ . The exact and asymptotical solutions, Mixed problem

Figure 9

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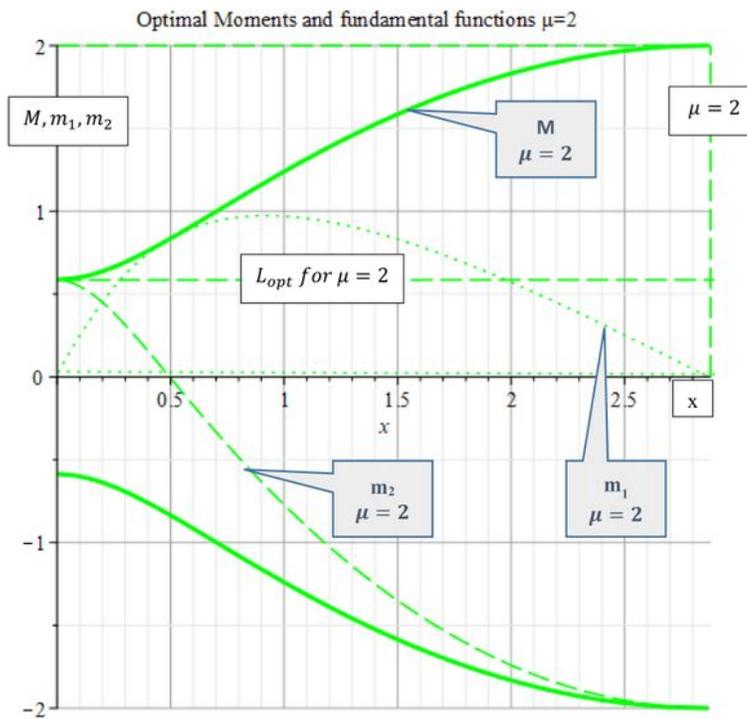


Figure 10 Optimal moments over the span for value $\mu = 2$, Mixed problem

Figure 10

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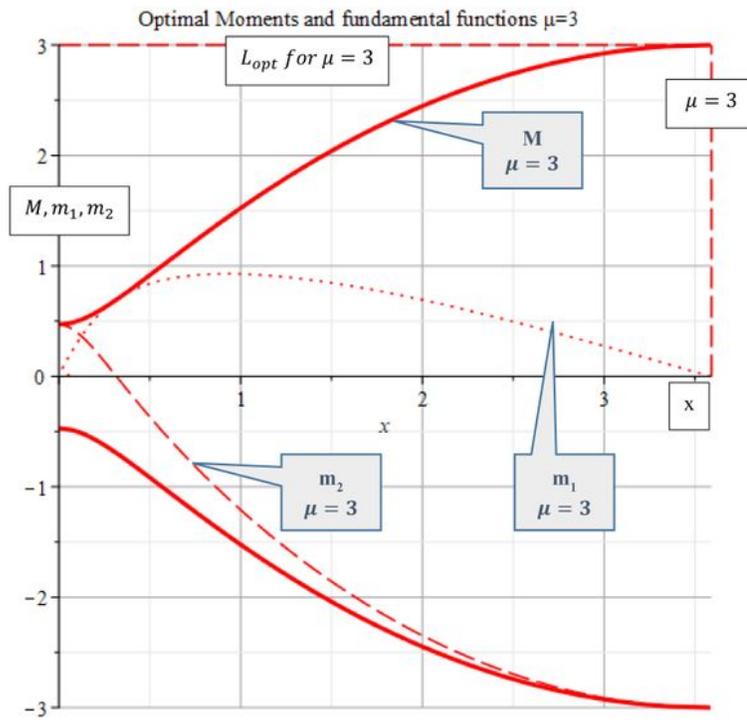


Figure 11 Optimal moments over the span for value $\mu = 3$, Mixed problem

Figure 11

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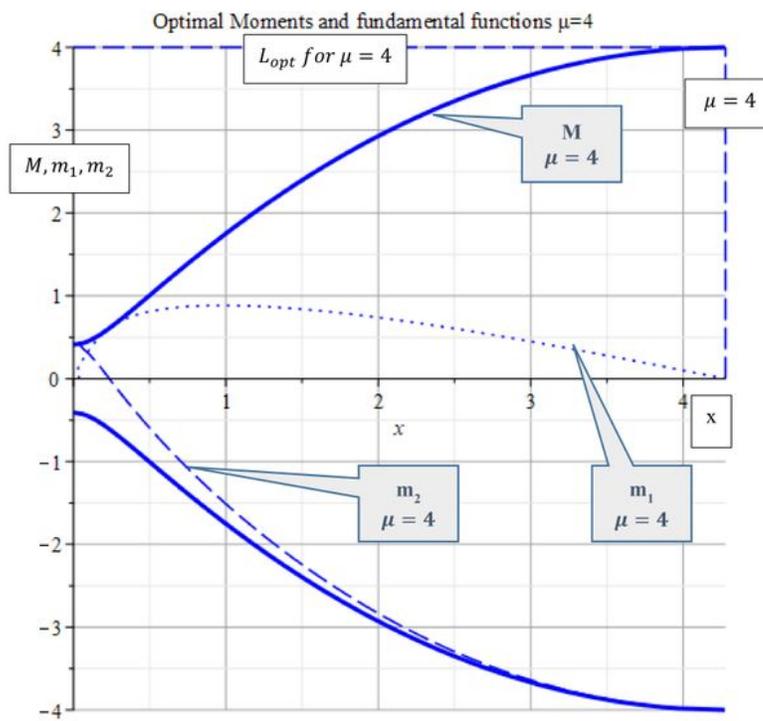


Figure 12 Optimal moments over the span for value $\mu = 4$ Mixed problem

Figure 12

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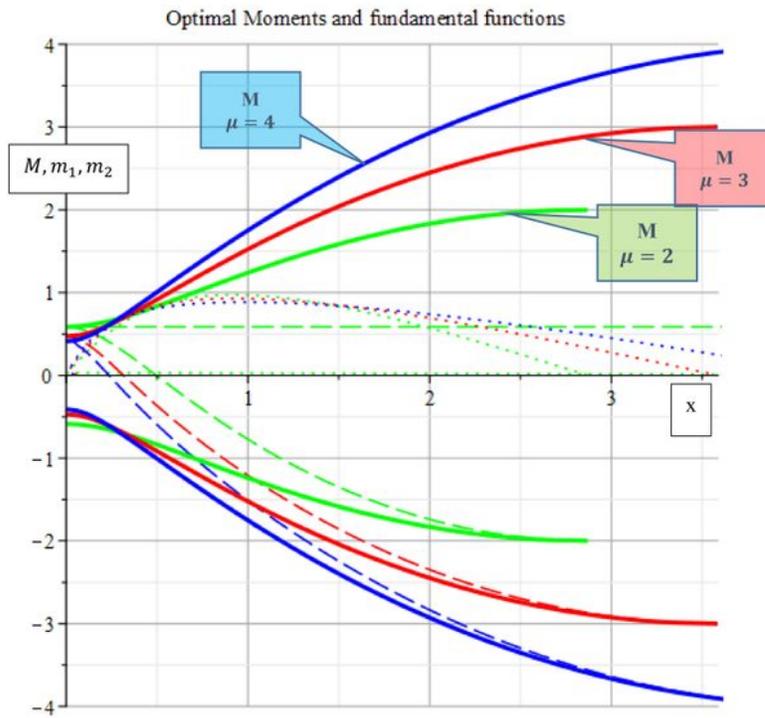


Figure 13 Optimal moments over the span for value $m=2,3,4$ Mixed problem

Figure 13

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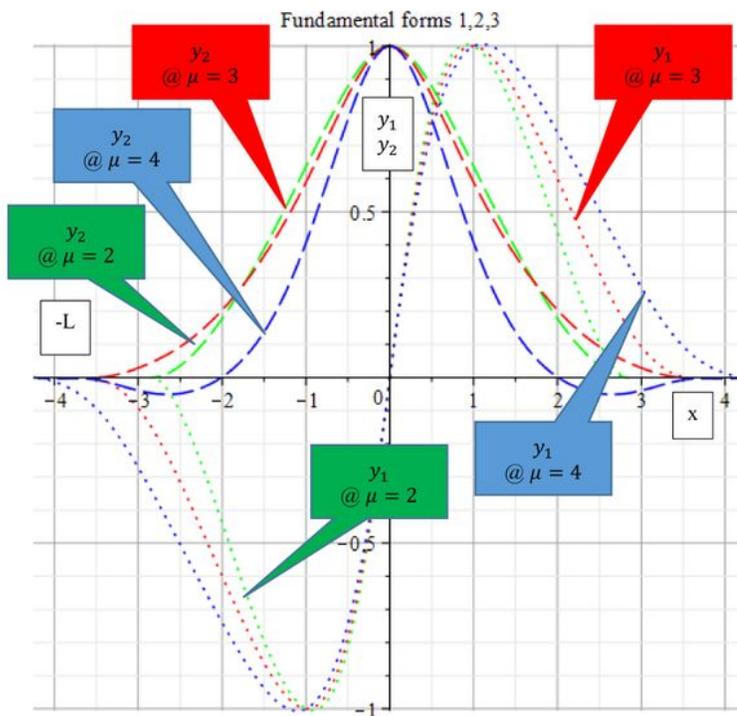


Figure 14 Optimal buckling displacements over the span for value $m=2,3,4$, Mixed problem

Figure 14

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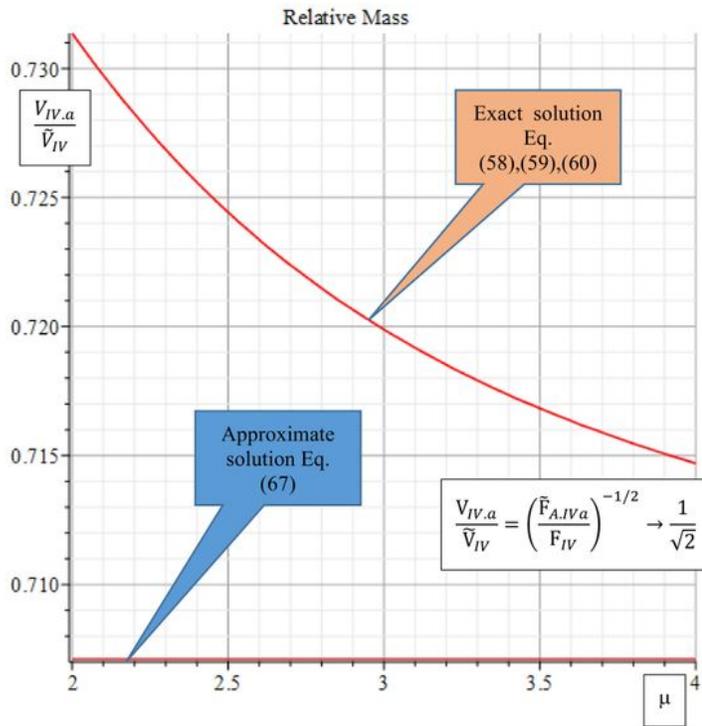


Figure 15 Relative mass, Mixed problem

Figure 15

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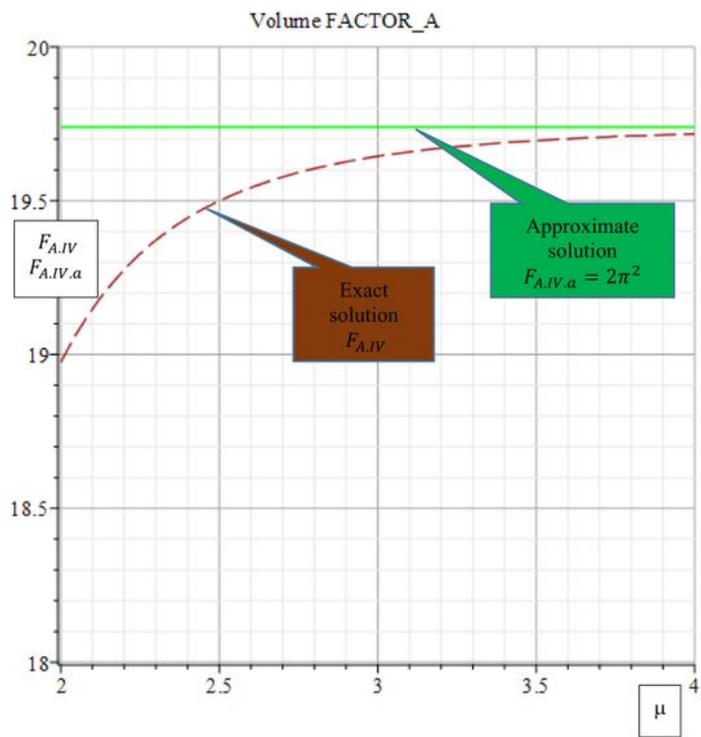


Figure 16 Factor A Mixed problem

Figure 16

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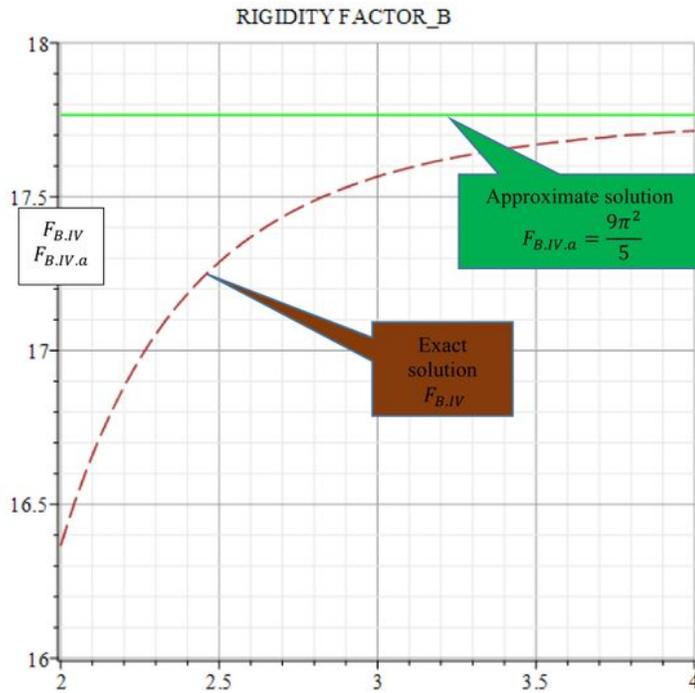


Figure 17 Factor B Mixed problem

Figure 17

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