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Physical Sciences - Article

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Posted Date: October 7th, 2020

DOI: <https://doi.org/10.21203/rs.3.rs-85734/v1>

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Elementary Quadrature for the Riccati equation

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Abstract: Using suitable transformation in combination with a specific Riccati-type equation solvable, the problem of solving Riccati equation can be transformed into that of a quasi-Abel equation of the second kind. By the extended Julia's integrability condition, the general solutions of Riccati equation in the form of elementary quadrature are obtained, which contains numerous. This method opens up a new prospect for the study of nonlinear differential equations by analytical method.

Keywords: Riccati equation; General solution; Elementary quadrature

MSC code: 34A05, 34A25, 34A34

1. Introduction

On the occasion of celebrating the founding of *Science* 125th anniversary, the magazine published 125 of the most challenging scientific issues, the 120th and 122th questions are “Is there a simple way to determine if there are infinitely many solutions to an elliptic curve?” and “mathematicians will eventually give the solution of the Navier-Stokes equation”, respectively. The Riccati equation is closely related to these two problems, and even the key to solve these problems [1-3].

The general Riccati equation

$$y'(x) = f_2(x)y^2 + f_1(x)y + f_0(x), y = y(x), f_2 f_0 \neq 0 \quad (1)$$

is one of the most simple nonlinear differential equations because it is of first order

and with quadratic nonlinearity. Considerable research effort has been attracted from of point of view of its analytic solutions as well as of its applications in many fields of the applied physics. However, the general solution of equation (1) has not been obtained in elementary quadrature form except some special cases so far [4-6]. This motivated the study, in which such an explicit general solution is possible.

2. Preliminaries

In this section, we introduce two existing results.

Lemma.1 [5] If the coefficients of Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x), \quad a(x)c(x) \neq 0 \quad (2)$$

satisfy conditions: (1) $c(x) = c_1(x) + c_2(x)$; (2) $b^2(x) - 4a(x)c(x) > 0$ and

$$(3) \quad c_2(x) = \left[\frac{-b(x) + \sqrt{b^2(x) - 4a(x)c_1(x)}}{2a(x)} \right]'$$

then the general solution of Riccati equation (2) can be exactly given as

$$y(x) = \frac{e^{\int \sqrt{b^2(x) - 4a(x)c_1(x)} dx}}{A_1 - \int a(x) e^{\int \sqrt{b^2(x) - 4a(x)c_1(x)} dx} dx} + \frac{-b(x) + \sqrt{b^2(x) - 4a(x)c_1(x)}}{2a(x)}$$

where A_1 is an integration constant (the proof see appendix).

Lemma.2 [6] For the general form of the Abel equation of second kind

$$[g_0(x) + g_1(x)y]y' = f_2(x)y^2 + f_1(x)y + f_0(x) \quad (3)$$

If there exists a constant λ such that

$$2B_1(x)g_0(x) = \lambda B_2(x)g_1(x)$$

Then equation (3) admits the general solution

$$B_1(x)y^2 + \lambda B_2(x)y = 2 \int \frac{B_1(x)f_0(x)}{g_1(x)} dx + A_2$$

where A_2 is an integration constant, $B_1(x) = e^{-2\int \frac{f_2(x)}{g_1(x)} dx}$ and $B_2(x) = e^{-\int \frac{f_1(x)}{g_0(x)} dx}$.

3. Main result and Proof

Based on above result, general solution of the Riccati equation is given.

Theorem Riccati equation (1) is integrable by elementary quadrature method and general solution is given as formula (13).

Proof: The substitution $y(x) = u(x)\phi(x) + v(x)$ brings equation (1) into

$$\phi' = f_2 u \phi^2 + \left(2f_2 v + f_1 - \frac{u'}{u} \right) \phi + \frac{f_0 + f_1 v + f_2 v^2 - v'}{u} \quad (4)$$

let

$$\frac{u'}{u} = 2f_2 v + f_1 \quad (5)$$

and equation (4) is reduced to

$$\phi' = f_2 u \phi^2 + \frac{f_0 + f_1 v + f_2 v^2 - v'}{u} \quad (6)$$

we introduce a differentiable function $\mathcal{G}(x)$ and rewrite equation (6) as

$$\phi' = f_2 u \phi^2 + \frac{1}{u} [f_0 + f_1 v + f_2 v^2 + \mathcal{G} v^4 - v' - \mathcal{G} v^4] \quad (7)$$

Equating the coefficients of similar items for equations (2) and (7), after setting

$a = f_2 u, b = 0, c_1 = -\frac{\mathcal{G} v^4}{u}$ and $c_2 = \frac{1}{u} (f_0 + f_1 v + f_2 v^2 + \mathcal{G} v^4 - v')$, by the **Lemma.1**, we

know if

$$\frac{1}{u} (f_0 + f_1 v + f_2 v^2 + \mathcal{G} v^4 - v') = \left(\frac{v^2}{u} \sqrt{\frac{\mathcal{G}}{f_2}} \right)' \quad (8)$$

we have

$$\phi(x) = \frac{e^{2\int f_2 \sqrt{\frac{g}{f_2}} v^2 dx}}{A_1 - \int f_2 u e^{2\int f_2 \sqrt{\frac{g}{f_2}} v^2 dx} dx} + \frac{v^2}{u} \sqrt{\frac{g}{f_2}} \quad (9)$$

Now we focus on equation (8), with the help of equation (5), it can be written as a quasi-Abel equation of the second kind:

$$\left(1 + 2\sqrt{\frac{g}{f_2}} v\right) v' = \left[f_2 \left(\sqrt{\frac{g}{f_2}} v + 1\right)^2 + f_1 \sqrt{\frac{g}{f_2}} - \left(\sqrt{\frac{g}{f_2}}\right)' \right] v^2 + f_1 v + f_0 \quad (10)$$

by the **Lemma.2**, we know if

$$\exp \left[-\int f_2 \left(\sqrt{\frac{g}{f_2}} v + 1\right) \left(\sqrt{\frac{g}{f_2}}\right)^{-1} dx \right] = \lambda \quad (11)$$

the general solution of equation (10) is given by quadratic algebraic equation

$$\sqrt{\frac{g}{f_2}} v^2 + v = \Phi(x) \quad (12)$$

where $\Phi(x) = e^{\int f_1 dx} \left(A_2 + \int f_0 e^{-\int f_1 dx} dx \right)$. Finally, by the transformation $y(x) = u\phi + v$,

we get

$$\begin{aligned} y(x) &= \frac{u e^{2\int f_2 \sqrt{\frac{g}{f_2}} v^2 dx}}{A_1 - \int f_2 u e^{2\int f_2 \sqrt{\frac{g}{f_2}} v^2 dx} dx} + \sqrt{\frac{g}{f_2}} v^2 + v \\ &= \frac{u e^{-\int 2f_2 v dx} e^{\int 2f_2 \Phi dx}}{A_1 - \int f_2 u e^{-\int 2f_2 v dx} e^{\int 2f_2 \Phi dx} dx} + \Phi = \frac{e^{\int (f_1 + 2f_2 \Phi) dx}}{A_1 - \int f_2 e^{\int (f_1 + 2f_2 \Phi) dx} dx} + \Phi \end{aligned} \quad (13)$$

Note that equation (5) can be rewritten to $u e^{-\int 2f_2 v dx} = e^{\int f_1 dx}$, which we used in the last step.

Remark: In this paper, we avoid solving the equations (11) and (12); in addition, because of the diversity of A_2 , the solutions of Riccati equation are also diverse.

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Acknowledgment

This research was supported by the Fund of the National Natural Science Foundation of China under Grant No.11972333, Zhejiang Provincial Natural Science Foundation of China under Grant No.R105248 and LZ19A020001. The author would like to express his sincere appreciation to these supports.

Appendix: Proof of the Lemma.1 [5]

We rewrite equation (2) as follow

$$\begin{aligned}
y' &= ay^2 + by + c \\
&= a \left(y + \frac{b + \sqrt{b^2 - 4ac_1}}{2a} \right) \left(y + \frac{b - \sqrt{b^2 - 4ac_1}}{2a} \right) + \left(\frac{-b + \sqrt{b^2 - 4ac_1}}{2a} \right)'
\end{aligned}$$

further, we can transform it into a *Bernoulli*-type equation

$$\left(y + \frac{b - \sqrt{b^2 - 4ac_1}}{2a} \right)' - \sqrt{b^2 - 4ac_1} \left(y + \frac{b - \sqrt{b^2 - 4ac_1}}{2a} \right) = a \left(y + \frac{b - \sqrt{b^2 - 4ac_1}}{2a} \right)^2$$

and this solution is easy to get.