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RESEARCH

Piecewise Sparse Approximation via Threshold P_OMP with Application to Surface Reconstruction

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Abstract
Sparse signal representations have gained extensive study in recent years. In applications, there are large amounts of signals that are structured. Motivated by signal decomposition and scattered data reconstruction applications, we consider a particular type of structured signals which can be represented by a union of several sparse vectors. We define this type of signal as piecewise sparse signals. To find a piecewise sparse representation of a signal, we propose a thresholding version of piecewise orthogonal matching pursuit(TP_OMP), which aims to overcome the disadvantages of P_OMP. We also establish the connection of piecewise sparsity and sampling over a union of subspaces. We evaluate the performance of the proposed greedy programs through simulations on surface reconstruction.

Keywords: piecewise sparse; greedy algorithm; thresholding; sampling

1 Introduction

Sparse recovery or sparse representation from a given signal(or image) $\mathbf{b}$ with linear noisy measurement

$$\mathbf{b}^{\sigma} = A\mathbf{x} + \mathbf{e} \tag{1}$$

is of interest in many different applications. The matrix $A \in \mathbf{R}^{m\times n}$ with $m \ll n$ and $\mathbf{e}$ is an additive noise. Large amounts of algorithms have been proposed to determine $\mathbf{x}$ in a stable and efficient manner, for detail refer to [1, 2, 3, 4, 5, 6, 7].

In standard sparse recovery problem, the undermined vector $\mathbf{x}$ is assumed to be sparse, i.e, contains at most s nonzero entries($s \ll m$) without any further information on the structure of $\mathbf{x}$. However, in real applications the nonzero elements of $\mathbf{x}$ are always correlative, thus $\mathbf{x}$ possess certain type of structure. Research in recent years study the sparsity properties of $\mathbf{x}$ together with its structure, which result in structured sparsity [8, 9, 10]. The structured sparsity has the merit of reducing the degree of freedom of the solution thereby better recovery capacity. Furthermore, one can use the structured sparsity of the signal to reduce the sampling rate, thus attain the goal of reduction of storage in applications. Particularly, we use "piecewise sparse" in [11, 12] with the following definition

Definition 1 *[12] For a given vector* $\mathbf{x} = (\underbrace{x_1,\ldots,x_{d_1}}_{\mathbf{x}_1^T},\underbrace{x_{d_1+1},\ldots,x_{d_1+d_2}}_{\mathbf{x}_2^T},\ldots,\underbrace{x_{n-d_N+1},\ldots,x_n}_{\mathbf{x}_N^T})^T$*, where* $n = \sum_{i=1}^{N} d_i$ *and* $s_i = \|\mathbf{x}_i\|_0$*,* $i = 1,\ldots,N$*. A vector* $\mathbf{x} = (\mathbf{x}_1^T,\ldots,\mathbf{x}_N^T)^T \in \mathbf{R}^n$ *is partitioned into N components and it is assumed that every* $\mathbf{x}_i^T \in \mathbf{R}^{n_i}$ *containing nonzero entries is sparse,* $s_i = \|\mathbf{x}_i\|_0$*,* $i = 1,\ldots,N$*. We call the vector* $\mathbf{x}$ *is* $(s_1,\ldots,s_N)$*-**piecewise sparse**.*

Piecewise sparsity describes a type of structured sparsity which emerges in the application of reconstructing a surface from scattered data in PSI space [13, 14, 15], data separation/signal decomposition[16, 17, 18, 19], etc. In these applications, data(including image, signal) are represented in different bases and frames, and each frame exhibits its particular geometric tendency or attribute, i.e. signal $\mathbf{b} = \mathscr{P} + \mathscr{C} = \Phi\mathbf{x} + \Psi\mathbf{y}$, where Φ and Ψ are different frames and data separation aims at obtaining sparse coefficients $\mathbf{x}$ and $\mathbf{y}$.

Instead of solving data separation problem, we discuss the approximation problem based on the special sparsity-piecewise sparse summarized from data separation: the dictionary is a union of basis $A = [A_1,\ldots,A_N]$ and we describe the piecewise sparse approximation as the following problem

$$\mathbf{b}^\sigma = \sum_{i=1}^{N} A_i\mathbf{x}_i + \mathbf{e}. \tag{2}$$

For each frame A_i, $\mathbf{x}_i$ is a sparse vector, thus $\mathbf{x} = (\mathbf{x}_1^T,\ldots,\mathbf{x}_N^T)^T$ is a piecewise sparse vector. It is shown in [12, 11] that better recovery and estimate is possible when A is in an union manner, which provide milder conditions for estimating the sparse recovery results. Sparse representations in unions of orthogonal bases are studied in [20, 21, 22]. We generalize the results in orthogonal cases to the cases of general bases in [12], improved bounds of $\|\mathbf{x}^*\|_0$ recovered by both l^0 and l^1 optimizations. Furthermore, piecewise inverse scale space method and piecewise sparse OMP(P_OMP) are provided in [11, 23] for piecewise sparse recovery based on l^1 and l^0 minimization,respectively, both aim at preserving the smallest magnitudes. P_OMP(Piecewise Orthogonal Matching Pursuit) is a type of greedy algorithm proposed by the motivation of a combination of the Compressive Sampling Matching Pursuit(CoSaMP) [5] and Generalized OMP(GOMP) [24]. However, lots of prior information such as piecewise sparsity should be provided before the algorithm started. The focus of the present paper is on developing a generalized greedy algorithm for piecewise sparse recovery which can be both applied to large amounts of signal decomposition applications and do not require prior information on the signal. Our algorithm is based on the P_OMP algorithm and the main idea of threshold greedy algorithms.

Organization The remaining of the paper is organized as follows. The P_OMP algorithm is described in Section 2. A thresholding version of P_OMP which do not require piecewise sparsity in prior named TP_OMP algorithm is also proposed in Section 2. We then show the connection of piecewise sparsity and sampling over a union of subspaces. In Section 3 we show a practical application of union sampling—surface reconstruction with scattered data and apply the results to surface reconstruction.

2 Method

2.1 P_OMP Algorithm

The main iteration of P_OMP algorithm in [23] are as follows:

Algorithm 1 P_OMP Recovery Algorithm

Input: Matrix $A = [A_1, \dots, A_N]$, noisy vector $\mathbf{b}$, piecewise sparsity level s_i for $i = 1, \dots, N$
Output: An $(s_1, \dots, s_N)$-piecewise sparse approximation $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)^T$
Initialization: $\mathbf{x}^0 \leftarrow 0$, $\mathbf{res} \leftarrow \mathbf{b}$, $k \leftarrow 0$
Repeat $k \leftarrow k+1$
For $i = 1, \dots, N$ **in parallel**
(Form piecewise signal proxy) $\mathbf{y}_i = A_i^T \mathbf{res}$
(Identify s_i large components of every piece) $\Omega_i \leftarrow supp(\mathbf{y}_{i(s_i)})$
(Merge supports) $\Lambda \leftarrow \bigcup \Omega_i \bigcup supp(\mathbf{x}^{k-1})$
(Signal estimation by least-squares) $\tilde{\mathbf{x}}|_\Lambda \leftarrow A_\Lambda^\dagger \mathbf{b}$ and $\tilde{\mathbf{x}}|_{\Lambda^c} \leftarrow 0$
(Select the first x_i for each piece and form together) $\mathbf{x}^k \leftarrow (\tilde{\mathbf{x}}_{1(s_1)}, \dots, \tilde{\mathbf{x}}_{N(s_N)})^T$
(Update current residual) $\mathbf{res} \leftarrow \mathbf{b} - A\mathbf{x}^k$
Until Stopping criterion *true*

It is obvious the major difference between P_OMP and CoSaMP in [5] lie in the **Identification** and **Pruning** steps. It is noticed that the P_OMP locate s entries of a vector in step **Identification** similar to the first step of GOMP. However, the P_OMP locate the largest s_i entries of $\mathbf{x}_i$ $(i = 1, \dots, N)$ and form all the components together as the $s = (s_1 + \dots + s_N)$ entries, which is different from both CoSaMP and GOMP. Instead of selecting s-largest components of least-square solution, the P_OMP sort the components of the piecewise vector $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_N)^T$ solved by least-square by magnitudes of each piece. The **piecewise pruning** process make progress in preserving small-scaled entries in each piece from false chosen cause of noise perturbation.

Though the P_OMP is designed especially for piecewise sparse recovery, it requires prior information–piecewise sparsity level s_i for $i = 1, \dots, N$ which is often unknown in applications. Actually, the majority of the greedy algorithms for sparse recovery require the global sparsity s before algorithms started. However, the purpose of signal decomposition applications is recover sparse representation for each components(sub-vector) without known sparsity level. Thus it is necessary to develop a generalized piecewise greedy algorithm for piecewise sparse recovery.

2.2 Threshold P_OMP Algorithm

In order to avoid the disadvantage P_OMP, we develop a threshold P_OMP algorithm which use threshold to adaptively select atoms in iterations as follows:

Notice that the length of $\mathbf{x}_i$ for each piece are known in prior in signal decomposition problem, we denote the length of $\mathbf{x}_i$ as d_i.

Thresholding Strategy In practice, hard thresholding leads to better results. Furthermore, Bobin etc. showed that the use of hard thresholding is likely to provide the l^0-sparsest solution in [25].

In algorithm 2 the $TH_\gamma(\mathbf{u})$ denotes component-wise thresholding with threshold γ: hard thresholding $TH_\gamma(\mathbf{u}) = u$ if $|\mathbf{u}| > \gamma$ and zero elsewhere.

Algorithm 2 Threshold P_OMP(TP_OMP) Recovery Algorithm

Input: Matrix $A = [A_1, \ldots, A_N]$, noisy vector $\mathbf{b}$,
Output: A piecewise sparse approximation $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)^T$
Initialization: $\mathbf{x}^0 \leftarrow 0$, $\mathbf{res} \leftarrow \mathbf{b}$, $k \leftarrow 0$
Repeat $k \leftarrow k+1$ compute t as the mean value of absolute value of the product $A^T\mathbf{res}$;
(Form piecewise signal proxy) **For** $i = 1, \ldots, N$ **in parallel** $\mathbf{y}_i = A_i^T \mathbf{res}$
(Identify large components of every piece) $\Omega_i \leftarrow supp(TH_{\beta_i}(\mathbf{y}_i))$
(Merge supports) $\Lambda \leftarrow \bigcup \Omega_i \bigcup supp(\mathbf{x}^{k-1})$
(Signal estimation by least-squares) $\tilde{\mathbf{x}}|_\Lambda \leftarrow A_\Lambda^\dagger \mathbf{b}$ and $\tilde{\mathbf{x}}|_{\Lambda^c} \leftarrow 0$
(Select entries for each piece and form together) $\mathbf{x}^k \leftarrow (TH_{\alpha_1}(\tilde{\mathbf{x}}_1), \ldots, TH_{\alpha_N}(\tilde{\mathbf{x}}_N))^T$
(Update current residual) $\mathbf{res} \leftarrow \mathbf{b} - A\mathbf{x}^k$
Until Stopping criterion *true*

Thresholding Parameters The thresholding parameters in Algorithm 2 are set as follows: $\beta_i = \frac{d_i t}{C_1}$ where $C_1 > 0$ is a constant(which is often set as $C_1 \geq 50$), $\alpha_i = C_2 tt_i$ where $tt_i (i = 1, \ldots, N)$ are the mean values of $|\tilde{\mathbf{x}}_i|$ and $C_2 > 0$ is a constant.

By using the thresholding strategy, the TP_OMP algorithm do not require the prior information on piecewise sparsity or global sparsity, thus it is more practical.

2.3 Sampling Over a Union of Subspaces

In this section we describe the connection of piecewise sparsity and sampling over a union of subspaces. Therefore we show the application to sparse approximation of fitting surface to scattered points, which is an application of sampling over a union of subspaces.

Large amounts of reference such as [26, 27] have illustrate in detail on the traditional sampling theory which recover a unknown signal $\mathbf{x} \in \mathscr{H}$ from m samples. It is common assumed that $\mathbf{x}$ lie in a particular subspace $\mathscr{A}$ of $\mathscr{H}$. The general linear sampling model is: $b_l = \langle f_l, x \rangle, \quad 1 \leq l \leq m$ with f_l is a series of function in $\mathscr{H}$. When $\mathscr{H} = \mathbf{R}^n$ the sampling system can be rewritten as:

$$\mathbf{b} = F^T \mathbf{x}, \tag{3}$$

where the matrix F is formed by columns of f_l. It is generally assumed that in [26, 27, 28] $\mathbf{x}$ lie in a given subspace $\mathscr{A}$ of $\mathscr{H}$. However, in applications $\mathbf{x}$ often do not lie in particular subspace but in a union of subspaces, such as piecewise polynomials, sparse representation, etc.[29] with the following description:

$$\mathbf{x} \in \mathscr{U} = \bigcup_l \mathscr{V}_l \tag{4}$$

where $\mathscr{V}_l$ are subspaces.

Connection of Block sparsity and Sampling Over a Union of Subspaces Eldar and Mishali considered a particular case of (4) in [30], i.e., $\mathscr{V}_i = \bigoplus_{l=s} \mathscr{A}_l$ where $\{\mathscr{A}_l,\ 1 \leq l \leq N\}$ is a series of disjoint subspaces, $|l| = s$ represents the sum of s indices which corresponds to the block sparsity in [30]. Then a signal in a union of subspace can be written as $\mathbf{x} = \sum_{|l|=s} D_l c_l$ where c_l are the basis coefficients of subspace $\mathscr{A}_l$. Denote matrix $D : \mathbf{R}^n \to \mathscr{H}$ as the operator formed by D_l, then $D\mathbf{c} = \sum_{l=1}^{N} D_l c[l]$ with $c[l] = (c_{l1}, \ldots, c_{ld_l})$. If subspace $\mathscr{A}_j$ is not included in the union, set $c[j] = 0$, thus the representation $\mathbf{x} = D\mathbf{c}$ contains at most s nonzero blocks $c[l]$, then $\mathbf{b}$

$$\mathbf{b} = F^T D\mathbf{c}. \tag{5}$$

is block sparse represented since $\mathbf{c}$ is s-block sparse.

Connection with Piecewise Sparse Recovery We consider another type of sampling:$\mathscr{V}_i = \bigoplus \mathscr{A}_l$ where $\{\mathscr{A}_l,\ 1 \le l \le N\}$ is a series of disjoint subspaces. Denote $d_l = dim(\mathscr{A}_l)$, then $n = \sum_{l=1}^{N} d_l$. We describe the recovered $\mathbf{x}$ as $\mathbf{x} = \sum_{l=1}^{N} D_l \mathbf{c}_l$ Furthermore,$D\mathbf{c} = \sum_{l=1}^{N} D_l \mathbf{c}_l$ with $\mathbf{c}_l = (c_{l1}, \ldots, c_{ld_l})$ are vector formed by the basis coefficients of $\mathscr{A}_l$. Rewritten the above equation as

$$D\mathbf{c} = \sum_{l=1}^{N} \left(\sum_{k \in I_i} D_{l(k)} c_{l(k)}\right), \quad |I_i| \le s_i.$$

Thus $\mathbf{x}$ is an $(s_1, \ldots, s_N)$ piecewise sparse vector, and the $\mathbf{b}$ can be piecewise sparse represented by

$$\mathbf{b} = F^T D\mathbf{c}.$$

with a piecewise sparse vector $\mathbf{c}$. It is obvious the system is equivalent to piecewise sparse recovery problem in Definition 1 by setting $A = F^T D$.

3 Experiments

In science and engineering fields, scattered data reconstruction refers to the problem of fitting a smooth surface through a scattered, or nonuniform, distribution of data samples. In a typical scattered data reconstruction problem, there is a set of scattered data sites $\mathscr{X} = \{x_1, x_2, \ldots, x_n\} \subseteq \Omega \subseteq \mathbf{R}^d$ and corresponding function values $f|_{\mathscr{X}} = \{f_1, f_2, \ldots, f_n\}$, and we seek a function g belonging to space $\mathscr{H}$ which fits the given data $\{(x_k, f_k)\}_{k=1}^{n}$. A classical approach to scattered data approximation is the smoothing spline for multi-dimensional data (i.e. $d \ge 2$), which solves the minimization:

$$\min \sum_{k=1}^{n} (g(x_k) - f_k)^2 + \alpha |g|_{H^m}^2.$$

where g belongs to the Beppo$-$Levi space H^m[15]. However, this method can be computationally expensive as the size of data grows large. Johnson *et al.*[15] proposed a regularized least square framework in an approximation space spanned by the shifts and dilates of a single compactly supported function (PSI space) using the uniform B-spline tool. By using the space $\mathscr{H} = \bigcup_{i=1}^{N} H_i$ as the approximation space, where $H_i(\subseteq$

H_{i+1}) is a principle shift invariant (PSI) space generated by a single B-spline function ϕ_i. The fitting surface g is represented as:

$$g = \sum_{i=1}^{N} g_i, \;\; g_i \in H_i, \;\; g_i = \sum_{j=1}^{m} c_j^i \phi_j^i.$$

where c_j^i are coefficients components, and $H_i = span\{\phi_j^i, j = 1, ..., m\}$. In accordance to the piecewise sparse model, the matrix $A = [A_1, \dots, A_N]$ is the observation matrix defined by ϕ_j^i

$$A_i(s,k) = \phi(\frac{2^i x_s}{h} - k); \;\; k \in K$$

where $h > 0$ is a scaling parameter that control the refinement of the PSI space, and K is the index set

$$K = \{k \in \mathbf{Z}^2 : \; supp(\phi(\frac{2^s \cdot}{h} - k)) \cap \Omega \neq \emptyset\}.$$

$b = (b_1, ..., b_n)$ is the vector of the noisy data $\{f_i\}$. It is obvious this problem is a typical scheme of sampling in a union of spaces.

In another word, reconstruction of a fitting surface $g(x,y)$ with a sparse representation for a given set of scattered points $\{(x_k, y_k)\}$ and the corresponding data $\{f(x_k, y_k)\}$ with noise. The approximate function is represented by the multi-level B-spline bases $\phi_j^i(x,y), \; i = 1, \dots, N; \; j = 1, \cdots, n_i$, where N means the number of levels of B-splines, and $n = n_1 + \cdots + n_N$ is the total number of the bases. That is

$$g(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} c_j^i \phi_j^i(x,y), \tag{6}$$

where the coefficients

$$c = (c_1^1, \dots, c_{n_1}^1, c_1^2, \dots, c_{n_2}^2, \dots, c_1^N, \dots, c_{n_N}^N)^T$$

is the N piecewise sparse vector to be determined. Thus the problem is also a piecewise sparse recovery probelm.

We use the 2D tensor product quadratic B-spline with uniform knots as the function ϕ in this paper. For a given set of scattered points $(x_i, y_i)_{i=1}^n \subseteq \Omega \subseteq \mathbf{R}^2$ and the corresponding noisy data $\{f_i\}_{i=1}^n$. Define the 2D tensor product quadratic B-spline function with uniform knots is:

$$S(x,y) = \sum_{i=0}^{2} \sum_{j=0}^{2} P_{ij} N_{i,2}(x) N_{j,2}(y);$$

where P_{ij} are the control points and $N_{i,2}(x), \; N_{j,2}(y)$ are the B-spline basis function corresponding to the uniform grid on the x and y axis. Given a grid $U : \{u_i\}_{-\infty}^{+\infty}, \; u_i \leq u_{i+1}, \; i = 0, \pm 1, \dots,$ the B-spline basis function is defined as:

$$N_{i,0}(u)=\begin{cases}1, & u\in[u_i,u_{i+1})\\ 0, & else.\end{cases}$$

and

$$N_{i,p}(u)=\frac{u-u_i}{u_{i+p}-u_i}N_{i,p-1}(u)+\frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}}N_{i+1,p-1}(u),\ p\geq 1.$$

3.1 Numerical Examples

In this part, we apply TP_OMP to reconstruct a fitting surface $g(x,y)$ with a sparse representation for a given set of 30×30 scattered points $\{(x_k,y_k)\}_{k=1}^{900}\subseteq\Omega=[-1,1]\times[-1,1]$ and the corresponding data $\{f(x_k,y_k)\}_{k=1}^{900}$ with noises ($\varepsilon\in\mathbf{N}(0,\sigma^2\mathbf{I}_n)$), compared with the approximation surface reconstructed by Multilevel Least Absolute Shrinkage and Selection Operator(MLASSO) in [31] and CoSaMP in [5]. The approximate function is represented by the multi-level B-spline bases $\phi_j^i(x,y),\ i=1,\ldots,N;\ j=1,\cdots,n_i$, where N means the number of levels of B-splines, and $n=n_1+\cdots+n_N$ is the total number of the bases. That is

$$g(x,y)=\sum_{i=1}^{N}\sum_{j=1}^{n_i}c_j^i\phi_j^i(x,y),$$

where the coefficients

$$c=(c_1^1,\ldots,c_{n_1}^1,c_1^2,\ldots,c_{n_2}^2,\ldots,c_1^N,\ldots,c_{n_N}^N)^T$$

is the N block vector to be determined. Since all bases $\phi_j^i(x,y)$ are linear dependent, so we want to find a sparse solution among all solutions which minimize the fitting error $\sum_{k=1}^{900}|f(x_k,y_k)-g(x_k,y_k)|^2$. Furthermore, due to the different scaled supports of the multi-level B-spline bases, the different scaled features of $g(x,y)$ can be expressed by the corresponding multiple coefficients c.

We test the following functions, with $N=4$ levels of B-spline bases.

$$\begin{cases}f_1(x,y)=0.75\exp(-(9x-2)^2/4-(9y-2)^2/4)\\ \qquad\qquad +0.75\exp(-(9x+1)^2/49-(9y+1)^2/10)\\ \qquad\qquad +0.5\exp(-(9x-7)^2/4-(9y-3)^2/4)\\ \qquad\qquad -0.2\exp(-(9x-4)^2-(9y-7)^2);\\ f_2(x,y)=(1.25+\cos(5.4y))/(6+6(3x-1)^2);\\ f_3(x,y)=\exp((-81(x-0.5)^2-81(y-0.5)^2)/4)/3.\end{cases}$$

[width=2.5in]F1scattered [width=2.5in]F1POMPth

(a) Scattered Data (b) TP_OMP

[width=2.5in]F1MLASSO [width=2.5in]F1COSAMP

(c) MLASSO (d) CoSaMP

Figure 1 Approximation Surface for $f1$: (a) shows the given scattered data; (b) shows the approximation surface reconstructed by TP_OMP; (c) and (d) show the surface reconstructed by MLASSO and CoSaMP,respectively

[width=2.5in]F2scattered.eps [width=2.5in]F2POMPth

(a) Scattered Data (b) TP_OMP

[width=2.5in]F2MLASSO [width=2.5in]F2COSAMP

(c) MLASSO (d) CoSaMP

Figure 2 Approximation Surface for $f2$: (a) shows the given scattered data; (b) shows the approximation surface reconstructed by TP_OMP; (c) and (d) show the surface reconstructed by MLASSO and CoSaMP,respectively

The difference between $f(x,y)$ and $g(x,y)$ is measured by the normalized root mean square (RMS) error [13]:

$$RMS = \frac{\sqrt{\sum_{k=1}^{M_1}\sum_{l=1}^{M_2}(f(x_k,y_l)-g(x_k,y_l))^2}}{M_1M_2},$$

where $x_k = -1+\frac{2k}{M_1-1}, k=0,1,\cdots,M_1-1,\ y_l=-1+\frac{2l}{M_2-1}, l=0,1,\cdots,M_2-1$ with $M_1=M_2=50$.

Parameters Setting In this part, the parameters in TP_OMP are set as $C_1=50$ and $C_2=1.2$. We use the regularization parameter selection rule in [31] to set $\lambda_1=0.1$, $\lambda_2=0.02$, $\lambda_3=0.01$, and $\lambda_4=0.1$ for the MLASSO model

$$\min_{\mathbf{x}_1,\ldots,\mathbf{x}_4}\sum_{i=1}^{4}\lambda_i\|\mathbf{x}_i\|_1+\frac{1}{2}\|\sum_{i=1}^{4}A_i\mathbf{x}_i-\mathbf{b}\|_2^2.$$

Since the global sparsity s is required before the CoSaMP started(which is the disadvantage of CoSaMP), we set $s=50$ in general for CoSaMP, which is equivalent to the maximum iteration number of TP_OMP.

4 Results and Discussion

Remark 1 *We claim that we do not compare the performance of P_OMP with TP_OMP since the latter is an thresholding version of P_OMP with the merit of less prior information, the numerical performance of TP_OMP is similar to P_OMP.*

[width=2.5in]F3scattered [width=2.5in]F3POMPth [width=2.5in]F3MLASSO [width=2.5in]F3COSAMP

Figure 3 Approximation Surface for $f3$: (a) shows the given scattered data; (b) shows the approximation surface reconstructed by TP_OMP; (c) and (d) show the surface reconstructed by MLASSO and CoSaMP,respectively

Table 1 Numerical Data of Comparison

Function	Measurements	TP_OMP	MLASSO	CoSaMP
	RMS	0.326	0.329	0.342
f_1	Piecewise Sparsity	(5,16,28,0)	(25,15,89,0)	(15,22,13,0)
	Time(sec.)	0.664	20.370	15.623
	RMS	0.107	0.106	0.107
f_2	Piecewise Sparsity	(7,18,33,0)	(25,20,92,0)	(16,23,11,0)
	Time(sec.)	0.693	20.793	15.580
	RMS	0.062	0.062	0.063
f_3	Piecewise Sparsity	(4,9,13,0)	(25,1,16,0)	(1,9,19,21)
	Time(sec.)	0.041	7.581	14.475

Figure 1-3 shows the approximation performance with respect to function f_1, f_2 and f_3. It is observed that the approximation surface recovered by MLASSO misses some details in Figure 1, and the approximation surface recovered by CoSaMP is more sensitive to noise than that of MLASSO and TP_OMP. Both the approximation surfaces recovered by MLASSO and CoSaMP fluctuate due to the noise in Figure 2. The peak of MLASSO in Figure 3 can not reach the original peak. For three examples, the approximation surface recovered by TP_OMP approximate the scattered data well.

It is observed from Tab.1 that TP_OMP's superiority in terms of running time. We use piecewise sparsity in order to show the piecewise sparse structure recovery results by different algorithms. Smaller but not all zeros piecewise sparsity implies better recovery quality. Due to the highly dependency on λ_i, MLASSO always fail to result good piecewise sparsity. Experimental results demonstrate that the TP_OMP algorithm exhibits competitive performance in the surface reconstruction, while with fast running time and better piecewise sparsity.

5 Conclusion

In this paper, we exploit the piecewise structure of signal in sparse recovery problem, based on which we propose a threshold greedy algorithm motivated by the P_OMP algorithm. The proposed thresholding algorithm enjoys the merits of less required prior information and less computational cost. We then present numerical experiments in application to reconstructing a surface with scattered data in order to demonstrate the efficiency of our algorithm.

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Abbreviations
OMP:Orthogonal matching pursuit;P_OMP:Piecewise orthogonal matching pursuit; TP_OMP:threshold piecewise orthogonal matching pursuit; PSI:principal shift invariant; CoSaMP:Compressive sampling matching pursuit; GOMP:Generalized orthogonal matching pursuit; MLASSO: Multilevel least absolute shrinkage and selection operator; RMS:Root mean square

Availability of data and materials
Please contact the author for data requests.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Zhong and Li conceived of the algorithm and designed the experiments. All authors read and approved the manuscript.

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Figures

[width=2.5in]F1scattered [width=2.5in]F1POMPth

(a) Scattered Data (b) TP_OMP

[width=2.5in]F1MLASSO [width=2.5in]F1COSAMP

(c) MLASSO (d) CoSaMP

Figure 1

Approximation Surface for f 1: (a) shows the given scattered data; (b) shows the approximation surface reconstructed by TP OMP; (c) and (d) show the surface reconstructed by MLASSO and CoSaMP, respectively

[width=2.5in]F2scattered.eps [width=2.5in]F2POMPth

(a) Scattered Data (b) TP_OMP

[width=2.5in]F2MLASSO [width=2.5in]F2COSAMP

(c) MLASSO (d) CoSaMP

Figure 2

Approximation Surface for f 2: (a) shows the given scattered data; (b) shows the approximation surface reconstructed by TP OMP; (c) and (d) show the surface reconstructed by MLASSO and CoSaMP, respectively

[width=2.5in]F3scattered [width=2.5in]F3POMPth [width=2.5in]F3MLASSO [width=2.5in]F3COSAMP

Figure 3

Approximation Surface for f 3: (a) shows the given scattered data; (b) shows the approximation surface reconstructed by TP OMP; (c) and (d) show the surface reconstructed by MLASSO and CoSaMP, respectively