Partial-limit solutions and rational solutions with parameter for the Fokas-Lenells equation

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September 2, 2021

Abstract

A partial-limit procedure is applied to soliton solutions of the Fokas-Lenells equation. Multiple-pole solutions related to real repeated eigenvalues are obtained. For the envelop $|u|^2$, the simplest solution corresponds to a real double eigenvalue, showing a solitary wave with algebraic decay. Two such solitons allow elastic scattering but asymptotically with zero phase shift. Single eigenvalue with higher multiplicity gives rise to rational solutions which contain an intrinsic parameter, live on a zero background, and have slowly-changing amplitudes.

Key Words: Fokas-Lenells equation, partial-limit, rational solution, real eigenvalue

1 Introduction

The well-known Kaup-Newell (KN) spectral problem reads [1, 2]

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} \frac{i}{2}\lambda^2 & \lambda q \\ \lambda r & -\frac{i}{2}\lambda^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1}$$

where λ is a spectral parameter (eigenvalue), $i^2 = -1$, and q, r are functions of $(x, t) \in \mathbb{R}^2$. The potential equation of the first member in the negative KN hierarchy is

$$u_{xt} + u - 2iuvu_x = 0, \quad v_{xt} + v + 2iuvv_x = 0, \tag{2}$$

which is also known as the Mikhailov model [3, 4], where $q = u_x, r = v_x$. With reduction $v = \delta u^*$, where $*$ stands for complex conjugate and $\delta = \pm 1$, it follows that

$$u_{xt} + u - 2i\delta |u|^2 u_x = 0, \quad \delta = \pm 1. \tag{3}$$

This equation has been used in 1970s to generate solutions to the massive Thirring model (in light-cone coordinates) [5–7]

$$\chi_{1,x} + i\chi_2 + i|\chi_2|^2\chi_1 = 0, \quad \chi_{2,t} + i\chi_1 + i|\chi_1|^2\chi_2 = 0, \tag{4}$$

by the transformation [1, 3, 8]

$$\chi_2 = u_x^* e^{-i\beta}, \quad \chi_1 = -iu^* e^{-i\beta}, \quad \beta = \int_{-\infty}^{x} |u_x|^2 \mathrm{d}y. \tag{5}$$

Later, the equation (3) (up to some transformations, cf.[9]) was derived by Fokas as a generalization of the nonlinear Schrödinger equation using bi-Hamiltonian structures of the Ablowitz-Kaup-Newell-Segur system[10], and was also derived by Lenells as a model of propagation of

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nonlinear pulses in monomode optical fibers [9]. Now the equation is known as the Fokas-Lenells (FL) equation. Several classical methods have been devoted to seeking solutions for the FL equation, e.g.[11–19]. It is notable that in the analytic approaches, such as the inverse scattering transform and Riemann-Hilbert method [11, 17], soliton solutions to the equation (3) were obtained by assuming the discrete eigenvalues $\{\lambda_j\}$ are distinct and do not locate on the coordinate axes of the complex plane. In a recent paper [20], the FL equation (3) was solved via bilinear approach and solutions were given in terms of double Wronskians. In [20], apart from solitons, solutions related to real eigenvalues were found for the FL equation (3) with $\delta = 1$. The simplest case yields a periodic solution, and it is notable that the solution related to two distinct real double eigenvalues yields two nonsingular solitary waves with algebraic decay. Such a solution of the FL equation was not reported before paper [20].

In the present paper, to understand more about the solutions related to real eigenvalues of the FL equation (3) with $\delta = 1$, we will apply a partial-limit procedure to the solitons of the FL equation and in this way we can recover those solutions related to real repeated eigenvalues. This will enable us to have a better understanding on the scattering of the solution related to two distinct real double eigenvalues which has been obtained in [20]. In addition, a new result we will get is rational solutions (in terms of the envelop $|u|^2$) for the FL equation. Different from the rogue waves of the FL equation [13], our rational solution contains an intrinsic real parameter and lives on a zero background.

Conventionally, multiple-pole solutions mean those solutions related to the eigenvalues as multiple poles of the transmission coefficients (compared with the simple poles, which generate solitons). Therefore, in principle, multiple-pole solutions can be understood as a result of some limits taking from solitons. Early examples of multiple-pole solutions can be referred to the double-pole solutions of the sine-Gordon equation [21] and the nonlinear Schrödinger equation [22]. It is more convenient to implement a limit procedure on a Wronskian [23] due to its special structure. In many cases, lower triangular Toeplitz matrices play useful roles in expressing multiple-pole solutions [24–27]. The partial-limit procedure we will show in the paper can also be extended to other integrable equations.

The paper is organized as the following. In Sec.2 we will briefly recall the formulae of the solutions presented in [20]. Then, in Sec.3, first we will implement a partial-limit procedure to the one-soliton solution of the FL equation and get a new rational solution that is a solitary wave with algebraic decay. Next, we will present general formulae of double Wronskians for the solutions related to real eigenvalues with higher multiplicity and investigate their scattering property. Finally, conclusions are given in Sec.4.

2 Double Wronskian solutions of the FL equation

In [20], coupled system (2) was solved by using bilinear approach and solutions were given in terms of double Wronskians. By means of a reduction technique, solutions in double Wronskian form for the FL equation (3) were obtained. In this section, we will sketch some notations and main formulae obtained in [20] for the solutions of the FL equation.

Bilinear form of the FL equation (3) is

$$D_x D_t\ g \cdot f + gf = 0, \tag{6a}$$
$$D_x D_t\ f \cdot f^* + i\delta D_x\ g \cdot g^* = 0, \tag{6b}$$
$$D_t\ f \cdot f^* + i\delta g g^* = 0, \tag{6c}$$

where

$$u = \frac{g}{f}, \tag{7}$$

and D is Hirota's bilinear operator defined as [28]

$$D_x^m D_t^n\ f \cdot g \equiv (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x'=x,t'=t}. \tag{8}$$

Introduce double Wronskians (cf.[29])

$$|\widetilde{N};\widehat{N-1}| = |\partial_x\phi, \partial_x^2\phi, \cdots, \partial_x^N\phi; \psi, \partial_x\psi, \cdots, \partial_x^{N-1}\psi|,$$
$$|\widehat{N};\widetilde{N-1}| = |\phi, \partial_x\phi, \cdots, \partial_x^N\phi; \partial_x\psi, \partial_x^2\psi, \cdots, \partial_x^{N-1}\psi|,$$

where

$$\phi = (\phi_1, \phi_2, \ldots, \phi_{2N})^T, \quad \psi = (\psi_1, \psi_2, \ldots, \psi_{2N})^T, \tag{9}$$

and ϕ_j and ψ_j are functions of (x, t).

Proposition 1. *[20] The bilinear FL equation (6) admits solutions*

$$f = |\widetilde{N};\widehat{N-1}|, \quad g = |\widehat{N};\widetilde{N-1}|, \tag{10}$$

where

$$\phi = \exp\Big(\frac{i}{2}A^2x + \frac{i}{2}A^{-2}t\Big)C, \quad \psi = S\phi^*, \tag{11}$$

$A, S \in \mathbb{C}_{2N\times 2N}$, $|A| \neq 0$, *and* A *and* S *satisfy*

$$A^2 = \delta SS^*, \quad \delta = \pm 1. \tag{12}$$

The envelop $|u|^2$ *can be expressed as*

$$|u|^2 = i\delta\left(\ln\frac{f}{f^*}\right)_t = 2\delta\left(\arctan\frac{\mathrm{Re}[f]}{\mathrm{Im}[f]}\right)_t. \tag{13}$$

Note that the Wronskians f *and* g *also provide solutions to the massive Thirring model (4) by*

$$\chi_1 = -i\frac{g^*}{f}, \quad \chi_2 = \left(\frac{g^*}{f^*}\right)_x\frac{f^*}{f}. \tag{14}$$

A set of special solutions to the matrix equation (12) are given in terms of

$$S = AT = TA^*, \quad TT^* = \delta\mathbf{I}_{2N}, \tag{15a}$$

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \quad A = \begin{pmatrix} K_1 & \mathbf{0}_N \\ \mathbf{0}_N & K_4 \end{pmatrix}, \tag{15b}$$

where $|\mathbf{I}_N|$ is the identity matrix of order N. For $\delta = 1$, it allows solutions

case	T	A
(1)	$T_1 = T_4 = \mathbf{0}_N, T_2 = T_3 = \mathbf{I}_N$	$K_1 = K_4^* = \mathbf{K}_N \in \mathbb{C}_{N\times N}$
(2)	$T_1 = \pm T_4 = \mathbf{I}_N, T_2 = T_3 = \mathbf{0}_N$	$K_1 = \mathbf{K}_N \in \mathbb{R}_{N\times N}, K_4 = \mathbf{H}_N \in \mathbb{R}_{N\times N}$

Table 1: T and A for (15) with $\delta = 1$

Explicit formula for ϕ corresponding to Table 1 is given by

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix}, \tag{16}$$

where $\phi^{\pm} = (\phi_1^{\pm}, \phi_2^{\pm}, \cdots, \phi_N^{\pm})^T$ take the forms

$$\phi^+ = \exp\Big[\frac{i}{2}(K_1^2x + K_1^{-2}t)\Big]C^+, \quad \phi^- = \exp\Big[\frac{i}{2}(K_4^2x + K_4^{-2}t)\Big]C^-, \tag{17}$$

and $C^{\pm} \in \mathbb{C}_{N\times 1}$. For case (1), when

$$\mathbf{K}_N = \mathrm{D}[k_j]_{j=1}^N \doteq \mathrm{Diag}(k_1, k_2, \cdots, k_N), \quad k_j \in \mathbb{C}, \tag{18}$$

one has

$$\phi^+ = (c_1^+ e^{\eta(k_1)}, c_2^+ e^{\eta(k_2)}, \cdots, c_N^+ e^{\eta(k_N)})^T, \tag{19a}$$

$$\phi^- = (c_1^- e^{\eta(k_1^*)}, c_2^- e^{\eta(k_2^*)}, \cdots, c_N^- e^{\eta(k_N^*)})^T, \tag{19b}$$

where

$$\eta(k) = \frac{i}{2}(k^2 x + k^{-2} t). \tag{20}$$

For case (2), note that $\mathbf{K}_N$ and $\mathbf{H}_N$ are independent real matrices. When

$$\mathbf{K}_N = \mathrm{D}[k_j]_{j=1}^N, \quad k_j \in \mathbb{R},$$

there is

$$\phi^+ = (c_1^+ e^{\eta(k_1)}, c_2^+ e^{\eta(k_2)}, \cdots, c_N^+ e^{\eta(k_N)})^T, \tag{21}$$

where $c_j^+ \in \mathbb{C}$. When

$$\mathbf{K}_N = \mathrm{J}_N[k] \doteq \begin{pmatrix} k & 0 & \cdots & 0 \\ 1 & k & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & k \end{pmatrix}_{N\times N}, \quad k \in \mathbb{R}, \tag{22}$$

ϕ^+ takes the form

$$\phi^+ = \Big(c^+ e^{\eta(k)}, \partial_k (c^+ e^{\eta(k)}), \frac{1}{2!}\partial_k^2 (c^+ e^{\eta(k)}), \cdots, \frac{1}{(N-1)!}\partial_k^{N-1}(c^+ e^{\eta(k)})\Big)^T, \tag{23}$$

where $c \in \mathbb{C}$. When $\mathbf{H}_N = \mathrm{D}[h_j]_{j=1}^N$, $h_j \in \mathbb{R}$, ϕ^- takes the form of (21) with replacement (k_j, c_j^+) by (h_j, c_j^-). When $\mathbf{H}_N = \mathrm{J}_N[h]$, $h \in \mathbb{R}$, ϕ^- takes the form of (23) with replacement (k, c^+) by (h, c^-).

Since in this paper we will only focus on solitons and those solutions related to real eigenvalues, we skip the case of $\mathbf{K}_N = \mathrm{J}_N[k]$ of case (1) and the general mixed case. For more details about these formulae one can refer to [20].

3 Partial-limit solutions

In this section, first, we will implement a partial-limit procedure on double Wronskians of one-soliton solution of the FL equation (3) with $\delta = 1$. The one-soliton solution is associated to a pair of complex eigenvalues, k and k^*, and to get a nonzero solution, neither the real part nor imaginary part of k can be zero. In the partial-limit procedure, we will naively take the imaginary part of k to approach to zero so that k becomes a real double eigenvalue. It is surprised that the resulted solution is nonzero and nonsingular. Such a partial-limit procedure will be the starting point of this section. It will not only provide a better understanding for the solution that was reported in [20] generated from two distinct real double eigenvalues, but also give rise to new rational solutions with an intrinsic real parameter for the FL equation.

3.1 Partial-limit solution and algebraic one-soliton

The one-soliton solution of the FL equation (3) with $\delta = 1$ is given by (cf.[20])

$$u = \frac{g}{f}, \tag{24}$$

where

$$f = \begin{vmatrix} c_1^+ (e^{\eta(k_1)})_x & k_1 (c_1^- e^{\eta(k_1^*)})^* \\ c_1^- (e^{\eta(k_1^*)})_x & k_1^* (c_1^+ e^{\eta(k_1)})^* \end{vmatrix}, \quad g = \begin{vmatrix} c_1^+ e^{\eta(k_1)} & c_1^+ (e^{\eta(k_1)})_x \\ c_1^- e^{\eta(k_1^*)} & c_1^- (e^{\eta(k_1^*)})_x \end{vmatrix}, \tag{25}$$

where $\eta(k)$ is defined by (20). Explicit formula of u is given by [20]

$$u = \frac{c_1^+ c_1^- (k_1^2 - k_1^{*2})}{|k_1|^2 \left[k_1^* |c_1^-|^2 \, \mathrm{e}^{-i(k_1^2 x + \frac{t}{k_1^2})} - k_1 |c_1^+|^2 \, \mathrm{e}^{-i(k_1^{*2} x + \frac{t}{k_1^{*2}})} \right]}, \tag{26}$$

and the envelop reads

$$|u|^2 = \frac{8a_1^2 b_1^2}{(a_1^2 + b_1^2)^3} \frac{1}{\cosh \left(4a_1 b_1 x - \frac{4a_1 b_1 t}{(a_1^2+b_1^2)^2} + 2 \ln \frac{|c_1^-|}{|c_1^+|} \right) - \frac{a_1^2 - b_1^2}{a_1^2 + b_1^2}}, \tag{27}$$

where $a_1 = \mathrm{Re}[k_1]$, $b_1 = \mathrm{Im}[k_1]$. Note that, obviously, the product $a_1 b_1$ must be NOT zero, otherwise, $|u|$ is null. This means, to generate usual solitons, the eigenvalues $\{k_j\}$ should not locate on coordinate axes.

To implement a partial-limit procedure, we take $c_1^{\pm} = 1$ and expand each element in f and g w.r.t. b_1 at $b_1 = 0$. The resulted f and g can be written as

$$f = b_1 \begin{vmatrix} \frac{ia_1^2}{2} e^{\eta(a_1)} + O(b_1) & a_1 e^{-\eta(a_1)} + O(b_1) \\ \frac{1}{a_1}(2a_1^2 - it + ia_1^4 x) e^{\eta(a_1)} + O(b_1) & -\frac{2}{a_1^2}(ia_1^2 - t + a_1^4 x) e^{-\eta(a_1)} + O(b_1) \end{vmatrix},$$

$$g = b_1 \begin{vmatrix} e^{\eta(a_1)} + O(b_1) & \frac{ia_1^2}{2} e^{\eta(a_1)} + O(b_1) \\ -\frac{2}{a^3}(t - a_1^4 x) e^{\eta(a_1)} + O(b_1) & \frac{1}{a_1}(2a_1^2 - it + ia_1^4 x) e^{\eta(a_1)} + O(b_1) \end{vmatrix}.$$

Then, substituting the above forms into (24) and taking a limit for $b_1 \to 0$, it turns out that

$$u = \frac{\begin{vmatrix} \frac{ia_1^2}{2} e^{\eta(a_1)} & a_1 e^{-\eta(a_1)} \\ \frac{1}{a_1}(2a_1^2 - it + ia_1^4 x) e^{\eta(a_1)} & -\frac{2}{a_1^2}(ia_1^2 - t + a_1^4 x) e^{-\eta(a_1)} \end{vmatrix}}{\begin{vmatrix} e^{\eta(a_1)} & \frac{ia_1^2}{2} e^{\eta(a_1)} \\ -\frac{2}{a_1^3}(t - a^4 x) e^{\eta(a_1)} & \frac{1}{a_1}(2a_1^2 - it + ia_1^4 x) e^{\eta(a_1)} \end{vmatrix}} = \frac{-2a_1 e^{\eta(a_1)}}{a_1^2 - 2it + 2ia_1^4 x}. \tag{28}$$

One can verify that this is a solution to the FL equation (3) with $\delta = 1$. The corresponding envelop is

$$|u|^2 = \frac{4a_1^2}{a_1^4 + 4(t - a_1^4 x)^2}. \tag{29}$$

This is a nonsingular solitary wave as depicted in Fig.1(a). Different from the usual soliton, for example, (27), which, for given t, decays exponentially as $|x| \to +\infty$, here for given t, $|u|^2$ decays algebraically as $|x| \to +\infty$.

By examining column vectors in (28), we find that (28) is equivalently generated by

$$\phi = \begin{pmatrix} e^{\eta(a_1)} \\ \partial_{a_1} e^{\eta(a_1)} \end{pmatrix}, \quad \psi = S\phi^*, \quad S = \begin{pmatrix} a_1 & 0 \\ 1 & a_1 \end{pmatrix}, \quad A = S. \tag{30}$$

This is a new solution to (11) and (12) where

$$A = S = \begin{pmatrix} a_1 & 0 \\ 1 & a_1 \end{pmatrix}, \tag{31}$$

since such A and S are not included in Table 1 in [20]. Note that (29) provides an algebraic one-soliton solution because it does behave like a "soliton" with a constant amplitude $\frac{4}{a_1^2}$ and constant speed $x'(t) = \frac{1}{a_1^4}$. Next, we will explore scattering of two such algebraic solitons.

3.2 Elastic scattering of two algebraic solitons

For convenience we introduce

$$A_j = S_j = \begin{pmatrix} a_j & 0 \\ 1 & a_j \end{pmatrix}, \quad a_j \in \mathbb{R}, \tag{32}$$

and define

$$A = \mathrm{Diag}(A_1, A_2, \cdots, A_N), \quad S = \mathrm{Diag}(\epsilon_1 S_1, \epsilon_2 S_2, \cdots, \epsilon_N S_N), \tag{33}$$

where ϵ_j takes either $+1$ or -1. Obviously, $A = SS^*$, satisfying (12) for $\delta = 1$. For such A, ϕ and ψ defined in (11) can be taken as the following explicit forms

$$\phi = (e^{\eta(a_1)}, \partial_{a_1} e^{\eta(a_1)}, e^{\eta(a_2)}, \partial_{a_2} e^{\eta(a_2)}, \cdots, e^{\eta(a_N)}, \partial_{a_N} e^{\eta(a_N)},)^T, \quad \psi = S\phi^*, \tag{34}$$

where $\eta(a)$ is defined by (20).

Consider an example where

$$A = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & \mathbf{0} \\ \mathbf{0} & -S_2 \end{pmatrix}. \tag{35}$$

Correspondingly,

$$u = \frac{g}{f} = \frac{|\phi, \partial_x \phi, \partial_x^2 \phi; \partial_x \psi|}{|\partial_x \phi, \partial_x^2 \phi; \psi, \partial_x \psi|}. \tag{36}$$

This is completely the same solution as discussed in Sec.4.2.2 in [20]. For the sake of completeness, we present the formula of $|u|^2$ below [20],

$$|u|^2 = \frac{4(a_2^2 - a_1^2)^2 G_2}{F_2}, \tag{37}$$

where

$$G_2 = M_1^2 + M_2^2 + M_3^2 + M_4^2 + 2(M_1M_3 + M_2M_4)\cos\vartheta_1 + 2(M_1M_4 - M_2M_3)\sin\vartheta_1,$$
$$F_2 = N_1^2 + N_2^2 + N_3^2 + N_4^2 + 2N_1N_2\cos 2\vartheta_1 - 2N_3(N_1 - N_2)\sin\vartheta_1 + 2N_4(N_1 + N_2)\cos\vartheta_1,$$

with

$$\vartheta_1 = (a_1^2 - a_2^2)\left(x - \frac{t}{a_1^2 a_2^2}\right),$$
$$M_1 = -2a_1(a_1^2 - a_2^2)(a_2^4 x - t), \quad M_2 = -a_1 a_2^2(3a_1^2 + a_2^2),$$
$$M_3 = 2a_2(a_1^2 - a_2^2)(a_1^4 x - t), \quad M_4 = -a_1^2 a_2(3a_2^2 + a_1^2),$$
$$N_1 = -4a_2^5 a_1^3, \quad N_2 = -4a_2^3 a_1^5, \quad N_3 = -2(a_1^2 - a_2^2)(a_1^4 - a_2^4)(a_2^2 a_1^2 x - t),$$
$$N_4 = -a_2^2 a_1^2(a_2^4 + 6a_2^2 a_1^2 + a_1^4) + 4(a_1^2 - a_2^2)^2(t - a_1^4 x)(t - a_2^4 x).$$

For the scattering property, one can refer to Proposition 3 in [20]. In the coordinate frames $(X_j = x - \frac{t}{a_j^4},\ t)$, asymptotically,

$$|u|^2 \sim \frac{4}{a_j^2(1+4a_j^4X_j^2)}, \quad (t \to \pm\infty), \tag{38}$$

for $j = 1, 2$.

We remark that such $|u|^2$ can be considered as the interaction of two single algebraic solitons that we derived in the previous subsection. In addition, $|u|^2$ allows either attractive interaction (see also Fig.6 in [20]) or repulsive interaction, see Fig.1(b,c), and asymptotically, both cases have no phase shift, although near the interacting point phase shifts can be observed (in other words, the two waves will be eventually back to their original tracks). Such a feather of phase shifts is different from the usual soliton interactions.

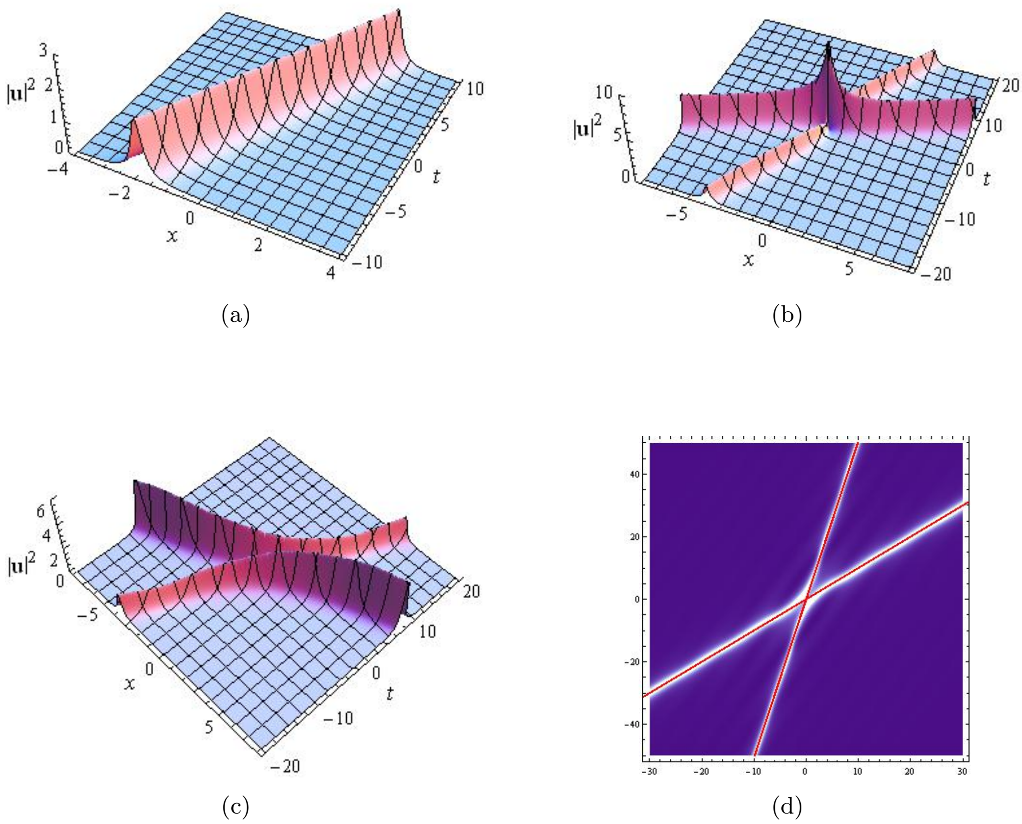

Fig. 1. Algebraic solitons and scattering. (a) $|u|^2$ given by (29) with $a_1 = 1.5$. (b) Attractive interaction of two algebraic solitons: $|u|^2$ given by (37) with $a_1 = 1.5, a_2 = -1$. (c) Repulsive interaction of two algebraic solitons: $|u|^2$ given by (37) with $a_1 = 1.5, a_2 = 1$. (d) Density plot of (c) with larger range $x \in [-30, 30]$, $t \in [-50, 50]$ and overlapped by red asymptotic lines $X_j = x - \frac{t}{a_j^4} = 0$ where $a_1 = 1.5, a_2 = 1$.

We also remark that, when N is odd, (33) will give rise to new solutions, since such a case can not be obtained from the assumption (15).

3.3 Rational solutions and new asymptotic property

3.3.1 Rational solutions

Consider

$$S=\mathrm{J}_{2N}[a],\quad A=S,\quad a\in\mathbb{R},\tag{39}$$

where $\mathrm{J}_{2N}[a]$ is a Jordan matrix defined as in (41). In this case, ϕ and ψ take the forms

$$\phi=(e^{\eta(a)},\partial_a e^{\eta(a)},\frac{1}{2!}\partial_a^2 e^{\eta(a)},\cdots,\frac{1}{(2N-1)!}\partial_a^{2N-1}e^{\eta(a)},)^T,\quad \psi=S\phi^*,\tag{40}$$

where η is defined as in (20). Consequently, $f=|\widetilde{N};\widehat{N-1}|$ composed by the above ϕ and ψ is a pure complex polynomial of (x,t), while $g=|\widehat{N};\widetilde{N-1}|$ is a complex polynomial of (x,t) multiplied by $e^{2\eta(a)}$. As a result, although $u=g/f$ is not a form of rational solution, the carrier wave $|u|^2$ is. In this sense, we say the solution is a rational solution of the FL equation in terms of $|u|^2$. In addition, the structure of $u=g/f$ also indicates that $e^{-2\eta(a)}u$ has a form of rational solution, which implies that one can assume a proper form of u and directly calculate low-order rational solutions (see [30] for the modified Korteweg-de Vries equation).

In such a solution, a is the single parameter and it cannot be zero. We can introduce more parameters by the trick of employing lower triangular Toeplitz matrices, which is defined as the following,

$$\mathcal{T}_{2N}=\begin{pmatrix} s_1 & 0 & 0 & \cdots & 0\\ s_2 & s_1 & 0 & \cdots & 0\\ s_3 & s_2 & s_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ s_{2N} & s_{2N-1} & s_{2N-2} & \cdots & s_1\end{pmatrix}_{2N\times 2N},\quad s_j\in\mathbb{C},\tag{41}$$

where $s_j\in\mathbb{C}$. Lower triangular Toeplitz matrices commute with the same type of matrices and play useful roles in expressing multiple-pole solutions (cf.[24–27]). For the above ϕ given in (40), noting that $A\mathcal{T}_{2N}=\mathcal{T}_{2N}A$, obviously, the double Wronskians $f=|\widetilde{N};\widehat{N-1}|$ and $g=|\widehat{N};\widetilde{N-1}|$ composed by $\phi'=\mathcal{T}_{2N}\phi$ and $\psi'=S\phi'^*$ are still solutions to the bilinear equations (6). In this case, $\{s_j\}$ will appear in the solution u as parameters. However, a is special as it is the parameter inheriting from the eigenvalue of the KN spectral problem. We call a is an intrinsic parameter, compared with those $\{s_j\}$ in $\mathcal{T}_{2N}$.

3.3.2 New asymptotic property

Let us consider solution

$$u=\frac{g}{f}=\frac{|\phi,\partial_x\phi,\partial_x^2\phi;\partial_x\psi|}{|\partial_x\phi,\partial_x^2\phi;\psi,\partial_x\psi|},\tag{42}$$

where

$$\phi=(e^{\eta(a)},\partial_a e^{\eta(a)},\frac{1}{2!}\partial_a^2 e^{\eta(a)},\frac{1}{3!}\partial_a^3 e^{\eta(a)})^T,\quad \psi=S\phi^*,\quad S=\mathrm{J}_4[a],\quad A=S.\tag{43}$$

The envelop $|u|^2$ reads

$$|u|^2=\frac{G}{F},\tag{44a}$$

where

$$G=16a^2[9a^{12}+64X^6+36a^8(4t-3X)(4t-5X)+48a^4X^2(4t-3X)(12t-7X)],\tag{44b}$$

$$F=9a^{16}+256X^8-256a^4X^4(24t^2-24tX-X^2)+144a^{12}(8t^2-8tX+5X^2)$$
$$+288a^8(128t^4-256t^3X+160t^2X^2-32tX^3+7X^4),\tag{44c}$$

and

$$X = t - a^4 x. \tag{45}$$

The solution is depicted in Fig.2(a), which exhibits an interaction of two curved waves with different amplitudes.

In the following, by means of asymptotic analysis we will explore an unusual asymptotic property of such a rational solution.

For convenience, we equivalently consider (44) in the coordinate frame (X, t), as depicted in Fig.2(c). Note that (45) is an linear transformation which does not change asymptotic property of $|u|^2$. Then, we rewrite $|u|^2$ in the coordinate frame

$$\left(X, T_1 = t - \Big(\frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} + \frac{\sqrt{3}}{4}a^3\Big)\right), \tag{46}$$

and we have

$$|u|^2 = \frac{G'}{F'}, \tag{47a}$$

where

$$\begin{aligned} G' &= 16[117a^{12} + 144\sqrt{3}a^{10}(2T_1 - X) + 96\sqrt{3}a^6X^2(14T_1 - 3X) + 128\sqrt{3}a^2X^4(6T_1 - X) \\ &\quad + 256X^6 + 96a^4X^2(24T_1^2 - 8T_1X + 7X^2) + 36a^8(16T_1^2 - 16T_1X + 19X^2)], \end{aligned} \tag{47b}$$

$$\begin{aligned} F' &= a^2[1521a^{12} + 7488\sqrt{3}a^{10}T_1 + 3072\sqrt{3}a^2T_1X^2(8T_1^2 + X^2) + 144a^8(296T_1^2 + 17X^2) \\ &\quad + 384\sqrt{3}a^6T_1(96T_1^2 + 25X^2) + 192a^4(192T_1^4 + 240T_1^2X^2 + 11X^4) + 1024X^4(12T_1^2 + X^2)]. \end{aligned} \tag{47c}$$

The curve $T_1 = 0$ is a parabola $t = t(X)$ opening upward, on which we have

$$|u|^2 = \frac{16(117a^{12} - 144\sqrt{3}a^{10}X + 684a^8X^2 - 288\sqrt{3}a^6X^3 + 672a^4X^4 - 128\sqrt{3}a^2X^5 + 256X^6)}{a^2(1521a^{12} + 2448a^8X^2 + 2112a^4X^4 + 1024X^6)}, \tag{48}$$

which is not a constant. Roughly speaking, when $|X| >> 0$, from Fig.3(a) one can see that the value of $|u|^2$ slowly increases w.r.t. X when $X < 0$ and also slowly increases w.r.t. X when $X > 0$. On the other hand, consider $|u|^2$ (44) in the coordinate frame

$$\left(X, T_2 = t - \Big(-\frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} - \frac{\sqrt{3}}{4}a^3\Big)\right). \tag{49}$$

We find that, on the curve $T_2 = 0$, $|u|^2$ reads

$$|u|^2 = \frac{16(117a^{12} + 144\sqrt{3}a^{10}X + 684a^8X^2 + 288\sqrt{3}a^6X^3 + 672a^4X^4 + 128\sqrt{3}a^2X^5 + 256X^6)}{a^2(1521a^{12} + 2448a^8X^2 + 2112a^4X^4 + 1024X^6)}, \tag{50}$$

which, when $|X| >> 0$, slowly decreases w.r.t. X when $X < 0$ and slowly decreases w.r.t. X when $X > 0$. Eventually, in both coordinate frames (46) and (49), we have

$$|u|^2 \sim \frac{4}{a^2}, \quad (|X| \to +\infty).$$

For the waves depicted in Fig.2(c), Fig.3(d) compares their amplitudes at different time.

Let us summarize the above analysis.

Theorem 1. *Consider the envelop $|u|^2$ given in (44) in the coordinate frame (X, t). Asymptotically, the waves travel along the curves*

$$t = \frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} + \frac{\sqrt{3}}{4}a^3 \tag{51}$$

and

$$t = -\frac{\sqrt{3}X^2}{6a^2} + \frac{X}{2} - \frac{\sqrt{3}}{4}a^3. \tag{52}$$

The amplitudes of the waves are not constants but slowly changing and finally they approach to $\frac{4}{a^2}$ *when* $|X| \to +\infty$.

Theorem 2. *In the coordinate frame* (x,t)*, for the envelop* $|u|^2$ *given in* (44)*, when* $|x|$ *is large enough, the two waves travel respectively along the curves (see Fig.2(b))*

$$t = a^4 x + \frac{\sqrt{3}}{2}a^2 \pm a^2\sqrt{8\sqrt{3}a^2 x - 3}, \quad (x >> 0) \tag{53}$$

and

$$t = a^4 x - \frac{\sqrt{3}}{2}a^2 \pm a^2\sqrt{-8\sqrt{3}a^2 x - 3}, \quad (x << 0). \tag{54}$$

The amplitudes of the waves are not constants but slowly changing and finally they approach to $\frac{4}{a^2}$ *when* $|t| \to +\infty$.

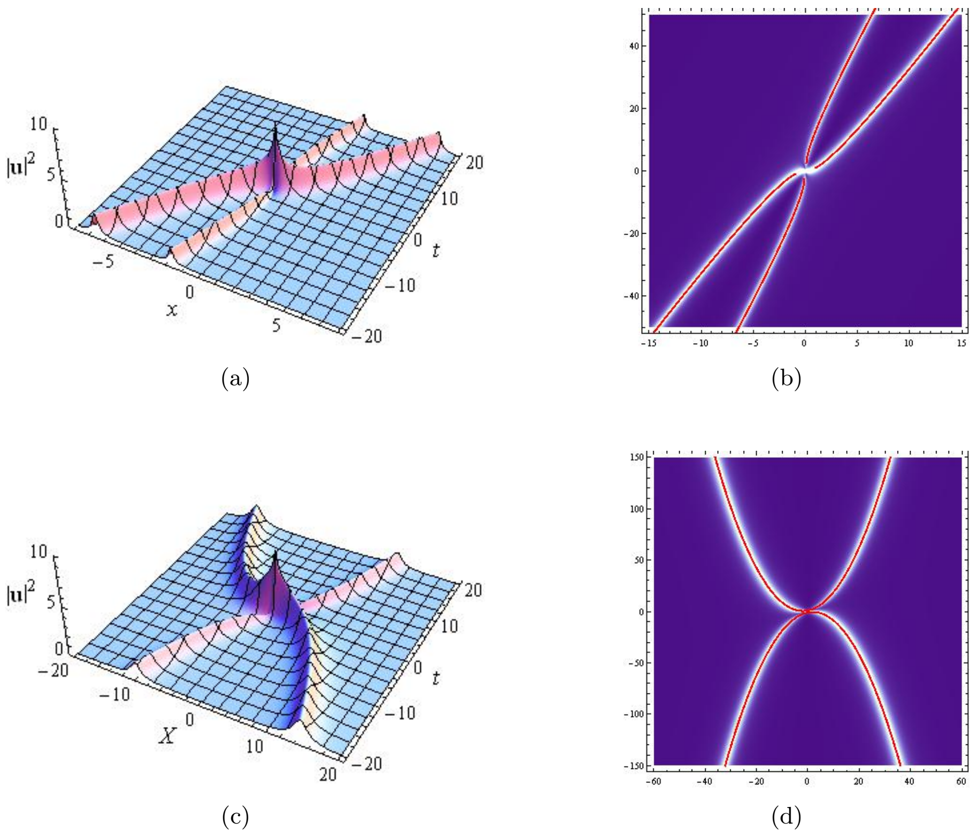

Fig. 2. Rational solutions and asymptotic property. (a) $|u|^2$ given by (44) in coordinates (x,t) with $a = 1.5$. (b) Density plot of (a) with larger range $x \in [-15, 15]$, $t \in [-50, 50]$ and overlapped with red asymptotic curves (53) and (54) where $a = 1.5$. (c) $|u|^2$ given by (44) in coordinates (X,t) with $a = 1.5$. (d) Density plot of (c) with larger range $X \in [-60, 60]$, $t \in [-150, 150]$ and overlapped with red asymptotic curves (51) and (52) where $a = 1.5$.

Note that in the rational solution the two waves asymptotically travel along parabolas, which is different from the asymptotic tracks of double-pole solutions of solitons which usually governed by logarithmic functions (cf.[22, 27, 31, 32].

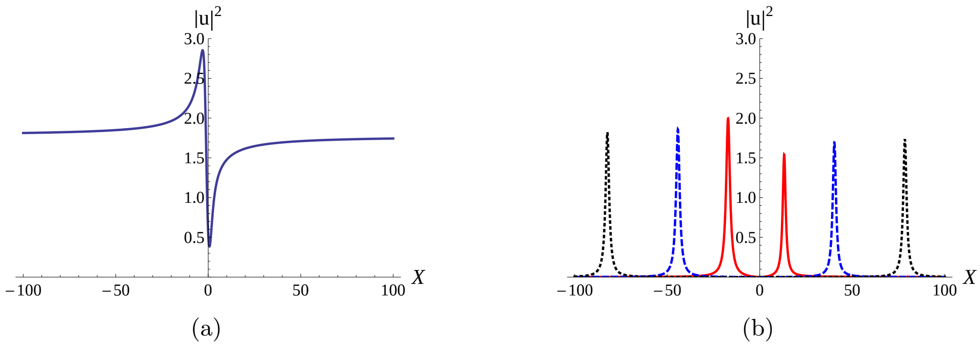

Fig. 3. Slowly-changing amplitudes of rational solution (47). (a) Plot of $|u|^2$ on $T_1 = 0$, given by (48) with $a = 1.5$. (b) Amplitudes changing of $|u|^2$, w.r.t. time t, given by (44) in coordinates (X, t): red curve, blue dashed curve and black dotted curve stand for the shapes of $|u|^2$ at $t = 30,\ t = 230$ and $t = 830$, respectively.

4 Conclusions

In this paper, by implementing a partial-limit procedure we investigated solutions related to real repeated eigenvalues for the FL equation (3) with $\delta = 1$. The simplest solution is generated by a real double eigenvalue of the KN spectral problem and the envelop $|u|^2$ of the solution is a rational solution and provides a solitary wave with algebraic decay. Interaction of two such waves, described by the solution generated by two distinct real double eigenvalues, coincides with the solution given in §4.2.2 in [20]. Thus we have given a better understanding for such a solution. Note that when N is odd, the solution generated from N distinct real double eigenvalues are new since this case is not based on the assumption (15) and not included in Table 1. In addition, as new solutions we derived rational solutions (in terms of $|u|^2$) for the FL equation (3) with $\delta = 1$. Note also that in [20] solutions related to real simple eigenvalues were obtained, but such solutions cannot be obtained by taking partial-limit.

Such type of solutions (related to real repeated eigenvalues) exhibit new asymptotic properties. For example, for the solution (36) with envelop (37), which is related to two distinct real double eigenvalues a_1 and a_2, as demonstrated in [20] the two waves do have apparent phase shifts near the interaction point, but for large time t (i.e. asymptotically), they travel along two straight lines without any phase shifts. This is different from the interaction of usual solitons. Another new feature is the asymptotic property of rational solutions. As we have analyzed and illustrated in Sec.3.3, the envelop $|u|^2$ contains two curved waves with different amplitudes that are slowly changing and finally approach to a same value as $|t| \to +\infty$.

For the FL equation (3) with $\delta = -1$, there are solutions related to pure imaginary eigenvalues and the partial limit should be implemented on $k = a + ib$ by taking $a \to 0$ in instead of $b \to 0$. The vector ϕ will be composed by $e^{\eta^*(b)}$ and $\frac{1}{j!}\partial_b^j e^{\eta^*(b)}$ where b is real and $\eta(b)$ is defined by (20). Corresponding to (31), in the simplest case, one has $A = S = i\begin{pmatrix} b & 0 \\ 1 & b \end{pmatrix}$. However, these solutions can be obtained as complex conjugates of those solutions given in Sec.3. In fact, if u is a solution of the FL equation (3) with $\delta = 1$, then u^* solves the FL equation with $\delta = -1$.

In addition, mixed solutions can be obtained from a more general form of matrix A, e.g., a combined matrix $A = \mathrm{Diag}(A_1, \cdots, A_s, \mathrm{J}_{N-2s}[a_{s+1}])$, where A_j is defined in (32). Using formulae (5) and (14) one can get new solutions to the massive Thirring model (4) from our solutions to the FL equation. The rational solution can also be obtained using a special Wronskian technique (cf.[33, 34]). The partial-limit procedure should be available as well to Hirota's form of soliton solutions (cf.[35, 36]) and may be extended to other integrable equations.

Acknowledgments

The author is grateful to Prof. Da-jun Zhang for discussion. This project is supported by the NSF of China (Nos.12171308 and 11875040).

Declarations

Funding: This project is supported by the NSF of China (Nos.12171308 and 11875040).

Conflicts of interest/Competing interests: The author declares that there is no conflict of interests regarding the publication of this paper.

Availability of data and material: The manuscript has no associated data.

Code availability: Not applicable.

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