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Generalized Exponential Distribution

Estimation of parameters using modifications in methods of Ranked Set Sampling

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Abstract In this study, we estimate the parameters of the Generalized Exponential Distribution using Moving Extreme Ranked Set Sampling (MERSS). Using the maximum likelihood estimation method, we derive the expressions. MERSS estimates are compared with estimates obtained by simple random sampling (SRS) using a real data set. We also study the other variations of the methods of Ranked Set Sampling like Quartile Ranked Set Sampling(QRSS), Median Ranked Set Sampling(MRSS) and Flexible Ranked Set Sampling(FLERSS) (a scheme based on QRSS and MRSS). For known shape parameter values, we present coefficients for linear combinations of order statistics for least squares estimates. Here, the expressions are derived through maximum likelihood, and the estimates are calculated numerically. Simulated results indicate that estimates generated using least-squares and the maximum likelihood method for Ranked Set Sampling (RSS) perform better than those generated using Simple Random Sampling (SRS). Asymptotically, MERSS outperforms SRS, QRSS, MRSS, and FLERSS.

Keywords ranked set sampling · moving extremes ranked set sampling · maximum likelihood estimator · Fisher information number · Mean Square Error

Mathematics Subject Classification (2020) 62F07 · 62F10 · 62D05

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1 Introduction

Consider a continuous random variable X follows three parameter Generalized Exponential distribution (GE(α,μ,σ))

$$G(x;\alpha,\mu,\sigma) = (1-e^{-(x-\mu)/\sigma})^{\alpha}, \quad \alpha,\sigma,x>\mu>0 \tag{1}$$

The probability density function is

$$g(x;\alpha,\mu,\sigma) = \frac{\alpha}{\sigma}(1-e^{-(x-\mu)/\sigma})^{\alpha-1}e^{-(x-\mu)/\sigma} \quad \alpha,\sigma,x>\mu>0 \tag{2}$$

Where α is the shape parameter,μ is the location parameter and σ is the scale parameter. Substituting $\mu=0$ and $\beta=1/\sigma$ in (1) and (2) we have

$$G(x;\alpha,\beta) = (1-e^{-\beta x})^{\alpha}, \quad \alpha,\beta,x>0 \tag{3}$$

$$g(x;\alpha,\beta) = \alpha\beta(1-e^{-\beta x})^{\alpha-1}e^{-\beta x} \quad \alpha,\beta,x>0 \tag{4}$$

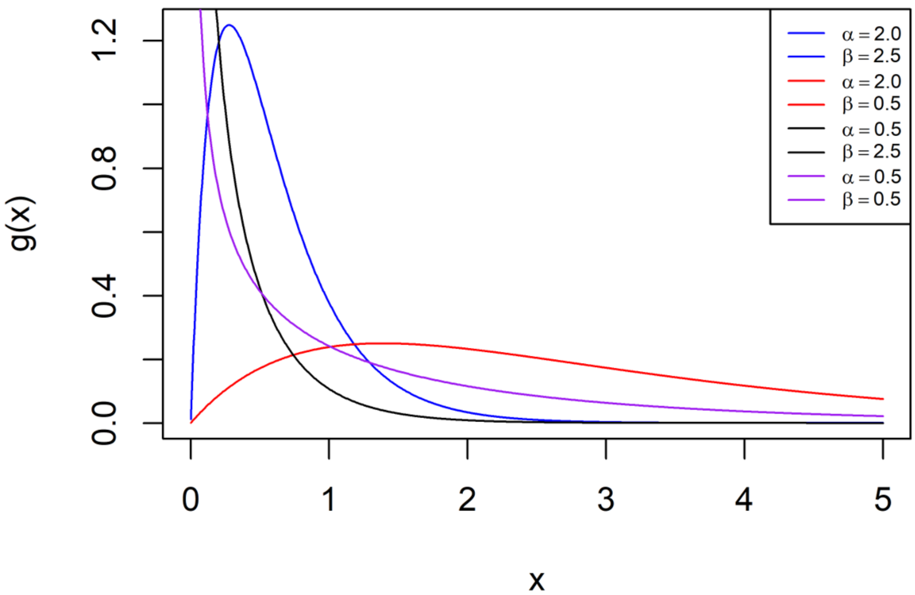

Fig. 1: α shape parameter, β scale parameter, $\mu=0$ location parameter

As seen in (3), the distribution is of the type $[F(\cdot)]^{[}/alpha]$ where $F(\cdot)$ is an absolutely continuous probability distribution function. The distribution (3) is introduced and studied in detail by Gupta and Kundu (1999). According to (Gupta and Kundu (2001)) when fitting positive life time data, the generalized exponential distribution is used as an alternative to the two parameter Weibull distribution and two parameter gamma distribution. Mean and variance of the distribution with density function given in (4) are

$$E(X) = \frac{1}{\beta}(\psi(\alpha+1) - \psi(1)) \tag{5}$$

$$V(X) = -\frac{1}{\beta^2}(\psi'(\alpha+1) - \psi'(1)) \tag{6}$$

Where $\psi(\cdot)$ is the digamma function and $\psi'(\cdot)$ is the derivative of $\psi(\cdot)$. The skewness and kurtosis both are independent of the scale parameter and they are decreasing function of the shape parameter α.(Gupta and Kundu (2007))

The generalized exponential has increasing or decreasing hazard rate based on shape parameter α. If $\alpha = 1$, the distribution in (1) is two parameter exponetial distribution function. The survival function and hazard function are given by

$$S(x;\alpha,\mu,\sigma) = 1 - (1 - e^{(x-\mu)/\sigma})^{\alpha}, \quad x > \mu \tag{7}$$

$$h(x;\alpha,\mu,\sigma) = \frac{g(x;\alpha,\mu,\sigma)}{S(x;\alpha,\mu,\sigma)} = \frac{\alpha}{\sigma}\frac{(1-e^{-(x-\mu)/\sigma})^{\alpha-1}e^{-(x-\mu)/\sigma}}{1-(1-e^{(x-\mu)/\sigma})^{\alpha}}, \quad x > \mu \tag{8}$$

Khan (1987) proposed the generalized exponential distribution and showed it to be a good fit to the survival data of black-headed gulls.

In reliability study, the system reliability is defined by a probability $R = P(Y < X)$, where random variable X refers to the strength of a component and random variable Y refers to the stress on that component. Additionally, Kundu and Gupta (2005) examined the estimation of R when X and Y are independent and distributed as generalized exponential distributions with common scale parameter and different shape parameters. Raqab et al. (2008) found the same solution using a three-parameter generalized exponential distribution. Using modified signed log-likelihood ratio statistics, Hajebi et al. (2012) have studied reliability intervals for R.

Chen and Lio (2010) studied the estimation of parameters of a generalized exponential distribution under progressive type-I interval censoring and derived estimates from real patient data.

Kundu and Gupta (2011) have discussed bivariate generalized exponential distributed whose marginal distributions are generalized exponential distributions and show it is a better fit than a bivariate exponential distribution.

Raqab Mohammad and Ahsanullah (2001) investigated the estimation of location and scale parameters using order statistics for a generalized exponential distribution (1).

2 Ranked Set Sampling

Ranked set sampling (RSS) is a method that is cost-effective when the measurement of the sampling unit is expensive but would facilitate ranking. (see McIntyre (1952),Zehua Chen et al. (2004))

A ranked set sample is obtained by randomly selecting m^2 units from an infinite population. This units are then partitioned randomly into m equal samples of m units each. The units in the sample are ranked by judgement or visual inspection or by cheap (law cost) way, or by using auxiliary variables but without actual measurements. The unit with lowest rank is measured with respect to variable X of interest and remaining units are discarded; a second sample of m units are ranked without actual measurements. As a result, the second lowest unit in this set is measured with respect to variable X of interest, while the rest are discarded. Once the largest unit in the last sample of size m has been measured, the entire procedure is repeated h times, yielding $n = mh$ measured units from m^2h selected units. Compared to simple random sample, this ranked set sample also represents the entire population and is spread throughout the population. As the case in general with the RSS method, where the sampling units are selected without replacement, the estimation of population mean using RSS, and the relative precision of RSS estimate by comparison with SRS estimate, are discussed in Patil et al. (1995).

In Tahmasebi and Jafari (2014) the authors discuss estimation of the mean of a bivariate generalized exponential distribution using ranked set sampling.

Samuh and Qtait (2015) has examined the estimation of parameters in an Exponentiated exponential distribution based on Median ranked set sampling(MRSS) and found that MRSS provides more efficient estimates than SRS.

There are different modifications of the methods of RSS avialble in literature. Al-Odat and Al-Saleh (2001) introduced the concept of varied size RSS, which is known as Moving Extreme Ranked Set Sampling.

Eftekharian and Razmkhah (2016) discussed the unified ranked set sampling scheme to estimate the population mean and Eftekharian et al. (2021) discussed the flexible ranked set sampling scheme to estimate scale parameters of the exponential and normal distributions.

Our discussion assumes the sampling units are ranked without error for the characteristic of interest.In section 5 we discuss the estimation of shape and scale parameters of distribution given in (3) using MERSS method. In section 6 we discuss the unified ranked set sampling method to estimate shape and scale parameters of distribution given in (3). We consider the estimation of shape, location and scale parameters using the RSS method in section 7. The steps for numerical computations are presented in section 9. Section 10 presents our findings.

3 Quartile Ranked Set Sampling

According to Muttlak (2003), we note the Quartile Ranked Set Sampling as:
If the sample size is even,

Step 1. Select for measurement from the first $m/2$ samples the $[q_1(m+1)]$th (the nearest integer) smallest rank. Where $q_1 = 0.25$
Step 2. Select for measurement from the second $m/2$ samples the $(q_u(m+1))$th (the nearest integer) smallest rank. Where $q_u = 0.75$

If the sample size is odd,

Step 1. Select for measurement from the first $(m-1)/2$ samples the $[q_1(m+1)]$th smallest rank. Where $q_1 = 0.25$
Step 2. Select for measurement from the second $(m-1)/2$ samples the $[q_u(m+1)]$th smallest rank. Where $q_u = 0.75$
Step 3. From one sample the median for that sample for actual measurement.

4 Median Ranked Set Sampling

As mentioned by Muttlak (1997) we consider,

Step 1. If the sample size m is odd, from each sample select for measurement the $((m+1)/2)$th smallest rank (the median of the sample)
Step 2. If the sample size m is even, select for measurement from the first $m/2$ samples the $(m/2)$th smallest rank and from the second $m/2$ samples the $((m+2)/2)$th smallest rank.

5 Moving Extreme Ranked Set Sampling

Al-Odat and Al-Saleh (2001) has discussed moving extreme ranked set (MERSS) sampling to reduce the error of ranking . The procedure of MERSS is

Step 1. Select m simple random samples of size $1, 2, 3, \ldots, m$ respectively.
Step 2. Order the sampling units of each of the samples by eye or by some other relatively inexpensive method, without actual measurement.
Step 3. Measure accurately the maximum order observation from the first set, the maximum ordered observation from the second set. The process continues in this way until the maximum ordered observation from the last mth sample is measured.
Step 4. Repeat the Step 1. to Step 3. again, and measure the minimum ordered observation instead maximum ordered observation.
Step 5. The prcedure above is one cycle.The entire cycle can be repeated h times to obtain a MERSS of size $n = 2mh$

Let $Y_{ji}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, h$ denotes a moving extreme ranked set sample with cumulative distribution function given in (3) and probability density function given in 2. Where h is number of cyclels and m is the set size.

$$f_{Y_{ji}}(y) = \begin{cases} f_{X^j_{i(i:i)}}(y) & \text{Maximum} \\ f_{X^j_{i(1:i)}}(y) & \text{Minimum} \end{cases} \tag{9}$$

Where $X^j_{i(i:i)} = \{X_{j1(1:1)}, \ldots, X_{jm(m:m)}\}$ and $X^j_{i(1:i)} = \{X_{j1(1:1)}, \ldots, X_{jm(1:m)}\}$

$$\begin{aligned} f_{X^j_{i(i:i)}}(y) &= i\,[G(y)]^{i-1}\,g(y) \\ &= i\left[(1-e^{-\beta y})^{\alpha}\right]^{i-1}\alpha\beta\left(1-e^{-\beta y}\right)^{\alpha-1}e^{-\beta y} \\ &= \alpha\beta i\left(1-e^{-\beta y}\right)^{i\alpha-1}e^{-\beta y} \end{aligned} \tag{10}$$

and

$$\begin{aligned} f_{X^j_{i(1:i)}}(y) &= i\,[1-G(y)]^{i-1}\,g(y) \\ &= i\left[1-(1-e^{-\beta y})^{\alpha}\right]^{i-1}\alpha\beta\left(1-e^{-\beta y}\right)^{\alpha-1}e^{-\beta y} \\ &= \alpha\beta i\left(1-e^{-\beta y}\right)^{\alpha-1}\left[1-(1-e^{-\beta y})^{\alpha}\right]^{i-1}e^{-\beta y} \end{aligned} \tag{11}$$

5.1 Likelihood Function

Let u_{ji} represents the observed values of $X^j_{i(i:i)}$ and v_{ji} represents the observed values of $X^j_{i(1:i)}$, then

$$\begin{aligned} L_{MERSS,\text{minimum}} &= \prod_{j=1}^{h}\prod_{i=1}^{m} f_{Y_{ji}}(v_{ji}) \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m} \alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left[1-(1-e^{-\beta v_{ji}})^{\alpha}\right]^{i-1}e^{-\beta v_{ji}} \\ L_{MERSS,\text{maximum}} &= \prod_{j=1}^{h}\prod_{i=1}^{m} f_{Y_{ji}}(u_{ji}) \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m} \alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1}e^{-\beta u_{ji}} \end{aligned}$$

Define, for $i = 1,2,\dots,m, \quad j = 1,2,\dots,h$

$$a_{ji} = \begin{cases} 1, & \text{for} \quad \text{observed minimum} \\ 0, & \text{for} \quad \text{observed maximum} \end{cases}$$

$$b_{ji} = \begin{cases} 0, & \text{for} \quad \text{observed minimum} \\ 1, & \text{for} \quad \text{observed maximum} \end{cases}$$

Therefore,

$$\begin{aligned} L_{MERSS} &= \prod_{j=1}^{h}\prod_{i=1}^{m}\left\{f_{Y_{ji}}(v_{ji})\right\}^{a_{ji}}\left\{f_{Y_{ji}}(u_{ji})\right\}^{b_{ji}} \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m}\left[\alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left\{1-(1-e^{-\beta v_{ji}})^{\alpha}\right\}^{i-1}e^{-\beta v_{ji}}\right]^{a_{ji}} \\ &\qquad \left[\alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1}e^{-\beta u_{ji}}\right]^{b_{ji}} \end{aligned}$$

$$\begin{aligned} \log L_{MERSS} &= \sum_{j=1}^{h}\sum_{i=1}^{m}\left[\log\left\{f_{Y_{ji}}(v_{ji})\right\}^{a_{ji}} + \log\left\{f_{Y_{ji}}(u_{ji})\right\}^{b_{ji}}\right] \\ &= \sum_{j=1}^{h}\sum_{i=1}^{m}\left[a_{ji}\log\left\{\alpha\beta i\left(1-e^{-\beta v_{ji}}\right)^{\alpha-1}\left\{1-(1-e^{-\beta v_{ji}})^{\alpha}\right\}^{i-1}e^{-\beta v_{ji}}\right\}\right. \\ &\qquad \left. + b_{ji}\log\left\{\alpha\beta i\left(1-e^{-\beta u_{ji}}\right)^{i\alpha-1}e^{-\beta u_{ji}}\right\}\right] \\ &= C_6 + \sum_{j=1}^{h}\sum_{i=1}^{m}\left[a_{ji}\left\{\log i + (\alpha-1)\log\left(1-e^{-\beta v_{ji}}\right) + (i-1)\log\left(1-(1-e^{-\beta v_{ji}})^{\alpha}\right) - \beta v_{ji}\right\}\right. \\ &\qquad \left. + b_{ji}\left\{\log i + (i\alpha-1)\log\left(1-e^{-\beta u_{ji}}\right) - \beta u_{ji}\right\}\right] \end{aligned} \tag{12}$$

Where $C_6 = 2mh\{\log\alpha + \log\beta\}$

5.2 Estimates

$$\begin{aligned} \frac{\partial \log L_{MERSS}}{\partial\alpha} = \frac{2mh}{\alpha} + \sum_{j=1}^{h}\sum_{i=1}^{m}\left[a_{ji}\log\left(1-e^{-\beta v_{ji}}\right)\left\{\frac{1-i\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{1-\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}\right\}\right. \\ \left. + b_{ji}\left\{i\log\left(1-e^{-\beta u_{ji}}\right)\right\}\right] \end{aligned} \tag{13}$$

The mle of α which depends upon β is the solution of equation
$\frac{\partial \log L_{MERSS}}{\partial \alpha} = 0$

$$\frac{\partial \log L_{MERSS}}{\partial \beta} = \frac{2mh}{\beta} + \sum_{j=1}^{h}\sum_{i=1}^{m}\left[a_{ji}v_{ji}\left\{ \frac{e^{-\beta v_{ji}}}{\left(1-e^{-\beta v_{ji}}\right)} \frac{(\alpha-1)-(i\alpha-1)\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{\left(1-\left(1-e^{-\beta v_{ji}}\right)^{\alpha}\right)} - 1 \right\} - b_{ji}u_{ji}\left\{ \frac{(i\alpha-1)e^{-\beta u_{ji}}}{(1-e^{-\beta u_{ji}})} - 1 \right\} \right] \tag{14}$$

The mle of β which depends upon α is the solution of equation
$\frac{\partial \log L_{MERSS}}{\partial \beta} = 0$

5.3 The Fisher Information (FI) matrix

We compute numerically the elements of Fisher Information Matrix

$$I_{MERSS}(\alpha,\beta) = -\frac{1}{2mh}\begin{bmatrix} E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) \end{bmatrix} \tag{15}$$

Where,

$$\frac{\partial^2 \log L_{MERSS}}{\partial \alpha^2} = -\frac{mh}{\alpha^2} - \sum_{j=1}^{h}\sum_{i=1}^{m} a_{ji}\left[\frac{(i-1)(1-e^{-\beta v_{ji}})^{\alpha}\log(1-e^{-\beta v_{ji}})}{\left(1-(1-e^{-\beta v_{ji}})\right)^2} \right] \tag{16}$$

$$\frac{\partial^2 \log L_{MERSS}}{\partial \beta^2} = -\frac{mh}{\beta^2} + \sum_{j=1}^{h}\sum_{i=1}^{m}\left[a_{ji}\left\{ \frac{e^{-\beta v_{ji}}v_{ji}}{\left(1-e^{-\beta v_{ji}}\right)} \right\}^2 \left\{ \frac{(\alpha-1)-(\alpha i(i+1)-\alpha-1)\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{\left(1-\left(1-e^{-\beta v_{ji}}\right)^{\alpha}\right)^2} \right\} - b_{ji}\left\{ \frac{(i\alpha-1)e^{-\beta u_{ji}}u_{ji}^2}{(1-e^{-\beta u_{ji}})^2} \right\} \right] \tag{17}$$

$$\frac{\partial \log L_{MERSS}}{\partial\alpha\partial\beta} = \sum_{j=1}^{h}\sum_{i=1}^{m}\left[a_{ji}\left\{\frac{e^{-\beta vji}v_{ji}}{1-e^{-\beta v_{ji}}}\right\}\right.$$
$$\left\{\frac{1-i\left(1-e^{-\beta v_{ji}}\right)^{\alpha}}{1-\left(1-e^{-\beta v_{ji}}\right)^{\alpha}} - \frac{\alpha(i-1)\left(1-e^{-\beta v_{ji}}\right)^{\alpha}\log\left(1-e^{-\beta v_{ji}}\right)}{\left(1-(1-e^{-\beta v_{ji}})^{\alpha}\right)^{2}}\right\}$$
$$\left.-b_{ji}\left\{i\frac{e^{-\beta uji}u_{ji}}{1-e^{-\beta u_{ji}}}\right\}\right] \tag{18}$$

The components of Information Matrix (15) can also be obtained as

$$I_{MERSS}(\alpha,\beta) = I_{MERSS,\text{maximum}}(\alpha,\beta) + I_{MERSS,\text{minimum}}(\alpha,\beta) \tag{19}$$

Denoting $G(\beta v_{ji}) = (1-e^{-\beta v_{ji}})$ and $g(\beta v_{ji}) = e^{-\beta v_{ji}}$ the components of matrix $I_{MERSS,\text{maximum}}(\alpha,\beta)$ are, $\begin{bmatrix} E(a_{11}) & E(a_{12}) \\ E(a_{21}) & E(a_{22}) \end{bmatrix}$
where

$$a_{11} = -\frac{hm}{\alpha^2}$$
$$a_{21} = a_{12} = \sum_{j=1}^{h}\sum_{i=1}^{m} iu_{ji}\frac{g(\beta u_{ji})}{G(\beta u_{ji})}$$
$$a_{22} = -\frac{hm}{\beta^2} + \sum_{j=1}^{h}\sum_{i=1}^{m} u_{ji}^2\left[(i\alpha-1)\left\{\frac{G(\beta u_{ji})g'(\beta u_{ji}) - (g(\beta u_{ji}))^2}{(G(\beta u_{ji}))^2}\right\}\right]$$

Therefore

$$\begin{aligned} E(a_{11}) &= -\frac{hm}{\alpha^2} \\ E(a_{22}) &= -\frac{-hm}{\beta^2} - \sum_{j=1}^{h}\sum_{i=1}^{m}\frac{i\alpha(i\alpha-1)}{(i\alpha-2)} \\ &\quad \left[(\psi(i\alpha+1)+\gamma)^2 - \psi'(i\alpha+1) + \frac{\pi^2}{6}\right] \\ E(a_{12}) &= \sum_{j=1}^{h}\sum_{i=1}^{m}\frac{i}{i\alpha-1}(\psi(i\alpha+1)+\gamma), i\alpha \neq 1, i\alpha \neq 2 \end{aligned} \tag{20}$$

Also the components of matrix $I_{MERSS,\text{minimum}}(\alpha,\beta)$ are,
$\begin{bmatrix} E(b_{11}) & E(b_{12}) \\ E(b_{21}) & E(b_{22}) \end{bmatrix}$ where

$$b_{11} = -\frac{hm}{\alpha^2} - \sum_{j=1}^{h}\sum_{i=1}^{m}(i-1)\left(\log G(\beta v_{ji})\right)^2 \frac{G(\beta v_{ji})^{\alpha}}{1-G(\beta v_{ji})^{\alpha}}$$

$$b_{21} = b_{12} = \sum_{j=1}^{h}\sum_{i=1}^{m} v_{ji}\left[\frac{(G(\beta v_{ji}))^{\alpha-1} g(\beta v_{ji})}{1-(G(\beta v_{ji}))^{\alpha}}\right.$$
$$\left.\left\{-(i-1) - \alpha(i-1)\frac{\log G(\beta v_{ji})}{1-(G(\beta v_{ji}))^{\alpha}}\right\} + \frac{g(\beta v_{ji})}{G(\beta v_{ji})}\right]$$

$$b_{22} = -\frac{hm}{\beta^2} + \sum_{j=1}^{h}\sum_{i=1}^{m} v_{ji}^2\left[\left\{(\alpha-1) - (i-1)\alpha\frac{G(\beta v_{ji})^{\alpha}}{1-(G(\beta v_{ji}))^{\alpha}}\right\}\right.$$
$$\left\{\frac{G(\beta v_{ji})g'(\beta v_{ji}) - (g(\beta v_{ji}))^2}{(G(\beta v_{ji}))^2}\right\}$$
$$\left. -(i-1)\alpha\frac{(G(\beta v_{ji}))^{\alpha-2}(g(\beta v_{ji}))^2}{(1-(G(\beta v_{ji}))^{\alpha})^2}\right]$$

5.4 Data Set

In this section we demonstrate the performance of MERSS from the data (Table 1) used by Lieblein and Zelen (1956) and Gupta and Kundu (2001, 1999) representing the number of revolutions in millions before failure for each of the 23 ball bearings in the life test. When $m=2$, from the steps for MERSS we require $m(m+1)=6$ obervations, and we use number of cycles $h=3$. Therefore sample size of MERSS is $2mh=12$. A simple random sample of the same size is selected without replacement from the data.

Table 1
Data set by Lieblein and Zelen (1956) and Gupta and Kundu (2001, 1999)

17.88	28.92	33.00	41.52	42.12
45.60	48.40	51.84	51.96	54.12
55.56	67.80	68.64	68.64	68.88
84.12	93.12	98.64	105.12	105.84
127.92	128.04	173.40		

Table 2
The Estimates and Asymptotic efficiency using MERSS compared to SRS. The sample size $n=2mh$, set size $m=2$ and number of cycles $h=3$.

Estimates	SRS	MERSS	AEFF
$\hat{\alpha}$	7.06423	6.65388	1.073420
$\hat{\beta}$	0.03508	0.03440	1.072517

Table 3
Calculated confidence limits, Fisher information and Variance-Covariance matrix

	SRS: 95% CI		MERSS: 95% CI	
α	0	15.55126	0	14.2623972
β	0.01623	0.05392	0.0166774	0.0521169
	SRS:FI		MERSS:FI	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
$\hat{\alpha}$	0.43572	-107.84840	0.5329473	-136.844
$\hat{\beta}$	-107.84841	43501.52840	-136.844047	53163.388
	SRS: VC		MERSS: VC	
	$\hat{\alpha}$	b	$\hat{\alpha}$	$\hat{\beta}$
$\hat{\alpha}$	5.94012	0.01473	5.533824	0.0142442
$\hat{\beta}$	0.01473	0.0000595	0.014244	0.0000555

6 Unified (flexible) Ranked Set Sampling

A Unified (flexible) RSS (FLERSS) with $M = h_1 + h_2$ cycles is considered which containes h_1 cycles of Quartile Ranked Set Sampling (QRSS) and h_2 cycles of Median Ranked Set Sampling (MRSS). We consider in this section $h_1 = h_2 = h$.
Let $Y_{ji}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, h$ denotes a Quartile ranked set sample with cumulative distribution function given in (3) and probability density function given in (4).Where h is number of cyclels and m is the set size.

$$f_{Y_{ji}}(y) = \begin{cases} f_{X_{q_1(m+1)}}(y) & \text{for} \quad i = 1, 2, \ldots \frac{m-1}{2} \\ f_{X_{q_u(m+1)}}(y) & \text{for} \quad i = \frac{m-1}{2} + 1, \frac{m-1}{2} + 2, \ldots m-1 \\ f_{X_{0.5(m+1)}}(y) & \text{for} \quad i = m \end{cases} \tag{21}$$

$q_1 = 0.25, q_u = 0.75$
Then,

$$
\begin{aligned}
f_{X_{q_1(m+1)}}(y) &= C_3\left[\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{q_1(m+1)-1}\left[1-\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{m-q_1(m+1)} \\
&\quad \alpha\beta\left[1-e^{-\beta y}\right]^{\alpha-1}e^{-\beta y} \\
&= C_3\alpha\beta e^{-\beta y}\left[1-e^{-\beta y}\right]^{\alpha(q_1(m+1))-1} \\
&\quad \left[1-\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{m-q_1(m+1)} \\
f_{X_{(q_u(m+1))}}(y) &= C_4\left[\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{(q_u(m+1))-1}\left[1-\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{m-(q_u(m+1))} \\
&\quad \alpha\beta\left[1-e^{-\beta y}\right]^{\alpha-1}e^{-\beta y} \\
&= C_4\alpha\beta e^{-\beta y}\left[1-e^{-\beta y}\right]^{\alpha(q_u(m+1))-1} \\
&\quad \left[1-\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{m-(q_u(m+1))}
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
f_{X_{(0.5(m+1))}}(y) &= C_5\left[\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{(0.5(m+1))-1}\left[1-\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{m-(0.5(m+1))} \\
&\quad \alpha\beta\left[1-e^{-\beta y}\right]^{\alpha-1}e^{-\beta y} \\
&= C_5\alpha\beta e^{-\beta y}\left[1-e^{-\beta y}\right]^{\alpha(0.5(m+1))-1} \\
&\qquad\qquad \left[1-\left\{1-e^{-\beta y}\right\}^{\alpha}\right]^{m-(0.5(m+1))}
\end{aligned} \tag{23}
$$

Let $Y_{ji}, i = 1,2,\ldots,m; j = 1,2,\ldots,h$ denotes a Median ranked set sample with cumulative distribution function given in (3) and probability density function given in (4).Where h is number of cyclels and m is the set size. To derive the expressions for maximum likelihood equations, we consider general expressions using pdf of r^{th} order statistics. The PDf of rth order statistics is

$$
\begin{aligned}
f_{X_{(r)}}(y) &= \frac{1}{B(r,m-r+1)}F^{r-1}(y)(1-F(y))^{m-r}f(y) \\
&= \frac{1}{B(r,m-r+1)}\left((1-e^{-\beta y})^{\alpha}\right)^{r-1}\left(1-(1-e^{-\beta y})^{\alpha}\right)^{m-r} \\
&\qquad \alpha\beta(1-e^{-\beta y})^{\alpha-1}e^{-\beta y}, \alpha > 0, \beta > 0
\end{aligned} \tag{24}
$$

The Likelihood function

$$
\begin{aligned}
L &= \prod_{j=1}^{h}\prod_{i=1}^{m} f_{Y_{(r)}}(y_{ji}) \\
&= \prod_{j=1}^{h}\prod_{i=1}^{m} \frac{1}{B(r,m-r+1)}\left((1-e^{-\beta y_{ji}})^{\alpha}\right)^{r-1} \\
&\qquad \left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)^{m-r}\alpha\beta(1-e^{-\beta y_{ji}})^{\alpha-1}e^{-\beta y_{ji}} \\
&= \left(\frac{1}{B(r,m-r+1)}\right)^{mh}\prod_{j=1}^{h}\prod_{i=1}^{m}\left((1-e^{-\beta y_{ji}})^{\alpha}\right)^{r\alpha-1} \\
&\qquad \left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)^{m-r}\alpha\beta e^{-\beta y_{ji}}
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\log L &= \sum_{j=1}^{h}\sum_{i=1}^{m} \log f_{Y_{(r)}}(y_{ji}) \\
&= C_7 + mh(\log\alpha + \log\beta) \\
&\quad + \sum_{j=1}^{h}\sum_{i=1}^{m}\Big\{(r\alpha-1)\log\left(1-e^{-\beta y_{ji}}\right) \\
&\qquad + (m-r)\log\left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right) - \beta y_{ji}\Big\}
\end{aligned} \tag{26}
$$

Where $C_7 = \log\left(\frac{1}{B(r,m-r+1)}\right)$

6.1 The Likelihood Functions

$$
\frac{\partial \log L}{\partial\alpha} = \frac{2mh}{\alpha} + \sum_{j=1}^{h}\sum_{i=1}^{m}\log(1-e^{-\beta y_{ji}})\left\{\frac{r-m(1-e^{-\beta y_{ji}})}{\left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)}\right\} \tag{27}
$$

The mle of α which depends upon β is the solution of equation $\frac{\partial \log L}{\partial\alpha} = 0$

$$
\frac{\partial \log L}{\partial\beta} = \frac{2mh}{\beta} + \sum_{j=1}^{h}\sum_{i=1}^{m} y_{ji}\left[\frac{e^{-\beta y_{ji}}}{(1-e^{-\beta y_{ji}})}\left\{\frac{r-m(1-e^{-\beta y_{ji}})^{\alpha}}{1-(1-e^{-\beta y_{ji}})^{\alpha}}-1\right\}-1\right] \tag{28}
$$

The mle of β which depends upon α is the solution of equation $\frac{\partial \log L}{\partial\beta} = 0$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{2mh}{\alpha^2} - \sum_{j=1}^{h}\sum_{i=1}^{m}\left\{\frac{(m-r)(1-e^{-\beta y_{ji}})\left(\log(1-e^{-\beta y_{ji}})\right)^2}{\left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)^2}\right\} \tag{29}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{2mh}{\beta^2} - \sum_{j=1}^{h}\sum_{i=1}^{m} y_{ji}^2 \frac{e^{-\beta y_{ji}}}{(1-e^{-\beta y_{ji}})^2}&\left[\left\{\frac{r-m(1-e^{-\beta y_{ji}})^{\alpha}}{1-(1-e^{-\beta y_{ji}})^{\alpha}} - 1\right\}\right. \\ &\left. + e^{-\beta y_{ji}} \frac{\alpha(m+r)(1-e^{-\beta y_{ji}})^{\alpha}}{\left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)^2}\right]\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha \partial \beta} = \sum_{j=1}^{h}\sum_{i=1}^{m} \frac{y_{ji} e^{-\beta y_{ji}}}{(1-e^{-\beta y_{ji}})}&\left[\left\{\frac{r-m(1-e^{-\beta y_{ji}})^{\alpha}}{\left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)}\right\}\right. \\ &\left. - \frac{\alpha(m-r)(1-e^{-\beta y_{ji}})^{\alpha}\log(1-e^{-\beta y_{ji}})}{\left(1-(1-e^{-\beta y_{ji}})^{\alpha}\right)^2}\right]\end{aligned} \tag{31}$$

6.2 Likelihhod function (Unified(Flexible) Ranked Set Sampling

Define, for $j = 1, 2, \ldots, h$

$$a_{ji} = \begin{cases} 1, & \text{for} \quad i = 1, 2, \ldots, k \\ 0, & \text{for} \quad i = k+1, k+2, \ldots, m \end{cases}$$
$$b_{ji} = \begin{cases} 0, & \text{for} \quad i = 1, 2, \ldots, k \\ 1, & \text{for} \quad i = k+1, k+2, \ldots, m-1 \\ 0, & i = m \end{cases} \tag{32}$$
$$c_{ji} = \begin{cases} 0, & \text{for} \quad i = 1, 2, \ldots, m-1 \\ 1, & \text{for} \quad i = m \end{cases}$$

Let L_{FLERSS} ,L_{QRSS} and L_{MRSS} denote respectively the likelihood functions of the sampling scheme FLERSS, QRSS and MRSS. Then,

$$L_{FLERSS} = L_{QRSS} L_{MRSS} \tag{33}$$

Where

$$\begin{aligned} L_{QRSS;m \text{ Odd}}(\alpha, \beta) &= \prod_{j=1}^{h}\prod_{i=1}^{k} f_{Y_i}(y_{ji}) \prod_{j=1}^{h}\prod_{i=k+1}^{m-1} f_{Y_i}(y_{ji}) \prod_{j=1}^{h} f_{Y_m}(y_{jm}) \\ &= \prod_{j=1}^{h}\prod_{i=1}^{m} [f_{Y_i}(y_{ji})]^{a_{ji}} [f_{Y_i}(y_{ji})]^{b_{ji}} [f_{Y_i}(y_{ji})]^{c_{ji}} \end{aligned} \tag{34}$$

and

$$L_{MRSS;m\ \text{Odd}}(\alpha,\beta) = \prod_{j=1}^{h}\prod_{i=1}^{m} f_{Y_i}(y_{ji}) \tag{35}$$

The Log-Likelihood function is

$$\log L_{FLERSS} = \log L_{QRSS} + \log L_{MRSS} \tag{36}$$

The normal equations are,

$$\frac{\partial \log L_{FLERSS}}{\partial \alpha} = \frac{\partial \log L_{QRSS}}{\partial \alpha} + \frac{\partial \log L_{MRSS}}{\partial \alpha} \tag{37}$$

$$\frac{\partial \log L_{FLERSS}}{\partial \beta} = \frac{\partial \log L_{QRSS}}{\partial \beta} + \frac{\partial \log L_{MRSS}}{\partial \beta} \tag{38}$$

and the Fisher Information matrix is

$$I_{FLERSS} = I_{QRSS} + I_{MRSS} \tag{39}$$

By substituting $r = [q_1(m+1)]$,$r = [q_u(m+1)]$ and $r = [0.5(m+1)]$ in ((27) to (31)) to compute the estimates by solving equations (36) to (39) numerically using R maxLik package (Henningsen and Toomet (2011)). Where $[x]$ represent nearest ingeter.

6.3 Simulated Output

In this section, we generate 5000 random numbers from the Generalized exponential distribution given in (3) when shape parameter $\alpha = 0.5, \alpha = 4.0$ and scale parameter $\beta = 0.5$. The Tables (4) to (8) show the computatuions. The Authors have computed similar tables when $\alpha = 4.0$ and $\beta = 0.5$.

Table 4 Estimate of shape parameter $(\alpha = 0.5)$ for different cycles h

	$\hat{\alpha}$				
h	MERSS	FLERSS	SRS	QRSS	MRSS
2	0.53896	0.54630	0.54659	0.55437	0.55472
3	0.52533	0.54274	0.52727	0.54726	0.54693
4	0.52342	0.54262	0.52273	0.53064	0.53120
5	0.52049	0.52234	0.52378	0.52772	0.53414

Table 5
Estimate of scale parameter ($\beta = 0.5$) for different cycles h

		$\hat{\beta}$			
h	MERSS	SRS	FLERSS	MRSS	QRSS
2	0.53098	0.54355	0.56170	0.56254	0.56320
3	0.52868	0.52421	0.54429	0.55008	0.55307
4	0.51752	0.52166	0.52656	0.53107	0.53135
5	0.51453	0.51872	0.51908	0.53498	0.52426

Table 6
Efficiency of MERSS for different cycles h.
$\alpha = 0.5, \beta = 0.5$

Sampling	h	$\hat{\alpha}$	$\hat{\beta}$
SRS	2	1.0136194	1.0578518
	3	1.0331492	1.0295447
	4	1.0366661	1.0174703
	5	1.0035649	1.0088355
QRSS	2	1.0292501	1.0594382
	3	1.0411325	1.0404958
	4	1.0148621	1.0261800
	5	1.0262226	1.0397384
MRSS	2	1.0285954	1.0606792
	3	1.0417569	1.0461465
	4	1.0137888	1.0267224
	5	1.0138931	1.0189093
FLERSS	2	1.0141577	1.0236766
	3	1.0036985	0.9915609
	4	0.9986808	1.0079921
	5	1.0063299	1.0081332

Table 7
Asymptotic Efficiency of MERSS for different cycles h.
$\alpha = 0.5, \beta = 0.5$

Sampling	Cycles	$\hat{\alpha}$	$\hat{\beta}$
SRS	2	6.54024	5.74138
	3	7.01180	5.67670
	4	7.09303	5.64129
	5	6.72755	5.64903
QRSS	2	2.93797	2.85718
	3	3.20964	2.92172
	4	3.04382	2.94519
	5	3.19900	3.09289
MRSS	2	4.96470	5.42898
	3	5.22723	5.27204
	4	4.90125	5.24506
	5	4.96617	5.24216
FLERSS	2	3.75883	3.88354
	3	3.80268	3.73140
	4	3.71812	3.83538
	5	3.80284	3.87356

Table 8
$\alpha = 0.5$,$\beta = 0.5$:
Observed Fisher Information (FI),Variance- Covariance(VC) and 95% confidence limits for different cycles h.

Sampling	h	Est	FI		VC		95% CI (α, β)		Width
SRS	2	$\hat{\alpha}$	109.33888	-42.15818	0.01228	0.00812	0.30339	0.78920	0.48581
		$\hat{\beta}$	-42.15818	63.74626	0.00812	0.02106	0.23483	0.88857	0.65374
	3	$\hat{\alpha}$	158.35636	-62.75473	0.00862	0.00583	0.34632	0.73916	0.39284
		$\hat{\beta}$	-62.75473	92.87910	0.00583	0.01470	0.28507	0.80352	0.51844
	4	$\hat{\alpha}$	205.31954	-84.68729	0.00672	0.00449	0.37272	0.71251	0.33980
		$\hat{\beta}$	-84.68729	126.82636	0.00449	0.01088	0.30942	0.74370	0.43428
	5	$\hat{\alpha}$	270.42170	-107.83260	0.00509	0.00349	0.37693	0.66776	0.29084
		$\hat{\beta}$	-107.83260	157.34370	0.00349	0.00875	0.32535	0.71281	0.38746
QRSS	2	$\hat{\alpha}$	377.70090	-197.56690	0.00551	0.00548	0.38700	0.72244	0.33544
		$\hat{\beta}$	-197.56690	198.77450	0.00548	0.01048	0.33162	0.79347	0.46185
	3	$\hat{\alpha}$	546.98160	-289.46490	0.00395	0.00400	0.41213	0.68173	0.26960
		$\hat{\beta}$	-289.46490	285.32510	0.00400	0.00757	0.36300	0.73716	0.37416
	4	$\hat{\alpha}$	755.36690	-395.90440	0.00288	0.00298	0.41857	0.64384	0.22527
		$\hat{\beta}$	-395.90440	383.52160	0.00298	0.00568	0.37285	0.68929	0.31645
	5	$\hat{\alpha}$	917.76280	-483.69680	0.00242	0.00252	0.43266	0.63561	0.20295
		$\hat{\beta}$	-483.69680	463.77260	0.00252	0.00479	0.39236	0.67761	0.28525
MRSS	2	$\hat{\alpha}$	388.56400	-226.16270	0.00932	0.01159	0.34210	0.76664	0.42454
		$\hat{\beta}$	-226.16270	181.86150	0.01159	0.01991	0.25678	0.86962	0.61285
	3	$\hat{\alpha}$	576.44870	-337.98210	0.00643	0.00801	0.37722	0.71731	0.34009
		$\hat{\beta}$	-337.98210	271.39520	0.00801	0.01366	0.30588	0.80026	0.49438
	4	$\hat{\alpha}$	804.84740	-466.72150	0.00464	0.00587	0.38902	0.67226	0.28324
		$\hat{\beta}$	-466.72150	369.48410	0.00587	0.01012	0.32293	0.73978	0.41685
	5	$\hat{\alpha}$	1004.03010	-585.54920	0.00376	0.00473	0.40201	0.65343	0.25142
		$\hat{\beta}$	-585.54920	464.71140	0.00473	0.00812	0.33973	0.70879	0.36906
FLERSS	2	$\hat{\alpha}$	381.84520	-213.11500	0.00706	0.00795	0.36503	0.72815	0.36312
		$\hat{\beta}$	-213.11500	189.15390	0.00795	0.01424	0.28933	0.79778	0.50844
	3	$\hat{\alpha}$	587.64660	-326.04790	0.00468	0.00536	0.38559	0.66895	0.28336
		$\hat{\beta}$	-326.04790	284.36930	0.00536	0.00967	0.32061	0.72782	0.40721
	4	$\hat{\alpha}$	787.69010	-434.77300	0.00352	0.00408	0.40136	0.64411	0.24275
		$\hat{\beta}$	-434.77300	375.14250	0.00408	0.00740	0.34547	0.69784	0.35237
	5	$\hat{\alpha}$	973.95640	-540.90420	0.00288	0.00333	0.41494	0.63263	0.21768
		$\hat{\beta}$	-540.90420	467.15640	0.00333	0.00600	0.36196	0.67548	0.31352
MERSS	2	$\hat{\alpha}$	783.94270	-317.46380	0.00188	0.00149	0.44810	0.62982	0.18172
		$\hat{\beta}$	-317.46380	401.22460	0.00149	0.00367	0.40322	0.65875	0.25553
	3	$\hat{\alpha}$	1208.34490	-476.24120	0.00123	0.00102	0.45317	0.59748	0.14431
		$\hat{\beta}$	-476.24120	573.77240	0.00102	0.00259	0.42362	0.63373	0.21012
	4	$\hat{\alpha}$	1584.19900	-641.62640	0.00095	0.00078	0.46128	0.58557	0.12429
		$\hat{\beta}$	-641.62640	778.27960	0.00078	0.00193	0.42830	0.60674	0.17844
	5	$\hat{\alpha}$	1988.80690	-804.86960	0.00076	0.00063	0.46529	0.57569	0.11040
		$\hat{\beta}$	-804.86960	971.66730	0.00063	0.00155	0.43505	0.59401	0.15896

7 Best Linear Unbiased Estimates

Consider the transformation $Y=\beta X$ in (4) the probability density function of Y is

$$f(y;\alpha)=\alpha(1-e^{-y})^{\alpha}e^{-y},\quad y>0 \tag{40}$$

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample from the distribution (40. Then pdf of r^{th} order statistic $Y_{r:n},(r=1,2,\ldots,n)$ is

$$f_{r:n}(y)=\frac{1}{B(r,n-r+1)}F^{r-1}(y)\left[1-F(y)\right]^{n-r}f(y) \tag{41}$$

Where $B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$$\begin{aligned}
f_{r:n}(y)&=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\alpha\left[1-e^{-y}\right]^{(r+i)\alpha-1}e^{-y}\\
&=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\alpha(r+i)\frac{\left[1-e^{-y}\right]^{(r+i)\alpha-1}}{r+i}e^{-y}\\
&=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}\frac{(-1)^i\binom{n-r}{i}}{r+i}f(y;\alpha_{(r+i)})
\end{aligned}$$

For $(r=1,2,\ldots,n)$ the moment genrating function is

$$\begin{aligned}
M_{Y_{r:n}}(t)&=\int_0^{\infty}f_{r:n}(y)e^{ty}dy\\
&=\int_0^{\infty}\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}\frac{(-1)^i\binom{n-r}{i}}{r+i}f(y;\alpha_{(r+i)})e^{ty}dy \qquad (42)\\
&=\frac{\alpha}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\frac{\Gamma(\alpha(r+i))\Gamma(1-t)}{\Gamma(\alpha(r+i)+1-t)},\quad t<1
\end{aligned}$$

Setting $M'_{Y_{r:n}}(t)\mid_{t=0}$ we get

$$\begin{aligned}
E(Y_{r:n})&=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}\frac{(-1)^i\binom{n-r}{i}}{r+i}\left[-\psi(1)+\psi(\alpha(r+i)+1)\right]\\
&=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}\frac{(-1)^i\binom{n-r}{i}}{r+i}\left[\gamma+\psi(\alpha(r+i)+1)\right]
\end{aligned} \tag{43}$$

Setting $M''_{Y_{r:n}}(t)\mid_{t=0}$ we get

$$\begin{aligned}
E\{Y^2_{(i:n)}\}=\frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{j+i}\Big\{&\left[\psi((i+j)\alpha+1)+\gamma\right]^2\\
&-\psi'((i+j)\alpha+1)+\pi^2/6\Big\}
\end{aligned} \tag{44}$$

Raqab Mohammad and Ahsanullah (2001) have derived the product moment of r^{th} and s^{th} order statistics,

$$E\{Y_{(r:n)}Y_{(s:n)}\} = C_{r,s:n}\alpha^2 \sum_{k=1}^{\infty}\sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{s-r-1}{j}\binom{n-s}{i} \frac{[\psi((s+i)\alpha+k+1)+\gamma]}{k\,[(s+i)\alpha+k]\,[(r+j)\alpha+k]} \tag{45}$$

Where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

7.1 Estimation of scale parameter using SRS

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the pdf given in (2) and $X_{(1:n)}, X_{(2:n)}, X_{(3:n)}, \ldots, X_{(n:n)}$ are the order statistics then

$$E\{X_{(r:n)}\} = u_r\sigma, \quad V\{X_{(r:n)}\} = \sigma^2 w_{rr}, \quad Cov(X_{(r:n)}, X_{(s:n)}) = \sigma^2 w_{rs} \tag{46}$$

Define $\mathbf{u} = (u_1, u_2, \ldots, u_n)_{n\times 1}$ be the vector of u_r,
$\mathbf{X} = (X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)})_{n\times 1}$, $\mathbf{w}_{n\times n}$ is the variance covariance matrix that is symmetric positive definite. The equations (46) in matrix form is

$$E(\mathbf{X}) = \sigma\mathbf{u}, V(\mathbf{X}) = \sigma^2\mathbf{w}$$

if $\hat{\sigma}_{\text{BLUESRS}}$ denotes the BLUE of σ based on SRS, then

$$\hat{\sigma}_{\text{BLUESRS}} = (\mathbf{u}'\boldsymbol{\Omega}\mathbf{u})^{-\mathbf{1}}\mathbf{u}'\boldsymbol{\Omega}\mathbf{X} \tag{47}$$

where $\boldsymbol{\Omega} = \mathbf{w}^{-\mathbf{1}}, \Delta = |\mathbf{u}'\boldsymbol{\Omega}\mathbf{u}|$ and

$$V(\hat{\sigma}_{\text{BLUESRS}}) = (\mathbf{u}'\boldsymbol{\Omega}\mathbf{u})\sigma^{\mathbf{2}}/\boldsymbol{\Delta} \tag{48}$$

7.2 Estimation of Scale Parameter using RSS

We assume that the both parameters α and μ are known.
Let $X_{(11)}, X_{(22)}, X_{(33)}, \ldots, X_{(nn)}$ be a ranked set sample (cycle 1) from the pdf given in (3),then

$$E\{X_{(ii)}\} = c_{i:n}\sigma$$
$$V\{X_{(ii)}\} = d_{i:n}\sigma^2$$

where $c_{i:n} = E\{Y_{(ii)}\}$, $d_{i:n} = V\{Y_{(ii)}\}$ and the expressions for $E\{Y_{(ii)}\}$ and $E\{Y^2_{(ii)}\}$ are same as given in (43) and (45). If $\hat{\sigma}_{\text{RSS}}$ denote the least squares estimate of σ based on RSS, then

$$\hat{\sigma}_{\text{RSS}} = \frac{1}{\sum_{i=1}^{n} c^2_{i:n}/d_{i:n}} \sum_{i=1}^{n} \frac{c_{i:n} X_{ii}}{d_{i:n}} \tag{49}$$

The variance of $\hat{\sigma}_{\text{RSS}}$ is given by

$$V(\hat{\sigma}_{\text{RSS-BLUE}}) = \sigma^2 \frac{1}{\sum_{i=1}^{n} c^2_{i:n}/d_{i:n}} \tag{50}$$

The relative precision (RP) in estimating $\hat{\sigma}$ using RSS is

$$\text{RP}_1 = \frac{V(\hat{\sigma}_{BLUESRS})}{V(\hat{\sigma}_{\text{RSS-BLUE}})} \tag{51}$$

Table 9
Relative Precision in estimating the scale parameter using RSS over SRS

α \ n	5	8	10	12	15
0.25	2.028766	2.791312	3.301645	3.813705	4.584696
0.50	2.348881	3.353996	4.025434	4.697940	5.708382
1.50	2.692024	3.969610	4.824324	5.680652	6.967306
2.50	2.776810	4.126146	5.029660	5.935170	7.296008
3.00	2.797785	4.165401	5.081374	5.999447	7.379242
5.00	2.836099	4.238297	5.177844	6.119687	7.535336

7.3 Estimating location and scale parameters using RSS

Let $X_{(11)}, X_{(22)}, X_{(33)}, \ldots, X_{(nn)}$ be a ranked set sample (Cycle 1) from the pdf given in (3),then $E\{X_{(ii)}\} = \mu + c_{i:n}\sigma$ and $V\{X_{(ii)}\} = d_{i:n}\sigma^2$, where $c_{i:n} = E\{Y_{(ii)}\}$, $d_{i:n} = V\{Y_{(ii)}\}$. Following the method of David and Nagaraja (2003), we compute the least squares estimates $\hat{\mu}_{\text{RSSBLUE}}$ and $\hat{\sigma}_{\text{RSSBLUE}}$ of μ and σ respectively.The relative precision of estimates are computed and shown in Table (10).

Table 10
Relative Precision in estimating location and scale parameters using RSS over SRS

n	5	6	7	8	10	15
α	$V(\hat{\mu}_{\text{SRSBLUE}})/V(\hat{\mu}_{\text{RSSBLUE}})$					
0.25	0.93901217	0.969205	0.988601	1.001446	1.01617	1.029123
0.50	0.92971395	0.990456	1.038524	1.077219	1.134978	1.216394
2.50	0.9695873	1.094576	1.214898	1.331423	1.555482	2.079552
5.00	0.97758532	1.110036	1.239165	1.365691	1.612841	2.20861
α	$V(\hat{\sigma}_{\text{SRSBLUE}})/V(\hat{\sigma}_{\text{RSSBLUE}})$					
0.25	1.79328971	2.086714	2.371291	2.64963	3.194192	4.522285
0.50	1.70705093	2.040913	2.377683	2.716588	3.398704	5.116085
2.50	1.22983342	1.427199	1.622855	1.817114	2.202202	3.149992
5.00	1.16515492	1.343524	1.51944	1.69343	2.036978	2.878687

We propose the modified estimator $\hat{\sigma^*}$ of σ,

$$\hat{\sigma^*} = \sum_{i=1}^{n} X_{(ik)}/nc_{k:n} \tag{52}$$

The values of k are given in Table 11.

Table 11
The choice of kth order statistic in ith set of size n (RSS Cycle 1), when $n \in [a,b]$[1] for different values of α

k \ α	0.1	0.25	0.5	0.75	1	2	5	10
n	[2,30]	[2,14]	[2,8]	[2,6]	[2,5]	[2,3]	2[2]	—
$n-1$		[15,27]	[9,15]	[7,11]	[6,9]	[4,6]	[3,4]	[2,3]
$n-2$		[28,30]	[16,23]	[12,17]	[10,14]	[7,10]	[5,7]	[4,5]
$n-3$			[24,29]	[18,23]	[15,19]	[11,13]	[8,9]	[6,8]
$n-4$				[24,29]	[20,24]	[14,16]	[10,11]	[9,10]
$n-5$						[17,20]	[12,14]	[11,12]
$n-6$						[21,23]	[15,16]	[13,14]
$n-7$						[24,27]	[17,19]	[15,16]
$n-8$							[20,21]	[17,18]

[1] a and b are integers respectively.
[2] Single value

From (52),

$$V(\hat{\sigma^*}) = \frac{\sigma^2}{n\left(c_{k:n}^2/d_{k:n}\right)} \tag{53}$$

From (48) and (53) we have

$$RP_2 = \frac{V\left(\hat{\sigma}_{\text{BLUESRS}}\right)}{V(\hat{\sigma^*})} \tag{54}$$

Table 12
Relative Precision: RSS($n=5$,Cycle 1),SRS
($n(n+1)/2=15$), Formula (54)

α	0.1	0.25	0.5	1.0	2.0
RP	1.57871	1.47454	1.34702	1.18739	1.18406

7.4 Estimatiing shape parameter α

Suppose the distribution in (1) is known for parameters μ and σ. Assume that the distribution of $Y = \dfrac{X-\mu}{\sigma}$ is given by

$$F(y;\alpha) = (1-e^{-y})^{1/\alpha}, \quad \alpha, y > 0 \tag{55}$$

$$f(y;\alpha) = (1/\alpha)(1-e^{-y})^{(1/\alpha)-1}e^{-y} \quad \alpha, y > 0 \tag{56}$$

Then, distribution of $Z = -\ln F(Y)$ is

$$F_Z(z) = 1 - e^{-z/\alpha}, \quad \alpha > 0, z > 0 \tag{57}$$

Let $Y_{(1:n)}, Y_{(2:n)}, \ldots, Y_{(n:n)}$ be an order statistics from standard exponential distribution

$$F_Y(y) = 1 - e^{-y}, \quad y > 0 \tag{58}$$

Then

$$\begin{aligned} c_{i:n} &= E\{Y_{(i:n)}\} = \sum_{j=1}^{i} \frac{1}{(n-j+1)} \\ d_{i:n} &= V\{Y_{(i:n)}\} = \sum_{j=1}^{i} \frac{1}{(n-j+1)^2} \\ Cov(Y_{(r:n)}, Y_{(s:n)}) &= V\{Y_{(r:n)}\}, \quad r < s \end{aligned} \tag{59}$$

Let $Z_{(11)}, Z_{(22)}, \ldots, Z_{(nn)}$ be an RSS from the distribution given in (57),then we write an estimator of α based on RSS as

$$\hat{\alpha}_{\text{RSS-BLUE}} = \sum_{i=1}^{n} w_i Z_{(ii)} \tag{60}$$

Where,

$$w_i = (c_{i:n}/d_{i:n})/\sum_{i=1}^{n}(c_{i:n}^2/d_{i:n})$$

and

$$V(\hat{\alpha}_{\text{RSS-BLUE}}) = \sigma^2/\sum_{i=1}^{n}(c_{i:n}^2/d_{i:n}) \tag{61}$$

The Table 13 shows the relative precision of estimate $\hat{\alpha}_{\text{RSS}}$

Table 13
Relative Precision: $V(\hat{\alpha}_{\text{SRS}})/V(\hat{\alpha}_{\text{RSS}})$

n	5	6	7	8	9	10
RP	2.59255	2.9908	3.38957	3.7888	4.18844	4.58844

Also, we suggest the modified ranked set sampling scheme to obtain the relative precision of estimate of α using ordered least squares. The proposed estimate of α and the variance of estimate respectively are

$$\begin{aligned} \hat{\alpha^*} &= \sum_{i=1}^{n} Z_{(ik)}/nc_{k:n} \\ V(\hat{\alpha^*}) &= \frac{\sigma^2}{n\left(c_{k:n}^2/d_{k:n}\right)} \end{aligned} \tag{62}$$

Where the value of k is

$$k = \begin{cases} n & 2 \leq n \leq 3 \\ n-1 & 4 \leq n \leq 6 \\ n-2 & 7 \leq n \leq 10 \\ n-3 & 11 \leq n \leq 13 \\ n-4 & 14 \leq n \leq 16 \\ n-5 & 17 \leq n \leq 20 \end{cases}$$

Table 14
Relative Precision: $V(\hat{\alpha}_{\text{SRS-BLUE}})/V(\hat{\alpha^*})$

n	5	10	12	15	20
RP	3.562156	6.81178	8.16024	10.0530	13.2911

8 Simulated Data

Based on the computer program developed by the author, we generate RSS sample of size $n = 10$ from the density function given in (2) for known values of $\alpha = 0.5, 2.0$, and $\mu = 10, \sigma = 5$ respectively. We refer to the Table 20 (Page 35) of coefficients to estimate μ and σ.

Table 15
Ranked Set Sample of size $n = 10$ to estimate the parameters μ and σ of probability density given in (2)

$\alpha = 2.0$	1	2	3	4	5	$\hat{\mu} =$	10.17431
	11.04929	13.35352	13.85929	16.04259	18.13354		
	6	7	8	9	10	$\hat{\sigma} =$	4.951395
	12.5353	23.59186	18.56782	15.65142	33.38607		
$\alpha = 0.5$	1	2	3	4	5	$\hat{\mu} =$	10.1829
	10.32787	10.1165	11.31832	10.80349	10.61852		
	6	7	8	9	10	$\hat{\sigma} =$	5.346812
	11.05307	14.8025	12.53511	27.81479	14.38177		

9 Simulation Study

The estimates $(\hat{\alpha}, \hat{\beta})$ are determined numerically, the steps are

1. Generate 5000 random numbers from density function given in (4) with $\alpha = 0.5, 4.0$ and $\beta = 0.5$
2. In the first cycle select Moving Extreme Ranked Set Sample from $m(m+1)$ observations given in step (1). Where $m = 7$
3. In the first cycle select Quartile ranked set sample and Median ranked set sample from m^2 observations each given in step (1).
4. In the first cycle select Simple Random Sample of size m.
5. Repeat the steps (1) to (5) for $h = 2, 3, 4, 5$ cycles. The sample size of MERSS is $2hm$, the sample size of QRSS, MRSS and FLERSS respectively is $2hm$. The sample size of SRS is $2hm$.
6. Using R **nlm()** function and R maxLik package find the estimates $(\hat{\alpha}, \hat{\beta})$, the elements of observed Fisher Information matrix and 95% confidence limits of the parameters α and β numerically using 500 simulation runs.
7. From the numerical values obtained in step (6) for 500 simulation run determine average values of mse, bias, confidence limits, entries of observed Fisher information matrix respectively. Compute the observed variance- co-variance matrix from the observed Fisher information matrix.The efficiency and asymptotic efficiency of the estimates respectively are given in Tables (4) to (8) and and Tables (16) to (18).

10 Conclusion

1. We obtain numerically the estimates of parameters of generalized exponential distribution given in (1) and investigate their performances, using ranked set sampling scheme and simple random sampling scheme with the method of least squares
(a) When shape parameter α is unknown, the estimator of shape parameter using RSS is efficient than that of using SRS and (b) When shape parameter α is known and is more than 1, the estimator of location parameter using RSS is efficient than that of using SRS for sample size 6 or more. For any value of shape parameter $\alpha > 0$, the estimator of scale parameter is efficient than that of using Simple Random Sampling for sample size $n \geq 5$. (The numerical comutations are given in Tables (9) and (10))
2. We use real data (Table 1) to compare the performance of SRS and the MERSS scheme. MERSS is asymptotically more efficient than SRS.
3. In this paper, we examine different modifications of ranked set sampling and compare the estimated parameter values of the generalized exponential distribution given in (3). The MERSS scheme is asymptotically more efficient than other schemes such as SRS, QRSS, MRSS, and FLERSS for cycles $h = 2,3,4,5$ to estimate both shape and scale parameters of distribution given in (3).The numerical computations are supplied in Tables (4) to (8) and Tables (16) to (18). respectively.

Table 16
Estimate of shape and scale parameters
($\alpha = 4.0, \beta = 0.5$) for different cycles h

Sampling	h	$\hat{\alpha}$	$\hat{\beta}$	bias($\hat{\alpha}$)	bias($\hat{\beta}$)	MSE($\hat{\alpha}$)	MSE($\hat{\beta}$)
SRS	2	4.49995	0.52013	0.49995	0.02013	3.66020	0.01182
MERSS	2	4.19172	0.50895	0.19172	0.00895	1.36022	0.00520
QRSS	2	4.48234	0.51387	0.48234	0.01387	3.89591	0.00867
MRSS	2	4.44518	0.51261	0.44518	0.01261	2.90570	0.00750
FLERSS	2	4.27402	0.50665	0.27402	0.00665	1.88851	0.00589
SRS	3	4.25081	0.50958	0.25081	0.00958	1.76369	0.00619
MERSS	3	4.07897	0.50235	0.07897	0.00235	0.68847	0.00317
QRSS	3	4.24073	0.50557	0.24073	0.00557	1.63238	0.00545
MRSS	3	4.17018	0.50206	0.17018	0.00206	1.73647	0.00535
FLERSS	3	4.09946	0.50084	0.09946	0.00084	1.03079	0.00354
SRS	4	4.13877	0.50309	0.13877	0.00309	1.11079	0.00441
MERSS	4	3.95781	0.50054	-0.04219	0.00054	0.50547	0.00248
QRSS	4	4.17945	0.50469	0.17945	0.00469	1.38519	0.00483
MRSS	4	4.04921	0.49722	0.04921	-0.00278	1.09618	0.00395
FLERSS	4	4.08016	0.49969	0.08016	-0.00031	0.85204	0.00279
SRS	5	4.027300	0.499034	0.027300	-0.000966	0.803603	0.003417
MERSS	5	3.972811	0.498362	-0.027189	-0.001638	0.395034	0.002063
QRSS	5	4.098269	0.499719	0.098269	-0.000281	0.945214	0.003286
MRSS	5	3.991335	0.493423	-0.008665	-0.006577	0.827605	0.003141
FLERSS	5	3.969677	0.495252	-0.030323	-0.004748	0.544031	0.002092

Table 17
$\alpha = 4.0, \beta = 0.5$:
Observed Fisher Information (FI),Variance- Covariance(VC) and 95% Confidence limits for different cycles h.

Sampling	h	Est	FI		VC		95% CI(α,β)		Width
SRS	2	$\hat{\alpha}$	2.02741	-20.57385	1.20213	0.06986	1.43985	7.56004	6.12019
		$\hat{\beta}$	-20.57385	354.04789	0.06986	0.00688	0.33086	0.70939	0.37853
	3	$\hat{\alpha}$	2.91944	-30.81175	0.89139	0.05200	1.94695	6.55466	4.60771
		$\hat{\beta}$	-30.81175	528.12938	0.05200	0.00493	0.35771	0.66145	0.30374
	4	$\hat{\alpha}$	3.85482	-41.43064	0.69793	0.04080	2.21751	6.06002	3.84251
		$\hat{\beta}$	-41.43064	708.70738	0.04080	0.00380	0.37302	0.63315	0.26013
	5	$\hat{\alpha}$	4.92411	-52.37682	0.55172	0.03278	2.37133	5.68327	3.31194
		$\hat{\beta}$	-52.37682	881.64748	0.03278	0.00308	0.38345	0.61462	0.23117
QRSS	2	$\hat{\alpha}$	7.02543	-85.96232	0.57291	0.03519	2.20059	6.76408	4.56349
		$\hat{\beta}$	-85.96232	1399.54567	0.03519	0.00288	0.38238	0.64535	0.26298
	3	$\hat{\alpha}$	10.38784	-130.45630	0.42877	0.02648	2.52354	5.95793	3.43439
		$\hat{\beta}$	-130.45633	2112.67340	0.02648	0.00211	0.39953	0.61161	0.21208
	4	$\hat{\alpha}$	13.90917	-174.48610	0.32989	0.02057	2.71919	5.63972	2.92053
		$\hat{\beta}$	-174.48610	2798.84540	0.02057	0.00164	0.41280	0.59658	0.18378
	5	$\hat{\alpha}$	17.03452	-218.18920	0.28552	0.01771	2.82825	5.36829	2.54005
		$\hat{\beta}$	-218.18915	3518.02810	0.01771	0.00138	0.41824	0.58120	0.16296
MRSS	2	$\hat{\alpha}$	7.48840	-101.35460	0.81440	0.05030	1.43133	7.45903	6.02770
		$\hat{\beta}$	-101.35460	1640.88370	0.05030	0.00372	0.34355	0.68168	0.33813
	3	$\hat{\alpha}$	11.68423	-156.86330	0.59861	0.03821	1.91439	6.42597	4.51158
		$\hat{\beta}$	-156.86329	2457.24690	0.03821	0.00285	0.36608	0.63804	0.27196
	4	$\hat{\alpha}$	15.54629	-210.43350	0.48951	0.03141	2.17294	5.92547	3.75252
		$\hat{\beta}$	-210.43351	3279.33360	0.03141	0.00232	0.38018	0.61426	0.23407
	5	$\hat{\alpha}$	19.31565	-264.33100	0.42115	0.02699	2.34902	5.63365	3.28464
		$\hat{\beta}$	-264.33097	4124.32300	0.02699	0.00197	0.38955	0.59730	0.20775
FLERSS	2	$\hat{\alpha}$	7.27041	-94.72299	0.72635	0.04519	1.86360	6.68444	4.82083
		$\hat{\beta}$	-94.72299	1522.38802	0.04519	0.00347	0.36242	0.65088	0.28847
	3	$\hat{\alpha}$	10.70515	-142.76810	0.57097	0.03581	2.23884	5.96009	3.72125
		$\hat{\beta}$	-142.76812	2276.45010	0.03581	0.00269	0.38372	0.61796	0.23423
	4	$\hat{\alpha}$	15.54629	-210.43350	0.48951	0.03141	2.17294	5.92547	3.75252
		$\hat{\beta}$	-210.43351	3279.33360	0.03141	0.00232	0.38018	0.61426	0.23407
	5	$\hat{\alpha}$	17.93984	-240.66080	0.37109	0.02351	2.59104	5.34831	2.75727
		$\hat{\beta}$	-240.66081	3799.11100	0.02351	0.00175	0.40528	0.58522	0.17994
MERSS	2	$\hat{\alpha}$	14.22281	-148.76610	0.20742	0.01311	3.11274	5.27070	2.15796
		$\hat{\beta}$	-148.76607	2354.00030	0.01311	0.00125	0.43279	0.58511	0.15232
	3	$\hat{\alpha}$	21.02721	-225.12810	0.14748	0.00933	3.23159	4.92635	1.69476
		$\hat{\beta}$	-225.12807	3557.46340	0.00933	0.00087	0.44090	0.56380	0.12290
	4	$\hat{\alpha}$	29.04885	-304.14470	0.10863	0.00709	3.25053	4.66510	1.41457
		$\hat{\beta}$	-304.14467	4661.78460	0.00709	0.00068	0.44716	0.55392	0.10676
	5	$\hat{\alpha}$	35.46128	-379.29750	0.09042	0.00582	3.33762	4.60800	1.27039
		$\hat{\beta}$	-379.29746	5895.81370	0.00582	0.00054	0.45087	0.54586	0.09499

Table 18
Efficiency and Asymptotic Efficiency of MERSS when $\alpha = 4.0, \beta = 0.5$ for different cycles h.

		Efficiency		Asymptotic Efficiency	
Sampling	h	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
SRS	2	1.073533	1.021956	5.795746	5.493002
	3	1.042127	1.014394	6.043929	5.652466
	4	1.045721	1.005090	6.425024	5.608347
	5	1.013716	1.001348	6.101925	5.666158
QRSS	2	1.069332	1.009651	2.762103	2.294811
	3	1.039657	1.006412	2.907232	2.418418
	4	1.056001	1.008303	3.036885	2.422000
	5	1.031579	1.002723	3.157809	2.542178
MRSS	2	1.060468	1.007190	3.926413	2.965705
	3	1.022360	0.999426	4.058762	3.265155
	4	1.023092	0.993373	4.506344	3.428378
	5	1.004663	0.990090	4.657791	3.626820
FLERSS	2	1.019635	0.995476	3.501908	2.767956
	3	1.005023	0.996997	3.871371	3.080053
	4	1.030912	0.998312	4.144899	3.077274
	5	0.999211	0.993760	4.104131	3.222161

Table 19
Coefficients to estimate scale parameter σ when location parameter μ and shape parameter α is known in RSS scheme.

		α				
n	r	0.5	1.5	2.0	2.5	3.00
5	1	0.9825037182988920	0.2556071091522490	0.2015208067487440	0.1720787428379490	0.1535003720349740
	2	0.6989089804924450	0.2461837940162550	0.2036932203153850	0.1788968220464130	0.1624192456164390
	3	0.5168956386817980	0.2140367156373320	0.1801118147727140	0.1594475221370020	0.1453157801333100
	4	0.3659685036212270	0.1638585273477010	0.1382756996923510	0.1223196288902010	0.1112520842326400
	5	0.2049121323589960	0.0913781327275358	0.0762596281534183	0.0667976461552510	0.0602361605950205
6	1	0.9311736217394010	0.2145725854023740	0.1665457605716850	0.1409513498884230	0.1250306778058400
	2	0.6716113190940580	0.2138079344832530	0.1749630393445690	0.1527689231794230	0.1382267364215350
	3	0.5127666257143390	0.1960863455654850	0.1638574677071430	0.1446008466798310	0.1315923484106400
	4	0.3922014084520480	0.1668000110489090	0.1405019073211970	0.1243330645523190	0.1132114686857780
	5	0.2816758767168000	0.1258553741581200	0.1059254147683600	0.0934827625620745	0.0848514488318376
	6	0.1574633549216760	0.0691126264744145	0.0574578058936571	0.0501872908390753	0.0451583200074794
7	1	0.8913479418121770	0.1839444023442580	0.1407932975481710	0.1182142189328370	0.1043403025768700
	2	0.6478299686511880	0.1876525730492450	0.1519808831316030	0.1319711710504620	0.1190198271798910
	3	0.5030102294527440	0.1778314551215940	0.1475082529691740	0.1297049640670570	0.1178135435584730
	4	0.3986560904108510	0.1596012885126730	0.1338725573232550	0.1182911105502020	0.1076727807878570
	5	0.3108713858264440	0.1340291249065090	0.1128363865314360	0.0997409008344962	0.0907079048419786
	6	0.2250623376454290	0.1000990309584820	0.0840316130471202	0.0740022856681434	0.0670492816236821
	7	0.1254290631792020	0.0543092467410219	0.0450071038015761	0.0392208199138988	0.0352273199574955

Table 19 Continued
Coefficients to estimate scale parameter σ when location parameter μ and shape parameter α is known in RSS scheme.

		α				
n	r	0.5	1.5	2.0	2.5	3.00
8	1	0.8594417609252830	0.1603079562732850	0.1211648461707940	0.1010092080037990	0.0887560023340842
	2	0.6274096834852850	0.1664102789554650	0.1334867371918160	0.1153149386438300	0.1036791166631800
	3	0.4919305631409550	0.1612597154887360	0.1327867926046160	0.1163314501016120	0.1054525470481230
	4	0.3974591128375520	0.1495134807450960	0.1247779938933460	0.1100131065558040	0.1000422747332710
	5	0.3220719951279840	0.1325362408434100	0.1113174823217000	0.0983614558575003	0.0894878706423606
	6	0.2541729099659420	0.1103338912130860	0.0927824052355158	0.0819070407451697	0.0743953932362284
	7	0.1848632249102330	0.0817625741513732	0.0684755170300670	0.0601876926910047	0.0544470925239362
	8	0.1026453052595070	0.0439257554393521	0.0363038844017596	0.0315746552422584	0.0283169540866984
9	1	0.8332472256814700	0.1415767674313750	0.1057856138217920	0.0876174896653086	0.0766764681231556
	2	0.6098595449847390	0.1489609857350450	0.1184245878980550	0.1018113463055070	0.0912747980672424
	3	0.4810306429565350	0.1467201675363420	0.1199627403575010	0.1047172643604490	0.0947314184286794
	4	0.3931419097184600	0.1390499724449870	0.1154282731733830	0.1015167631464420	0.0922023127208415
	5	0.3253908535198030	0.1273467483790180	0.1065916408701980	0.0940695327982823	0.0855558415443944
	6	0.2674504642939120	0.1119683942761280	0.0940604320809770	0.0830688178839563	0.0755180223686782
	7	0.2127138957264320	0.0926123208286863	0.0777744080195515	0.0685668576859930	0.0622036865149020
	8	0.1551135225499820	0.0682030499598977	0.0569959960118946	0.0500123178637840	0.0451797807436336
	9	0.0857896494178625	0.0363389819212962	0.0299638116512422	0.0260168320342313	0.0233025273769646

Table 19 Continued
Coefficients to estimate scale parameter σ when location parameter μ and shape parameter α is known in RSS scheme.

		α				
n	r	0.5	1.5	2.0	2.5	3.00
10	1	0.8113215524265910	0.1264087980606120	0.0934613820461486	0.0769506579696280	0.0670915833934818
	2	0.5946876600342840	0.1344478246526180	0.1059967389269380	0.0907172873980787	0.0811095569533571
	3	0.4708312581727480	0.1340875337767470	0.1088937997126810	0.0947232418476488	0.0855194955954219
	4	0.3876129878137750	0.1291100074854660	0.1066036376392900	0.0935128472694942	0.0848174880287778
	5	0.3249696932803470	0.1208559813152780	0.1007629016534910	0.0887788276352873	0.0806891024445374
	6	0.2732354391569720	0.1098058714859300	0.0920362226305032	0.0812376098786507	0.0738634456771827
	7	0.2267495764688250	0.0959779401291645	0.0805973338403488	0.0711247714834190	0.0646055748182554
	8	0.1812871425763170	0.0789863320326817	0.0662377821137959	0.0583210558893095	0.0528493831480433
	9	0.1323794446157380	0.0578681256487973	0.0482643559403860	0.0422861155208278	0.0381534267130871
	10	0.0729278300615333	0.0306140807553395	0.0251921762434510	0.0218419019247569	0.0195413371578559

Table 20
Coefficients to estimate location parameter μ and scale parameter σ when shape parameter α is known in RSS scheme.

		α					
		0.5		1.5		2	
n	r	μ	σ	μ	σ	μ	σ
5	1	0.9371721962968540	-1.6127031939134200	1.0057659438473000	-0.7210562718329800	1.0475467128469800	-0.6363711322862210
	2	0.1263434017065880	0.3490402256179710	0.2575459954789230	-0.0039099223621939	0.2970699471333020	-0.0339214950762098
	3	-0.0054130425770306	0.5318853764989450	-0.0197185219683630	0.2331846679531410	-0.0271990698438764	0.2018672945923160
	4	-0.0319260776470728	0.4543778458071170	-0.1257766043928290	0.2859956949925340	-0.1627589150398060	0.2684602358523640
	5	-0.0261764777793396	0.2773997459893900	-0.1178168129650390	0.2057858312494980	-0.1546586750966030	0.1999650969177500
6	1	0.8910661046852050	-1.6947021377570000	0.9046755037157250	-0.6776499488495380	0.9313220026554380	-0.5871278698363740
	2	0.1453009235779740	0.2434251597847750	0.2930022050252430	-0.0751610591893862	0.3370282968246180	-0.0977775802196725
	3	0.0168993822365093	0.4629659679045680	0.0517069031523886	0.1450911942880840	0.0615076306597609	0.1140823317552750
	4	-0.0157542309863500	0.4386274289673030	-0.0622663221795602	0.2282092296531640	-0.0812223210578362	0.2062311839916600
	5	-0.0217610090525740	0.3458032247024160	-0.1035304055980280	0.2279606710025750	-0.1368280440628860	0.2166537003843740
	6	-0.0157511704607652	0.2038803563979370	-0.0835878841157687	0.1515499130951000	-0.1118075650190950	0.1479382339247370
7	1	0.8574088795703810	-1.7685119008103300	0.8261971798844730	-0.6401601205770590	0.8409585287539590	-0.5454541700173250
	2	0.1554914912691710	0.1654631733555850	0.3075025979756660	-0.1190711582484540	0.3518486392647340	-0.1351381951653390
	3	0.0306962323178790	0.4077841621614320	0.0956387772649416	0.0824349192465587	0.1155604365884180	0.0532074589965304
	4	-0.0040776608116843	0.4113058383937140	-0.0137658074072718	0.1733322288041690	-0.0188495255449254	0.1492543374321770
	5	-0.0142905843347982	0.3552037368265720	-0.0685814652860940	0.2024368816828560	-0.0909187846113281	0.1870288509162190
	6	-0.0150513330663156	0.2717546886986270	-0.0844587892944082	0.1843438964126500	-0.1136258078850500	0.1767536917764190
	7	-0.0101770249446339	0.1570003013744050	-0.0625324931373079	0.1166833526792770	-0.0849734865658070	0.1143480260613180

Table 20 Continued
Coefficients to estimate location parameter μ and scale parameter σ when shape parameter α is known in RSS scheme.

		α					
		0.5		1.5		2	
n	r	μ	σ	μ	σ	μ	σ
8	1	0.8318539418271510	-1.8357604816929600	0.7631546818576050	-0.6074637298358060	0.7683574485743200	-0.5097301192983480
	2	0.1613086987505680	0.1047703777540120	0.3117828068912690	-0.1472587805909710	0.3545280742415400	-0.1576147255082260
	3	0.0396780215119625	0.3633739894631240	0.1233934612602600	0.0371197369774019	0.1491660065825460	0.0103072321517541
	4	0.0041472758510251	0.3840219616594400	0.0212957637836304	0.1280888798668830	0.0260635306928206	0.1033773419871770
	5	-0.0079400321995763	0.3477976562263180	-0.0364395367056843	0.1691962328511350	-0.0485204283988120	0.1511573954454880
	6	-0.0114070131164776	0.2911315703595890	-0.0649716999369608	0.1756986629898270	-0.0875682728850678	0.1646843339544030
	7	-0.0107079065901383	0.2195567839647200	-0.0695791214739357	0.1517626418729000	-0.0950356018140243	0.1465088371157230
	8	-0.0069329860345161	0.1251081422657560	-0.0486363556761846	0.0928563558686282	-0.0669907569933237	0.0913097041520274
9	1	0.8118611638879660	-1.8976407856386800	0.7111771045842470	-0.5786873233195370	0.7085458944291370	-0.4787558950109810
	2	0.1647557598017850	0.0556643829893719	0.3107900259972790	-0.1658001251702980	0.3510550268573680	-0.1711914224351460
	3	0.0457844544416894	0.3270237470719360	0.1412659898269720	0.0036491672737088	0.1702908788881240	-0.0205251067507263
	4	0.0100349024977255	0.3593871297822360	0.0466055393783886	0.0918489359831133	0.0582343533529325	0.0673856593604142
	5	-0.0030023800551144	0.3354900725274880	-0.0103563997707529	0.1378354758932360	-0.0141650177552399	0.1182776035105750
	6	-0.0077947480855708	0.2936699523603260	-0.0435354606228329	0.1560601243934220	-0.0587198236817640	0.1425035528487390
	7	-0.0088838893286902	0.2425969692625900	-0.0588673757961712	0.1522318679561320	-0.0805130710022388	0.1441966854957710
	8	-0.0078344587017047	0.1814665867133270	-0.0581086839982711	0.1270542111895810	-0.0804135849292441	0.1233361987210830
	9	-0.0049208044580874	0.1023419449314040	-0.0389707395988595	0.0758076658006407	-0.0543146561590774	0.0747727242602696

Table 20 Continued
Coefficients to estimate location parameter μ and scale parameter σ when shape parameter α is known in RSS scheme.

		α					
		0.5		1.5		2	
n	r	μ	σ	μ	σ	μ	σ
10	1	0.7958462392636610	-1.9550420322037600	0.6674358571358950	-0.5531493597220230	0.6582851862546870	-0.4516292510171740
	2	0.1668383123968820	0.0147572549312275	0.3070042082090450	-0.1781323460581940	0.3444212325698740	-0.1791999989339170
	3	0.0500859663360415	0.2967323097361220	0.1528665051269020	-0.0215554085375671	0.1835224683784240	-0.0430713068377123
	4	0.0143477875255024	0.3377400412917350	0.0650516832826388	0.0628768233081763	0.0814199127936247	0.0391841776007862
	5	0.0007892339170436	0.3222263141448430	0.0101972203942329	0.1104735539531270	0.0127845950881987	0.0901766646006215
	6	-0.0047389122147200	0.2897079102635640	-0.0243735073986521	0.1346220625991930	-0.0329325589396476	0.1193059072225880
	7	-0.0067525202500726	0.2502213543106890	-0.0444389097856483	0.1412239714068260	-0.0607716212197738	0.1309190518049310
	8	-0.0069291160929808	0.2053727680245610	-0.0525816068895105	0.1325229521124310	-0.0728469161175405	0.1265584037111620
	9	-0.0058779657696051	0.1528112688231180	-0.0491931484068470	0.1079547443844730	-0.0688602560571009	0.1052838383027690
	10	-0.0036090251117531	0.0854728106778992	-0.0319683016680578	0.0631630065535555	-0.0450220427507480	0.0624725135459441

Table 20 Continued
Coefficients to estimate location parameter μ and scale parameter σ when shape parameter α is known in RSS scheme.

		α			
		2.5		3	
n	r	μ	σ	μ	σ
5	1	1.0872382569581200	-0.5884883010874330	1.1242102306770400	-0.5578140201811140
	2	0.3279278048431980	-0.0505019132853736	0.3530315493334840	-0.0609521622349368
	3	-0.0346507355073740	0.1836871101997120	-0.0418987264002011	0.1718260949220650
	4	-0.1944686972692280	0.2583583317885800	-0.2221336234717110	0.2518012796223490
	5	-0.1860466290247200	0.1969447723845140	-0.2132094301386150	0.1951388078716360
6	1	0.9593069427612370	-0.5366303951788260	0.9867306732177440	-0.5045735497458260
	2	0.3715872498757820	-0.1096921578285540	0.3999104252860040	-0.1169445086163610
	3	0.0683101182020424	0.0963517545134984	0.0731689957715207	0.0849053340977761
	4	-0.0978963498882259	0.1934796292608450	-0.1127598767432280	0.1851602760175990
	5	-0.1653751907940110	0.2102912681343360	-0.1902214704721620	0.2062262527485080
	6	-0.1359327701568250	0.1461999010987000	-0.1568287470598790	0.1452261954983040
7	1	0.8598640884094870	-0.4932742332298150	0.8799314747334550	-0.4604269610485820
	2	0.3866662343376750	-0.1430047702940110	0.4152825395163010	-0.1475213640476360
	3	0.1306270560895860	0.0368101247366515	0.1424797880196170	0.0263656043582659
	4	-0.0239996619358493	0.1353583614418230	-0.0290657801744380	0.1263280983956610
	5	-0.1102766697674500	0.1781636550984660	-0.1272935978623650	0.1724088770268220
	6	-0.1386675948744840	0.1726151177399660	-0.1604451511598200	0.1700279471043680
	7	-0.1042134522589650	0.1133317445069180	-0.1208892730727520	0.1128177982111000

Table 20 Continued
Coefficients to estimate location parameter μ and scale parameter σ when shape parameter α is known in RSS scheme.

		α			
		2.5		3	
n	r	μ	σ	μ	σ
8	1	0.7800578411708370	-0.4564977783708090	0.7943282550924520	-0.4232499209378230
	2	0.3880432150782540	-0.1620193717153570	0.4156321893110000	-0.1642279374613560
	3	0.1689387912985090	-0.0044090276522345	0.1847553664345540	-0.0136365592052521
	4	0.0290321284596696	0.0892638574728169	0.0308427860653742	0.0801617164698091
	5	-0.0592989524917786	0.1407423903908180	-0.0690076153296101	0.1339686096480720
	6	-0.1070786752847870	0.1584361215546180	-0.1241435763084830	0.1544155184157150
	7	-0.1169299355019300	0.1437574647233090	-0.1359648349582160	0.1420869321103940
	8	-0.0827644127287766	0.0907263435968392	-0.0964425703070708	0.0904816409604398
9	1	0.7144231573252750	-0.4249076985493890	0.7240422654241970	-0.3915217382592520
	2	0.3825691096640880	-0.1726426700925680	0.4085447869818260	-0.1729085868346150
	3	0.1926192081538450	-0.0334672204236396	0.2105753521479720	-0.0414360498147144
	4	0.0667964720184707	0.0535971596980941	0.0733215386732039	0.0447893169119851
	5	-0.0180682377874332	0.1070316363887960	-0.0219386118027872	0.0997423320120014
	6	-0.0719924380264943	0.1347159929107370	-0.0837341277951148	0.1296642653696980
	7	-0.0991901720487552	0.1397256115380750	-0.1154880192102520	0.1368834200539620
	8	-0.0996300092157938	0.1214866096775190	-0.1163350365694630	0.1204072337008760
	9	-0.0675270900832030	0.0744605788523750	-0.0789881478495829	0.0743798068600571

Table 20 Continued
Coefficients to estimate location parameter μ and scale parameter σ when shape parameter α is known in RSS scheme.

		α			
		2.5		3	
n	r	μ	σ	μ	σ
10	1	0.6593861356846680	-0.3974739192434810	0.6652180082421070	-0.3641296003187120
	2	0.3736563503750420	-0.1781263331393840	0.3977895763293510	-0.1767537147076290
	3	0.2070769547575250	-0.0542674502441825	0.2260509766365550	-0.0610158785477199
	4	0.0937929197895855	0.0260293745232052	0.1035106984663100	0.0177176720790071
	5	0.0141514891303595	0.0785969116314946	0.0147367982218488	0.0711361145599391
	6	-0.0406588746845919	0.1104914391423930	-0.0476791635013247	0.1047710054075340
	7	-0.0749541137339232	0.1250538303370390	-0.0874108149006616	0.1212687959032020
	8	-0.0903397422059310	0.1233199869891980	-0.1055849658932500	0.1212938230317690
	9	-0.0858302745414138	0.1040405102642260	-0.1005817831331440	0.1033546014334980
	10	-0.0562808445713217	0.0623356497394895	-0.0660493304677920	0.0623571811591102

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