New Soliton Solutions of The Nonlinear Radhakrishnan-Kundu-Lakshmanan Equation With the Beta-Derivative

Yusuf Pandir (✉ yusufpandir@gmail.com)

Bozok Universitesi

Yusuf Gurefe

Usak University: Usak Universitesi

Tolga Akturk

Ordu University: Ordu Universitesi

Research Article

Keywords: Radhakrishnan-Kundu-Lakshmanan equation, Atangana's conformable beta-derivative, the modified exponential function method, new soliton solutions

Posted Date: October 18th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-866518/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Version of Record: A version of this preprint was published at Optical and Quantum Electronics on March 13th, 2022. See the published version at <https://doi.org/10.1007/s11082-022-03585-z>.

New soliton solutions of the nonlinear Radhakrishnan-Kundu-Lakshmanan equation with the beta-derivative

Yusuf Pandir^{1,*}, Yusuf Gurefe², Tolga Akturk³

¹*Department of Mathematics, Faculty of Arts and Sciences, Yozgat Bozok University, 66100 Yozgat/Turkey*

²*Department of Econometrics, Usak University, 64100 Usak/Turkey*

³*Department of Mathematics and Science Education, Faculty of Education, Ordu University, 52200 Ordu/Turkey*

* Corresponding author, e-mail: yusufpandir@gmail.com

Abstract

In this article, the modified exponential function method is applied to find the exact solutions of the Radhakrishnan-Kundu-Lakshmanan equation with Atangana's conformable beta-derivative. The definition of the conformable beta derivative and its properties proposed by Atangana are given. With the proposed method, exact solutions of the nonlinear Radhakrishnan-Kundu-Lakshmanan equation which can be stated with the conformable beta-derivative of Atangana are obtained. The exact solutions found as a result of the application of the method seem to be 1-soliton solutions, dark soliton solutions, periodic soliton solutions and rational function solutions. According to the obtained results, we can say that the Radhakrishnan-Kundu-Lakshmanan equation with Atangana's conformable beta-derivative have different soliton solutions. Also, three-dimensional contour and density graphs and two-dimensional graphs drawn with different parameters are given of these new exact solutions.

Keywords: Radhakrishnan-Kundu-Lakshmanan equation; Atangana's conformable beta-derivative; the modified exponential function method; new soliton solutions.

1. Introduction

In recent years, many research articles have been made to find exact solutions of the problems that can be modeled mathematically by using fractional derivatives to understand some physical phenomena [1-7]. Such physical phenomena are often explained by nonlinear FPDEs. Fractional differential equations have applications in many fields such as physics, dynamics, signal processing, control theory, continuum mechanics, solid-state physics, engineering, chemistry, biology. There are different definitions of fractional derivative

operators in the literature. The most well-known of these are Jumarie, Caputo, Riemann-Liouville, Liouville-Caputo, Caputo-Fabrizio, Atangana-Baleanu [8-12].

Using these derivative operators, distinct methods have been improved to obtain exact, semi-analytical and approximate solutions of the nonlinear fractional equations such as first integral method [13], extended trial equation method [14], modified trial equation method [15], generalized Kudryashov method [16], finite difference method [17], Laplace transforms [18], local fractional Fourier series method [19], variational iteration method [20], homotopy perturbation method [21], Adomian decomposition method [22].

In 2014, a new fractional derivative was defined and called the conformable derivative [23]. All properties of this derivative have the fractional compound. Exact solutions of the many differential equations have been investigated using this new fractional derivative operator [24,25]. Recently, many scientists have contributed to the development of this fractional derivative by doing research on the conformable derivative. Atangana et al. gave a new definition to this fractional derivative and named it beta-derivative [26-28]. The function in this new derivative depends on the range from which it is derivated. Many equations involving this derivative are reported in some interesting studies [29-34].

In this article, the validity of the modified exponential function method was researched to determine exact solutions of equations containing Atangana's conformable beta-derivative. This method has been applied to various nonlinear physical problems and has been found to be effective.

The rest of the article is designed as follows: In sec. 2, some basic properties of the conformable beta-derivative of Atangana are given. Then, the modified exponential function method is explained in detail for fractional partial differential equations with beta-derivative. In sec. 4, the application of the method is given. This article is completed with a conclusion.

2. Beta-derivatives

The fractional conformable beta-derivative in the mathematical model in this study is widely used in physical problems. The definition of this derivative is as follows.

Definition: Beta-derivative was introduced to the literature by Atangana *et al.* The definition of this effective derivative is given below [26-28]:

$${}^A D_x^\beta \{f(x)\} = \lim_{\varepsilon \rightarrow 0} \frac{f\left(x + \varepsilon \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{\varepsilon}. \quad (1)$$

The most important reason for choosing the fractional derivative of Atangana for the mathematical model used in this study is that it can provide various properties of the fundamental derivatives. These properties are stated below:

- i) Let a and b the real numbers. f and $g \neq 0$ are differentiable functions with respect to β in the interval $(0,1]$. According to these conditions, the following feature is provided,

$${}_0^A D_x^\beta \{a f(x) + b g(x)\} = a {}_0^A D_x^\beta \{f(x)\} + b {}_0^A D_x^\beta \{g(x)\}. \quad (2)$$

- ii) Where c is any constant that satisfies the following equation,

$${}_0^A D_x^\beta \{c\} = 0. \quad (3)$$

iii) .
$${}_0^A D_x^\beta \{f(x)g(x)\} = g(x) {}_0^A D_x^\beta \{f(x)\} + f(x) {}_0^A D_x^\beta \{g(x)\} \quad (4)$$

iv)
$${}_0^A D_x^\beta \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) {}_0^A D_x^\beta \{f(x)\} - f(x) {}_0^A D_x^\beta \{g(x)\}}{g^2(x)}. \quad (5)$$

If $\varepsilon = \left(x + \frac{1}{\Gamma(\beta)} \right)^{\beta-1}$ h is substituted in equation (1) and $h \rightarrow 0$, when $\varepsilon \rightarrow 0$, we obtain,

$${}_0^A D_x^\beta \{f(x)\} = \left(x + \frac{1}{\Gamma(\beta)} \right)^{1-\beta} \frac{df(x)}{dx}, \quad (6)$$

together with

$$\eta = \frac{\delta}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta, \quad (7)$$

δ is any constant and, accordingly, the following equation can be written

$${}_0^A D_x^\beta \{f(\eta)\} = \delta \frac{df(\eta)}{d\eta}. \quad (8)$$

3. The Modified Exponential Function Method

In this section, obtaining exact solutions of mathematical models represented as Atangana derivatives using the modified exponential function method will be explained in detail [35].

According to the this method, first of all, the general form of the studied nonlinear fractional differential equations should be determined. The u function and its derivative terms used in the mathematical model analyzed in the study are written as follows:

$$P(u, {}_0^A D_x^\beta u, {}_0^A D_t^\beta u, {}_0^A D_x^{2\beta} u, L) = 0, \quad (9)$$

where x space and t represents time. According to the method, after the general form of the investigated equation is obtained, the operations are performed according to the following steps:

Step 1: In this part, the complex wave transform is arranged according to the independent variable that the solution function u in the nonlinear fractional differential equation depends on:

$$u(x, t) = u(\xi), \quad \xi = \left(\frac{1}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta - \frac{\nu}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta \right) e^{i \left(-\frac{\kappa}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{\omega}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta \right)}, \quad (10)$$

where ν , κ and ω are constants. According to the method, if the derivative terms in equation (9), which is considered as the general form of the mathematical model, are obtained by using wave transform (10) and written instead,

$$N(u, u', u'', L) = 0. \quad (11)$$

Step 2: In this part, the assumed function as the wave solution of the mathematical model according to the method;

$$u(\xi) = \frac{\sum_{j=0}^n A_j [e^{-g(\xi)}]^j}{\sum_{i=0}^m B_i [e^{-g(\xi)}]^i} = \frac{A_0 + A_1 e^{-g} + K + A_n e^{-ng}}{B_0 + B_1 e^{-g} + K + B_m e^{-mg}}, \quad (12)$$

where $A_j, B_i, (0 \leq j \leq n, 0 \leq i \leq m)$ are constants. The relation for $g(\xi)$ and its derivative in equation (12) is given as follows.

$$g'(\xi) = e^{-g(\xi)} + \mu e^{g(\xi)} + \lambda. \quad (13)$$

In equation (13), if it is arranged and integrated according to the states of the roots on the right side of the equation, the following situations are obtained [35]:

Family 1: If, $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$g(\xi) = \ln \left(\frac{-\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\eta + E) \right) - \frac{\lambda}{2\mu} \right). \quad (14)$$

Family 2: If, $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$g(\xi) = \ln \left(\frac{\sqrt{4\mu - \lambda^2}}{2\mu} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\eta + E) \right) - \frac{\lambda}{2\mu} \right). \quad (15)$$

Family 3: If, $\mu = 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$g(\xi) = -\ln \left(\frac{\lambda}{e^{\lambda(\eta+E)} - 1} \right). \quad (16)$$

Family 4: If, $\mu \neq 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu = 0$,

$$g(\xi) = \ln \left(-\frac{2\lambda(\eta + E) + 4}{\lambda^2(\eta + E)} \right). \quad (17)$$

Family 5: If, $\mu = 0$, $\lambda = 0$ and $\lambda^2 - 4\mu = 0$,

$$\mathcal{G}(\xi) = \ln(\eta + E), \quad (18)$$

where E is the integration constant obtained by integrating equation (13). The λ and μ terms are also constants that the method must provide to their family state.

Step 3: In this section, which is the last part, the limits of the symbols for the last part are determined in equation (13), which is accepted as the solution of the mathematical model. In order to determine this part, the balancing principle must be used. The operation of this technique is as follows: a relationship between n and m is obtained by balancing the term containing the highest order derivative in the nonlinear ordinary differential equation of the studied mathematical model with the term of the highest order. Then, the n value is obtained by giving an arbitrary value to the m term in the found relation. In this way, the limits of the solution function are determined. After all these operations, the u function required in equation (11) and the derivative terms of u with respect to ξ are formed from equation (13) and written in their place. Then, by arranging the $e^{\mathcal{G}(\xi)}$ term and its powers, a system of algebraic equations consisting of $A_0, A_1, A_2, \dots, A_n, B_0, B_1, B_2, \dots, B_m$ is obtained. The obtained coefficients are replaced in the solution function and it is checked with the help of the package program that it provides the equation. Finally, by giving appropriate values to the parameters in the solution function determined to provide the equation, taking into account the family conditions specified in the method, the three-dimensional, density, contour graphics representing the behavior of the mathematical model and the two-dimensional graphics within the appropriate t time value are obtained using the package program.

4. Applications of the nonlinear Radhakrishnan-Kundu-Lakshmanan (RKL) equation with beta-derivatives

In this section, the derivation of the wave solutions of the nonlinear Radhakrishnan-Kundu-Lakshmanan (RKL) equation using the modified exponential function method with the help of the β derivative is analyzed. First, let's physically comment on the RLK equation whose wave solution is being investigated because such equations are mathematical models and represent a physical event. For this reason, before analyzing the wave solutions of such equations, information about the physical phenomenon represented by the model should be given. The nonlinear RKL equation is a mathematical model describing the propagation and dynamics of light pulses.

The nonlinear Radhakrishnan-Kundu-Lakshmanan (RKL) equation with the preferred beta derivatives for the application of the method is as follows [31],

$$i_0^A D_t^\beta \{u\} + a_0^A D_x^{2\beta} \{u\} + b|u|^2 u = i \left[\rho_0^A D_x^\beta \{u\} + \kappa_0^A D_x^\beta \{|u|^2 u\} + \theta_0^A D_x^\beta \{|u|^2\} u - \nu_0^A D_x^{3\beta} \{u\} \right]. \quad (19)$$

The function $u(x, t)$ is a function of complex variables with arguments x and t . In addition, $a, b, \rho, \kappa, \theta, \nu$ given in the equation are constants.

According to the method used in the article, firstly, the following wave transform is applied by considering the independent variables in the RKL equation, which is investigated with beta derivatives. In this way, the nonlinear fractional differential equation is reduced to an ordinary differential equation.

$$u(x, t) = u(\xi), \quad \xi = \left(\frac{1}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta - \frac{\nu}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta \right) e^{i \left(\frac{\kappa}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{\omega}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta \right)}. \quad (20)$$

When the derivative concepts in equation (19) are obtained from the complex wave transformation equation (20) and substituted, the real and imaginary parts of equation (19) are get, respectively.

$$(a + 3\kappa\gamma)u'' - (\omega + \rho\kappa + a\kappa^2 + \gamma\kappa^3)u + (b - \nu\kappa)u^3 = 0, \quad (21)$$

$$3\gamma u'' - 3(\nu + \rho + 2a\kappa + 3\gamma\kappa^2)u - (3\zeta + 2\theta)u^3 = 0. \quad (22)$$

The balance procedure is applied considering the equations (21) and (22) obtained. In other words, by equalizing the term u^3 with the highest order derivative and the nonlinear term u'' in these equations, the relation $n = m + 1$ is found regarding the m and n limits in the equation (12). Then $n = 2$ is obtained by choosing $m = 1$.

Considering the m and n values obtained above, equation (12) is written as follows.

$$u(\xi) = \frac{\Upsilon}{\phi} = \frac{A_0 + A_1 e^{-\theta} + A_2 e^{-2\theta}}{B_0 + B_1 e^{-\theta}}. \quad (23)$$

Derivative terms required in equation (21) and (22) are obtained from equation (23) as follows;

$$u'(\xi) = \frac{\Upsilon' \phi - \Upsilon \phi'}{\phi^2}, \quad (24)$$

$$u''(\xi) = \frac{\left(\left(\Upsilon'' \phi^3 + \Upsilon' \phi' \phi^2 - (\phi^2 \Upsilon' \phi' + \phi^2 \Upsilon \phi'') \right) - 2\phi \phi' (\Upsilon' \phi - \Upsilon \phi') \right)}{\phi^4}. \quad (25)$$

Then the following steps are followed so that equations (19) and (20) can be combined into a single equation:

- The coefficients in equation (21) are determined first. Then, these coefficients obtained are substituted in equation (22).

- In the Mathematica package program, the recently formed equation (22) is written into the factor command, and its solutions are obtained.

Considering all these steps, the simplest form of equations (19) and (20) are as follows:

$$-\frac{a}{\kappa}u'' - 3(\nu + \rho + 2a\kappa - a\kappa)u - \left(3\frac{b}{k} + 2\theta\right)u^3 = 0. \quad (26)$$

If equation (23-25) is replaced in equation (26) and the equation obtained is arranged according to the powers of $e^{\theta(\xi)}$, the algebraic equation system is found. With the solution of this system, the following cases are obtained.

Case 1.

$$A_1 = \frac{\sqrt{a}(2B_0 + \lambda B_1)}{\sqrt{-6b - 4\theta\kappa}}, \quad A_0 = \frac{\sqrt{a}\lambda B_0}{\sqrt{-6b - 4\theta\kappa}}, \quad A_2 = \frac{\sqrt{2a}B_1}{\sqrt{-3b - 2\theta\kappa}}, \quad \nu = \frac{a(\lambda^2 - 6\kappa^2 - 4\mu) - 6\kappa\rho}{6\kappa}. \quad (27)$$

These coefficients, which are obtained by solving the algebraic system of equations, are first substituted in equation (23) then, according to the method, the solution function of equation (19) is obtained by substituting the phi function in equation (23) under family conditions.

Family 1: When $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$u_{1,1}(x, t) = \sqrt{\frac{a}{-6b - 4\theta\kappa}} \frac{(\lambda^2 - 4\mu + \lambda\psi)}{(\lambda + \psi)}, \quad (28)$$

$$\text{where } \psi = \sqrt{\lambda^2 - 4\mu} \tanh\left[\frac{\sqrt{\lambda^2 - 4\mu}}{2}(E + \xi)\right].$$

Let's construct the graphs of this solution function by determining the appropriate parameters according to the family condition.

Fig.1. Three-dimensional contour and density plots representing the real and imaginary parts of equation (28) for $b = -2$, $\mu = 1$, $\kappa = -0.2$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0.65$, $\rho = 2.5$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 10$, $A_0 = 1.62317$, $A_1 = 1.92321$, $A_2 = 0.319715$, $\nu = -62.35$, $\gamma = 1.25$, $\omega = 0.48$, $E = 0.75$, $\nu = 0.85$ values and two-dimensional $t = 1$.

Family 2: When $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$u_{1,2}(x,t) = \sqrt{\frac{a}{-6b-4\theta\kappa}} \frac{(\lambda^2 - 4\mu - \lambda\phi)}{(\lambda - \phi)}, \quad (29)$$

where $\phi = \sqrt{4\mu - \lambda^2} \tan\left[\frac{\sqrt{4\mu - \lambda^2}}{2}(E + \xi)\right]$.

Fig.2. Three-dimensional contour and density plots representing the real and imaginary parts of equation (29) for $b = -2$, $\mu = 4$, $\kappa = -4$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0.65$, $\rho = 2.5$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 0.5$, $A_0 = 0.0639042$, $A_1 = 0.318553$, $A_2 = 0.251744$, $\nu = 0.992188$, $\gamma = 0.0625$, $\omega = 2$, $E = 0.75$, $\nu = 0.85$ values and two-dimensional $t = 1$.

Family 3: When $\mu = 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$u_{1,3}(x,t) = \lambda \sqrt{\frac{a}{-6b-4\theta\kappa}} \left(1 + \frac{2}{e^{\lambda(E+\xi)} - 1} \right). \quad (30)$$

Fig.3. Three-dimensional contour and density plots representing the real and imaginary parts of equation (30) for $b = -2$, $\mu = 0$, $\kappa = -4$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0.65$, $\rho = 2.5$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 0.5$, $A_0 = 0.0639042$, $A_1 = 0.318553$, $A_2 = 0.251744$, $\nu = 0.492188$, $\gamma = 0.0625$, $\omega = 2$, $E = 0.75$, $\nu = 0.85$ values and two-dimensional $t = 1$.

Family 4: When $\mu \neq 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu = 0$,

$$u_{1,4}(x, t) = \beta \lambda \sqrt{\frac{2a}{-3b-2\theta\kappa}} \frac{e^{\frac{i\kappa\left(x+\frac{1}{\Gamma[\beta]}\right)^\beta}{\beta}}}{\beta(2+E\lambda)e^{\frac{i\kappa\left(x+\frac{1}{\Gamma[\beta]}\right)^\beta}{\beta}} + \lambda\varpi e^{\frac{i\omega\left(t+\frac{1}{\Gamma[\beta]}\right)^\beta}{\beta}}}, \quad (31)$$

$$\text{where } \varpi = \left(-\nu \left(t + \frac{1}{\Gamma[\beta]} \right)^\beta + \left(x + \frac{1}{\Gamma[\beta]} \right)^\beta \right).$$

Fig.4. Three-dimensional contour and density plots representing the real and imaginary parts of equation (31) for $b = -2$, $\mu = 1$, $\kappa = -1$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0.65$, $\rho = 2.5$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 2$, $A_0 = 0.305521$, $A_1 = 0.606412$, $A_2 = 0.300892$, $\nu = -1.75$, $\gamma = 0.25$, $\omega = 2$, $E = 0.75$, $\nu = 0.85$ values and two-dimensional $t = 1$.

Family 5: When $\mu = 0$, $\lambda = 0$ and $\lambda^2 - 4\mu = 0$,

$$u_{1,5}(x,t) = \beta \sqrt{\frac{2a}{-3b-2\theta\kappa}} \frac{e^{\frac{i\kappa\left(x+\frac{1}{\Gamma[\beta]}\right)^\beta}{\beta}}}{\frac{\beta E e^{\frac{i\kappa\left(x+\frac{1}{\Gamma[\beta]}\right)^\beta}{\beta}}}{\beta} + \frac{\varpi e^{\frac{i\omega\left(t+\frac{1}{\Gamma[\beta]}\right)^\beta}{\beta}}}{\beta}}. \quad (32)$$

Fig.5. Three-dimensional contour and density plots representing the real and imaginary parts of equation (32) for $b = 0$, $\mu = 0$, $\kappa = -1$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0.65$, $\rho = 2.5$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 0$, $A_0 = 0$, $A_1 = 0.808332$, $A_2 = 0.796084$, $\nu = -1.75$, $\gamma = 0.25$, $\omega = 2$, $E = 0.75$, $\nu = 0.85$ values and two-dimensional $t = 1$.

Case 2.

$$A_1 = \frac{i\sqrt{a}B_1(2\mu B_1 - \lambda B_0)}{\sqrt{6b + 4\theta\kappa B_0}}, \quad A_0 = -\frac{i\sqrt{a}(\lambda B_0 - 2\mu B_1)}{\sqrt{6b + 4\theta\kappa}}, \quad A_2 = 0, \quad \nu = \frac{(a(\lambda^2 - 6\kappa^2) - 6\kappa\rho)B_0^2 - 4a\lambda\mu B_0 B_1 + 4a\mu^2 B_1^2}{6\kappa B_0^2} \quad (33)$$

These coefficients obtained with the help of the package program are written in equation (23). Then, the derivative terms required for equation (26) are substituted in equation (23). According to all these cases, solution functions satisfying equation (19) are obtained under the following conditions.

Family 1: If $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$u_{2,1}(x,t) = \frac{A_2}{6B_0} \left(\lambda^2 - \frac{12\lambda\mu}{\lambda + \psi} + 2\mu \left(1 + \frac{12\mu}{(\lambda + \psi)^2} \right) \right), \quad (34)$$

where $\psi = \sqrt{\lambda^2 - 4\mu} \tanh \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2} (E + \xi) \right]$.

Fig.6. Three-dimensional contour and density plots representing the real and imaginary parts of equation (34) for $b = 0.9$, $\mu = 1$, $\kappa = -0.205379$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0$, $\rho = 6.34775$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 3$, $A_0 = 0.458333$, $A_1 = 0.75$, $A_2 = 0.25$, $\nu = -1.75$, $\gamma = 1.32$, $\omega = 1.6$, $E = 0.5$, $\nu = 0.85$ values and two-dimensional $t = 1$.

Family 2: When, $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$u_{2,2}(x,t) = \frac{A_2}{6B_0} \left(\lambda^2 - \frac{12\lambda\mu}{\lambda - \phi} + 2\mu \left(1 + \frac{12\mu}{(\lambda - \phi)^2} \right) \right), \quad (35)$$

where $\phi = \sqrt{4\mu - \lambda^2} \tan \left[\frac{\sqrt{4\mu - \lambda^2}}{2} (E + \xi) \right]$.

Fig.7. Three-dimensional contour and density plots representing the real and imaginary parts of equation (35) for $b = 0.9$, $\mu = 3$, $\kappa = -0.205379$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0$, $\rho = 11.2792$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 1$, $A_0 = 0.291667$, $A_1 = 0.75$, $A_2 = 0.25$, $\nu = -1.75$, $\gamma = 1.32$, $\omega = 1.6$, $E = 0.5$, $\nu = 0.6$ values and two-dimensional $t = 1$.

Family 3: When $\mu = 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$u_{2,3}(x,t) = \frac{\lambda^2 A_2}{6B_0} \left(1 + \frac{6}{(e^{\lambda\Lambda} - 1)^2} + \frac{6}{e^{\lambda\Lambda} - 1} \right), \quad (36)$$

where $\Lambda = (E + \xi)$.

Fig.8. Three-dimensional contour and density plots representing the real and imaginary parts of equation (36) for $b = 0.9$, $\mu = 0$, $\kappa = -0.205379$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0$, $\rho = 7.58061$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 1$, $A_0 = 0.041667$, $A_1 = 0.25$, $A_2 = 0.25$, $\nu = -1.75$, $\gamma = 1.32$, $\omega = 1.6$, $E = 0.5$, $\nu = 0.6$ values and two-dimensional $t = 1$.

Family 4: When $\mu \neq 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu = 0$,

$$u_{2,4}(x,t) = \frac{A_2 \left(4\mu + \lambda^2 \frac{12e^{\frac{2i\kappa \left(x + \frac{1}{\Gamma[\beta]}\right)^\beta}}}{\beta^2} \frac{\beta^2}{\beta(2+E\lambda)e^{\frac{i\kappa \left(x + \frac{1}{\Gamma[\beta]}\right)^\beta}} + \lambda\tau e^{\frac{i\omega \left(x + \frac{1}{\Gamma[\beta]}\right)^\beta}} - 1} \right)}{12B_0}, \quad (37)$$

where, $\tau = \left(-\nu \left(t + \frac{1}{\Gamma[\beta]} \right)^\beta + \left(x + \frac{1}{\Gamma[\beta]} \right)^\beta \right)$.

Fig.9. Three-dimensional contour and density plots representing the real and imaginary parts of equation (37) for $b = 0.9$, $\mu = 1$, $\kappa = -0.205379$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0$, $\rho = 7.88883$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 2$, $A_0 = 0.25$, $A_1 = 0.5$, $A_2 = 0.25$, $\nu = -1.75$, $\gamma = 1.32$, $\omega = 1.6$, $E = 0.5$, $v = 0.6$ values and two-dimensional $t = 1$.

Family 5: When $\mu = 0$, $\lambda = 0$ and $\lambda^2 - 4\mu = 0$,

$$u_{2,5}(x,t) = \frac{A_2}{B_0(\Lambda)^2}. \quad (38)$$

Fig.10. Three-dimensional contour and density plots representing the real and imaginary parts of equation (38) for $b = 0.9$, $\mu = 0$, $\kappa = -0.205379$, $a = 0.75$, $B_0 = 0.66$, $B_1 = 0$, $\rho = 7.88883$, $\theta = 0.5$, $\beta = 0.5$, $\Gamma[\beta] = 1.73$, $\lambda = 0$, $A_0 = 0$, $A_1 = 0$, $A_2 = 0.25$, $\nu = -1.75$, $\gamma = 1.32$, $\omega = 1.6$, $E = 0.5$, $\nu = 0.6$ values and two-dimensional $t = 1$.

When the exact solutions of the nonlinear Radhakrishnan-Kundu-Lakshmanan (RKL) equation with Atangana's conformable beta-derivative are compared with the solutions

obtained by Aguilar [31], we can say that there are different solutions. For this reason, obtained soliton solutions by the modified exponential function method are new exact solutions that are not included in the literature. The obtained exact solutions $u_{1,1}(x,t)$ and $u_{2,1}(x,t)$ are called the dark soliton solution. Solutions of $u_{1,2}(x,t)$ and $u_{2,2}(x,t)$ are described the periodic soliton solutions. Solutions of $u_{1,3}(x,t)$ and $u_{1,5}(x,t)$ are entitled 1-soliton solution and rational function solution respectively. At the same time, three-dimensional contour and density graphs and two-dimensional graphs are plotted in Fig. 1-10 which represent with different parameters. These graphs are shown to help us understand the complex phenomena such as the propagation and dynamics of light pulses.

5. Conclusions

In this study modified exponential function method has been successfully applied to attain the new exact solutions of the nonlinear Radhakrishnan-Kundu-Lakshmanan (RKL) equation with Atangana's conformable beta-derivative. With this method, different and new exact solutions of RKL equation were obtained. These solutions contain dark soliton solutions, rational function solutions, 1-soliton solutions, periodic soliton solutions. During the finding of the solutions belonging to the mathematical model, all the calculations performed in accordance with the method and the graphics representing the physical behavior of the solution functions were plotted using Mathematica 12 according to the appropriate parameters. In the literature research, it was determined that the exact solutions obtained in the study were not found in the literature.

References

1. S. Zhang, H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A* 375 (2011) 1069-1073.
2. S. Guo, L. Mei, Y. Li, Y. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, *Phys. Lett. A* 376 (2012) 407-411.
3. Y. Pandir, Y. Gurefe, E. Misirli, The extended trial equation method for some time-fractional differential equations, *Disc. Dyn. Nature Soc.* 2013 Article ID 491359 (2013) 13 pages.
4. S. T. Demiray, Y. Pandir, H. Bulut, Generalized Kudryashov method for time-fractional differential equations, *Abst. Appl. Anal.* 2014 Article ID 901540 (2014) 1-13.
5. S. T. Demiray, Y. Pandir, H. Bulut, The investigation of exact solutions of nonlinear time-fractional Klein-Gordon equation by using generalized Kudryashov method, *AIP Conf. Proceed.* 1637 (2014) 283-289.

6. H. Yepez-Martinez, J. F. Gomez-Aguilar, Fractional sub-equation method for Hirota-Satsuma-coupled KdV equation and coupled mKdV equation using the Atangana's conformable derivative, *Waves Rand. Comp. Media* 29(4) (2019) 678-693.
7. H. Yepez-Martinez, J. F. Gomez-Aguilar, Optical solitons solution of resonance nonlinear Schrodinger type equation with Atangana's conformable derivative using sub-equation method, *Waves Rand. Comp. Media* 31(3) (2021) 573-596.
8. I. Podlubny, *Fractional Differential Equations*, (Academic Press, 1999).
9. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* 1 (2015) 73-85.
10. A. Atangana, D. Baleanu, Theory and application to heat transfer model, *Therm. Sci.* 20 (2016) 763-769.
11. M. Caputo, M. Fabrizio, Damage and fatigue described by a fractional derivative model, *J. Comput. Phys.* 293 (2014) 400-408.
12. G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, *Comput. Math. Appl.* 51(9-10) (2006) 1367-1376.
13. B. Lu, The first integral method for some time fractional differential equations, *J. Math. Anal. Appl.* 395(2) (2012) 684-693.
14. Y. Pandir, Y. Gurefe, E. Misirli, The extended trial equation method for some time-fractional differential equations, *Disc. Dyn. Nat. Soc.*, 2013 (2013) 491359.
15. Y. Pandir, Y. Gurefe, E. Misirli, New exact solutions of the time-fractional nonlinear dispersive KdV equation, *Int. J. Model. Opt.*, 3 (2013) 349-352.
16. Y. Gurefe, The generalized Kudryashov method for the nonlinear fractional partial differential equations with the beta-derivative, *Rev. Mexicana de Fisica* 66(6) (2020) 771-781.
17. G. H. Gao, Z. Z. Sun, Y. N. Zhang, A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions, *J. Comput. Phys.*, 231(7) (2012) 2865-2879.
18. H. Jafari, H. K. Jassim, Numerical solutions of telegraph and Laplace equations on cantor sets using local fractional Laplace decomposition method, *Int. J. Adv. Appl. Math. Mech.* 2(3) (2015) 144-151.
19. M. S. Hu, R. P. Agarwal, X. J. Yang, Local fractional Fourier series with application to wave equation in fractal vibrating string, *Abst. Appl. Anal.*, 2012 (2012) 567401.
20. N. Das, R. Singh, A. M. Wazwaz, J. Kumar, An algorithm based on the variational iteration technique for the Bratu-type and the Lane-Emden problem, *J. Math. Chem.* 54 (2016) 527-551.
21. D. Kumar, J. Singh, S. Kumar, Numerical computation of fractional Black-Scholes equation arising in financial market, *J. Egyptian Math. Soc.*, 22 (2014) 373-378.
22. D. Rani, V. Mishra, Modification of Laplace Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations based on Newton Raphson formula, *Eur. J. Pure Appl. Math.* 11 (2018) 202-214.
23. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264 (2014) 65-70.
24. C. Chen, Y. L. Jiang, Simplest equation method for some time-fractional partial differential equations with conformable derivative, *Comput. Math. Appl.* 75 (2018) 2978-2988.
25. M. T. Darvishi, M. Najafi, A. R. Seadawy, Dispersive bright, dark and singular optical soliton solutions in conformable fractional optical fiber Schrödinger models and its applications, *Opt. Quantum Elect.* 50:181 (2018) 1-16.

26. A. Atangana, E. F. D. Goufo, On the mathematical analysis of ebola hemorrhagic fever: deathly infection disease in West African countries, *Bio. Med. Res. Int.* Article ID 261383, (2014) 1-7.
27. A. Atangana, S. C. Oukouomi Noutchie, Model of breakbone fever via beta-derivatives, *Bio. Med. Res. Int.* Article ID 523159, (2014) 1-10.
28. A. Atangana, A novel model for the lassa hemorrhagic fever: deathly disease for pregnant women, *Neural Comput. Appl.* 26(8) (2015) 1895-1903.
29. A. Atangana, D. Baleanu, A. Alsaedi, Analysis of time fractional Hunter-Saxton equation: a model of neumatic liquid crystal, *Open Phys.* 14 (2016) 145-149.
30. H. Yopez-Martinez, J.F. Gomez-Aguilar, A. Atangana, First integral method for non-linear differential equations with conformable derivative, *Math. Model. Nat. Phenom.* 13(1) (2018) 1-22.
31. B. Ghanbari, J. F. Gomez-Aguilar, The generalized exponential rational function method for Radhakrishnan-Kundu Lakshmanan equation with β -conformable time derivative, *Rev. Mex. Fis.*, 65(5) (2019) 503-518.
32. S. T. Demiray, New solutions of Biswas-Arshed equation with beta time derivative *Optik*, 222, (2020) 165405.
33. S. Tuluce Demiray, U. Bayrakci, Soliton solutions for space-time fractional Heisenberg ferromagnetic spin chain equation by generalized Kudryashov method and modified $\exp(-\Omega(\eta))$ -expansion function method, *Rev. Mex. Fis.* 67(3) (2021) 393-402.
34. T. Akturk, Modified exponential function method for nonlinear mathematical models with Atangana conformable derivative, *Rev. Mex. Fis.* 67 (2021) 040704 1-18.
35. M. A. Rahman, The $\exp(-\Phi(\eta))$ -expansion method with application in the (1+1)-dimensional classical Boussinesq equations, *Results in Phys.* 4, (2014) 150-155.