A deformed model for N-type four-level atom and a single mode field system in the presence of the Kerr medium

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Abstract: The aim of this paper is to study the interaction between a single mode field and four-level atom in N-configuration under nonlinear medium effect. The non-resonance case and the deformation forms in the coupling interaction between the field and the atom are included. The wave function of the proposed system is obtained when the atom is prepared initially in its excited state while the field is prepared in a coherent state. The effect of the deformation and nonlinear medium on the temporal behavior of collapse-revival, field entropy and geometric phase of the system are examined. The results show that the presence of the intensity of the coupling interaction and the non-linear medium have an important influence on the properties of these phenomena.

1 Introduction

In the past few years, the famous Jaynes-Cummings model (JCM) [1] has been went through various generalizations, including the multiplicity of cavity modes [2] and the multiplicity of atoms [3]. Moreover, the two-level has been expanded to the multilevel, one of the most important of these generalizations is the study of an atom with four levels [4]. The four-level atom model has a lot of configurations for example, cascade, bi-cascade, N-type, bi-$\Lambda(V)$, tripod and etc. In particular, many forms for the realization of four-level atom interacting with a cavity field have been studied and several of their aspects have been demonstrated [5]. The four-level atomic system is one of fundamental sources of nonclassical properties of quantum information. In particular, an all-optical switching in a bi-Λ four-level atomic system both theoretically and experimentally are presented [6] and the spontaneous emission behavior when the atom inside in an optical crystal with anisotropic dispersion is also investigated [7]. The Pump-probe spectroscopy of coherently-driven bi-Λ four-level, using the master equation approach has been demonstrated [8]. A four-state of a quantum system that absorbs two photons but does not absorb a single photon has been described [9,10].

Recently, lasing without population in the Y-diagram in a bi-quantum dot nanostructure has been discussed in [11]. Also, the optical aspects of this system, using the density matrix formalism are investigated [12]. On the other hand, many features of four-level quantum systems have been studied, as the electromagnetic induced absorption [13,14], the absorbtion spectrum and the squeezing [15] and the optical gain properties [16]. Also, the effect of a nonlinear medium on the interaction of a four-level moving atom with a cavity field has been studied in [17,18]. Moreover, the quantum behavior of an atomic system with a four-level tripod type [19-23] has been examined. Through those reviews, investigations into four-level atomic systems are very interesting and important in the field of quantum information.

The main objective in this paper is to generalize the proposed model [24] to study the interaction of a N-type four-level atom with a cavity field. The model consists of an atom with four levels of N-type and contains a non-resonant case with deformation functions in addition to the nonlinear medium. Depending on the solution of the time-dependent Schrödinger equation, the wave function is obtained when the atom and the field start from the excited and coherent states respectively.

Some properties for the considered system are calculated and investigated. The effect of the deformation function and the non-linear medium on the behavior of atomic population inversion, the quantum entropy and the geometric phase in the non-resonance case are investigated.

The article is arranged as follows: In Section 2, we present the proposed model and show some mathematical properties that are used in obtaining the exact solution for Schrödinger equation. We devote Section 3 to the wave function describing the proposed system. The atomic population inversion, quantum entropy and geometric phase are calculated in Sections 4, 5 and 6, respectively. We finish the paper by giving results and conclusions.

2 The model

Here, we assume that a four-level N-type atom interacting inside a cavity interacting with one-mode field. The model has the deformed operators for some kinds of nonlinearities of both the intensity dependent-coupling and the field. We denote that, the four levels $|j\rangle, (j = 1, 2, 3, 4)$ with energy $\omega_j (j = 1, 2, 3, 4)$ and the filed with frequency Ω. The Hamiltonian $\hat{H}$ of the atomic system in the rotating wave approximation can be written, as ($\hbar = 1$)

$$\begin{aligned}
\hat{H} &= \sum_{j=1}^{4} \omega_j \hat{\sigma}_{jj} + \Omega \hat{R}^\dagger \hat{R} + \chi \hat{R}^{\dagger 2} \hat{R}^2 + \left[\hat{R} \Big(\lambda_1 \hat{S}_{13} + \lambda_2 \hat{S}_{23} + \lambda_3 \hat{S}_{24} \Big) + h.c. \right], \\
\hat{R}^\dagger &= f(\hat{a}^\dagger \hat{a}) \hat{a}^\dagger, \qquad \hat{R} = \hat{a} f(\hat{a}^\dagger \hat{a}). \qquad (1)
\end{aligned}$$

where $\hat{a}^\dagger(\hat{a})$ is the creation (annihilation) operator, $\hat{S}_{k\ell} = |k\rangle\langle\ell|(\ell = 1, 2, 3, 4)$ are the population operators for $\ell = k$ and the raising (lowering) operators for $\ell \neq k$, χ describes the strength of the non-linearity modelling Kerr medium, $\lambda_s(s = 1, 2, 3)$ is the coupling constant between the atom and the field mode, and $\hat{a}^\dagger \hat{a} = \hat{m}$ is the photon number operator.

The model include functions of the photon number $f(\hat{m})$ and also the Kerr medium is taken into account. Also, choosing $f(\hat{m}) = 1$ leads to the model [24]. Now, we shall present some interesting properties of the operators of the model under consideration. The atomic operators satisfy the following commutation relations,

$$[\hat{S}_{jk}, \hat{S}_{\ell n}] = \hat{S}_{jn} \delta_{k\ell} - \hat{S}_{\ell k} \delta_{nj},$$

where δ_{jk} is the Kroniker symbol. Moreover, the deformed field operators satisfy,

$$\begin{aligned}
[\hat{R}, \hat{R}^\dagger] &= (\hat{m} + 1) f^2(\hat{m} + 1) - \hat{m} f^2(\hat{m}), \\
[\hat{R}, \hat{m}] &= \hat{R}, [\hat{R}^\dagger, \hat{m}] = -\hat{R}^\dagger. \qquad (2)
\end{aligned}$$

Now, we turn our attention to obtain the wave function of our system.

3 The Wave Function

In order to find the wave function of the atomic system at any time t, we consider that it takes the following form

$$\begin{aligned}
|\Psi(t)\rangle &= \sum_m q_m \Big[L_1(m, t) e^{-iA_1 t} |1, m\rangle + L_2(m, t) e^{-iA_2 t} |2, m\rangle \\
&\quad + L_3(m, t) e^{-iA_3 t} |3, m + 1\rangle + L_4(m, t) e^{-iA_4 t} |4, m + 1\rangle \Big] \qquad (3)
\end{aligned}$$

with

$$
\begin{aligned}
A_1 &= \omega_1 + \Omega\, m\, f^2(n) + \chi m\,(m-1)\, f^2(n)\, f^2(m-1),\\
A_2 &= \omega_2 + \Omega\, m\, f^2(n) + \chi\, m\,(m-1)\, f^2(m)\, f^2(m-1),\\
A_3 &= \omega_3 + \Omega\,(m+1)\, f^2(m+1) + \chi\, m\,(m+1)\, f^2(n)\, f^2(m+1),\\
A_4 &= \omega_4 + \Omega\,(m+1)\, f^2(m+1) + \chi\, m\,(m+1)\, f^2(m)\, f^2(m+1)
\end{aligned}
$$

where q_n describes the amplitude of state $|m\rangle$ and the coefficients L_j are the probability amplitudes which can be obtained by the solution of time dependent Schrodinger equation $i\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$ under the initial state $|\Psi(0)\rangle$. So, the coefficients L_j obeys the coupled system of differential equations (D. E.) :

$$
i\frac{d}{dt}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_1 e^{-i\delta_1 t} & 0 \\ 0 & 0 & f_2 e^{-i\delta_2 t} & f_3 e^{-i\delta_3 t} \\ f_1 e^{i\delta_1 t} & f_2 e^{i\delta_2 t} & 0 & 0 \\ 0 & f_3 e^{i\delta_3 t} & 0 & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix} \tag{4}
$$

where

$$
\begin{aligned}
f_s &= \lambda_s \sqrt{m+1}\ \ f(m+1), & \delta_1 &= A_3 - A_4,\\
\delta_2 &= A_3 - A_2, & \delta_3 &= A_4 - A_2.
\end{aligned} \tag{5}
$$

It is remarkable that, when $\chi = 0$ and $f(m) = 1$ the quantities δ_1, δ_2 and δ_3 are the usual detuning parameters Δ_1, Δ_2 and Δ_3, respectively.

Let us now consider the atom starts the intraction from the upper state $|1\rangle$ and the field in a coherent state $|\alpha\rangle$:

$$
|\alpha\rangle = \sum_m q_m |m\rangle = \sum_m \sqrt{\frac{\alpha^{2m} e^{-\langle \hat{m} \rangle}}{m!}} |m\rangle \tag{6}
$$

where $\langle \hat{m} \rangle = |\alpha|^2$ is the initial mean photon number, then the initial state vector of the system is given by,

$$
|\Psi(t=0)\rangle = \sum_m q_m |m, 1\rangle \tag{7}
$$

In this case, the coupled D. E. system (6) can be solved exactly, the probability amplitudes taken the following expressions:

$$
\begin{aligned}
L_1(m,t) &= \textstyle\sum_j \ \ c_j \ \ e^{i\mu_j t}\\
L_2(m,t) &= \tfrac{1}{f_1 f_2} \textstyle\sum_j \ \ c_j \ \ ({\mu^2}_j + \delta_1 \mu_j - f_1^2) \ \ e^{i(\mu_j - \delta_{12})t}\\
L_3(m,t) &= \tfrac{-1}{f_1} \textstyle\sum_j \ \ c_j \ \ \mu_j e^{i(\mu_j + \delta_1)t}\\
L_4(m,t) &= \tfrac{-1}{f_1 f_2 f_3} \ \ \textstyle\sum_j c_j \ \ ({\mu^3}_j + \Gamma_1 \mu_j^2 + \Gamma_2 \mu_j + \Gamma_3) \ \ e^{i(\mu_j + \Gamma_4)t},
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
&c_j=\frac{-[\Gamma_3+f_1^2(\Gamma_1+\mu_k+\mu_\ell+\mu_m)+\mu_k\mu_\ell mu_m]}{(\mu_j-\mu_k)(\mu_j-\mu_\ell)(\mu_j-\mu_m)}, && (j\neq k\neq \ell\neq m=1,2,3,4),\\
&\delta_{12}=\delta_1-\delta_2, && \Gamma_1=\delta_{12}+\delta_1,\\
&\Gamma_2=\delta_1\delta_{12}-f_1^2-f_2^2, && \Gamma_3=-\delta_{12}f_1^2, \Gamma_4=\delta_3+\delta_{12}.
\end{aligned} \tag{9}
$$

where μ_j satisfies the following fourth-order equation

$$
\mu^4+x_3\mu^3+x_2\mu^2+x_1\mu+x_0=0 \tag{10}
$$

where

$$
\begin{aligned}
&x_0=\Gamma_3\Gamma_4+f_1^2f_3^2, \quad x_1=\Gamma_2\Gamma_4+\Gamma_3-\delta_1f_1^2,\\
&x_2=\Gamma_1\Gamma_4+\Gamma_2, \qquad x_3=\Gamma_1+\Gamma_4.
\end{aligned} \tag{11}
$$

Based on Ferari method the four roots of (10) are:

$$
\begin{aligned}
&\mu_1=\frac{-x_3}{4}-\frac{y}{2}-\frac{y_-}{2}, \quad \mu_2=\frac{-x_3}{4}-\frac{y}{2}+\frac{y_+}{2},\\
&\mu_3=\frac{-x_3}{4}+\frac{y}{2}-\frac{y_+}{2}, \quad \mu_4=\frac{-x_3}{4}+\frac{y}{2}+\frac{y_+}{2},
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
&y=\sqrt{z_1+\frac{z_2}{3}+\frac{z_3}{3z_2}}, && y_\mp=\sqrt{z_4\mp\frac{z_5}{4z_6}},\\
&z_1=\frac{-2x_2}{3}+\frac{x_3^2}{4}, && z_2=12x_0+x_2^2-3x_1x_3,\\
&z_3=[\frac{z_7+\sqrt{-4z_2^3+z_1^2}}{2}]^{\frac{1}{3}}, && z_4=2z_1-\frac{z_2}{3z_3}-\frac{z_3}{3},\\
&z_5=-8x_1+4x_2x_3-x_3^3, && z_6=\sqrt{z2+\frac{z_2}{3z_3}+\frac{z_3}{3}},\\
&z_7=27(x_1^2+x_0x_3^2)-x_2(72x_0-2x_2^2+9x_1x_3).
\end{aligned} \tag{13}
$$

Now, the time-dependence properties of some statistical quantities of the considered system will be obtained by the atomic system wave function $|\Psi(t)\rangle$.

4 Atomic population inversion

The population inversion is one of important quantities in quantum information [25, 26, 27, 28]. Where through its results we can determine the periods of collapse and revival, which are useful in determining the periods of maximally entangled state and purity periods (separable state). The atomic population inversion (API) is defined in terms of the diagonal elements of the atomic density matrix $\hat{\rho}^{AB}(t)$ as

$$
\begin{aligned}
W(t) &= \rho_{11}(t)-\rho_{33}(t)\\
\rho_{11}(t) &= \sum_m |q_mL_1(m,t)|^2, \quad \rho_{44}(t)=\sum_n |q_mL_4(m,t)|^2
\end{aligned} \tag{14}
$$

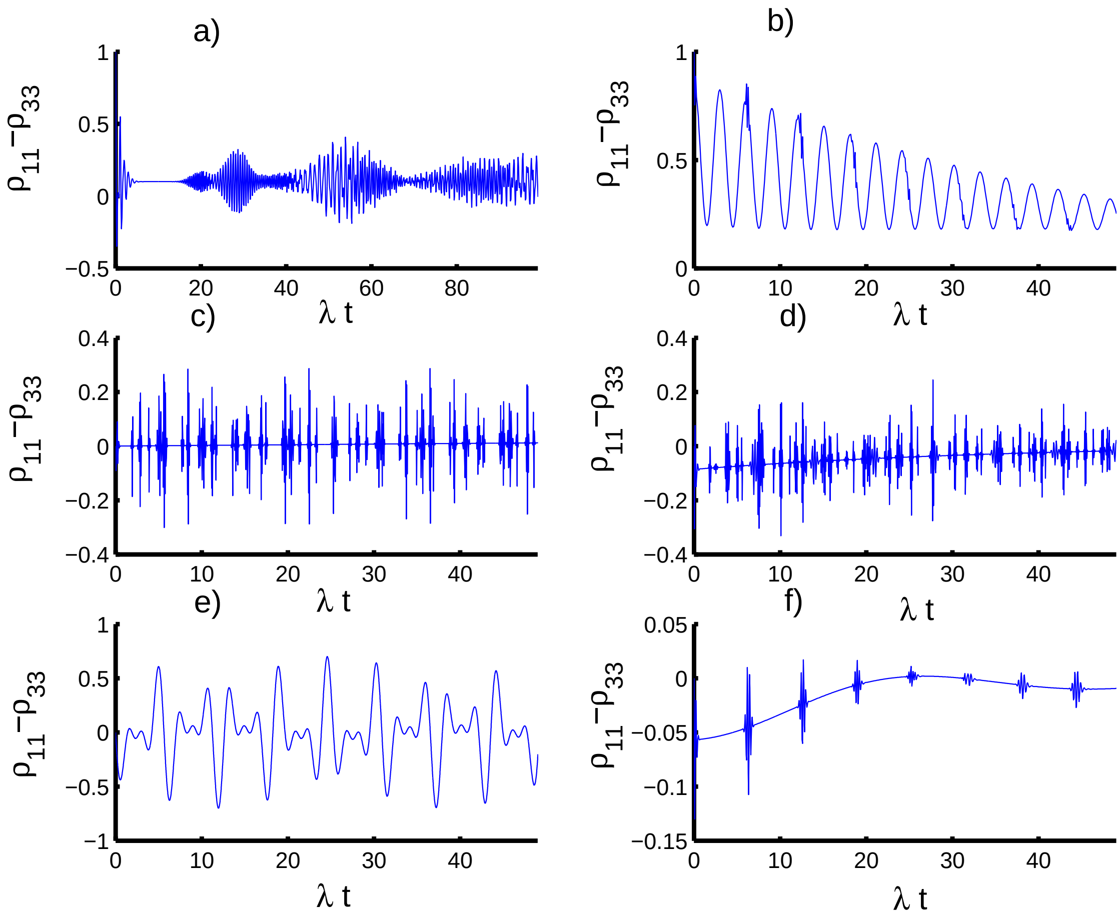

Figure 1: Time evolution of the *atomic inversion* $W(t)$.The field is initially in the identical pair coherent states ($\alpha = \sqrt{10}$) with $\chi = 0$ in (a,c,e) and $\chi = 0.5$ (b,d,f). $f(m) = 1$in (a,b), $f(m) = \sqrt{m}$ in (c,d) and $f(m) = \frac{1}{\sqrt{m}}$ in (e,f).

We neglect the effect of the deformation $f(\hat{m}) = \hat{I}$ and the nonlinear medium $\chi = 0$. The function $W(t)$ oscillates regularly and the revival occurs periodically every $2\pi\sqrt{10}$ and the collapse achieved once at the beginning of the interaction time. Moreover, the intensity of oscillations decreases with increasing time, as seen in the Fig.(1a). The regular behavior becomes chaotic and the phenomena of revival and collapse are erased after adding the Kerr medium into the interaction cavity. When the case of deformation $f(\hat{m}) = \sqrt{\hat{m}}$ is taken into account, the phenomena of revival and collapse are largely achieved while the intensity of the oscillations decreases. Moreover, the amplitude of oscillations and the revival times are reduced. The effect of the Kerr medium appears at the beginning of the interaction and this effect decreases gradually with increasing time. The axis of symmetry of the function $W(t)$ appears in the negative part and after a short period of time it reaches the horizontal axis, as observed in Fig.(1d). For another case of the deformation $f(\hat{m}) = \frac{1}{\sqrt{\hat{m}}}$ is taken into account, the function $W(t)$ fluctuates smoothly and the phenomena of revival and collapse are not realized as the intensity of the oscillations decreases. After the insertion of the Kerr medium of the interaction, the phenomena of revival and collapse is clearly generated in the negative part and gradually rises until the axis of symmetry reaches the horizontal axis.

5 Field entropy

The Shannon entropy measure is one of the most classical measures of quantum optics [29]. While the von Neumann entropy measure is the first to appear and suitable for closed systems [30]. In this case the field entropy $S_f(t)$ can be expressed in terms of the eigenvalues $\Lambda_j(t)$ for the reduced field density operator $Tr_{field}|\psi(t)\rangle\langle\psi(t)|$ as,

$$S_f(t) = -\sum_{j=1}^{4} \Lambda_j(t) \ln \Lambda_j(t), \tag{15}$$

We are now studying the entanglement between the cavity field and the atom by means of previous conditions in the atomic population. The interaction starts from the separate state, followed by a partial entangled state that improves in the collapse region and decreases in the middle of the revival region, these results are in agreement with [31, 32]. The entanglement is significantly reduced after the addition of the Kerr medium to the interaction and a weak interaction is generated between the field and the atom as shown in the Fig.(2b). After taking the deformation

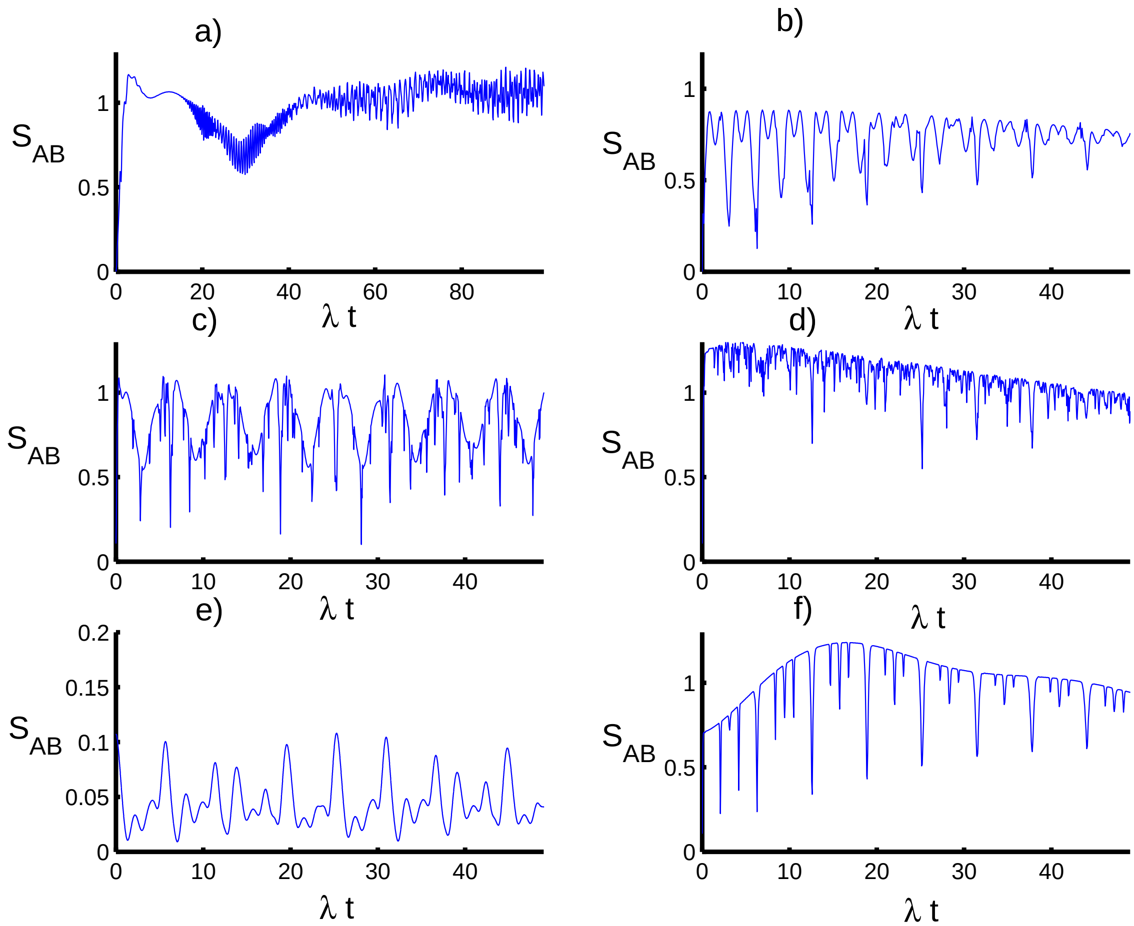

Figure 2: Time evolution of the *entropy* $S_{AB}(t)$ with same conditions as Fig. 1

$f(\hat{n}) = \sqrt{\hat{m}}$ into account, the entanglement improves and an orderly fluctuation appears while the intensity of the oscillations decreases compared to the previous case Fig.(2a,2c). The entanglement decreases after adding Kerr medium and weak correlation appears with increasing time. For the second case of the deformation $f(\hat{m}) = \frac{1}{\sqrt{\hat{m}}}$, the Rabi frequency does not depend on the number of photons, due to a weak entanglement is generated and the entanglement between the parts of the system is closer to separation state. After the inclusion of the Kerr medium for interaction, the Rabi frequency that depends on the number of photons and is proportional to $\hat{n}$, the entanglement improves clearly.

6 Geometric phase

One essential feature in quantum mechanics, namely, the Pancharatnam phase and the geometric phase has been studied by many physicists [33, 34, 35, 36]. The geometric phase is very sensitive to the initial state and different system parameters. The dynamical performance of the geometric phase between the initial and final states of the system is defined by [37, 38]

$$\Phi_G(t) = \arg\langle\Psi(0)|\Psi(t)\rangle \tag{16}$$

We now begin to study the geometric phase under the same conditions in the previous sections. In the absence of the deformation and the Kerr medium, the geometric phase function $\Phi_G(t)$ has symmetric and uniform oscillations around the horizontal axis. Also, these oscillations are erased in the period of collapse and return to the fluctuations in the period of revival see Fig.(1a,3a). The situation changes completely after adding the Kerr medium to the interaction cavity. The regular oscillations in the previous case become random and the amplitude of the oscillations increases. The Rabi frequency becomes more effect when the deformation $f(\hat{m}) = \sqrt{\hat{m}}$ taken into account. The function $\Phi_G(t)$ has more oscillation and the intensity of the oscillations increases. After insertion of the Kerr medium, the behavior of the function $\Phi_G(t)$ becomes chaotic and increases both the intensity and amplitude of the oscillations. The second case of deformation $f(\hat{m}) = \frac{1}{\sqrt{\hat{m}}}$ reduces the effect of the Rabi frequency (almost constant). The function $\Phi_G(t)$ becomes regular and the intensity of the oscillations decreases significantly. The intensity of the oscillations increases again after the insertion of the Kerr medium (Rabi frequency dependence

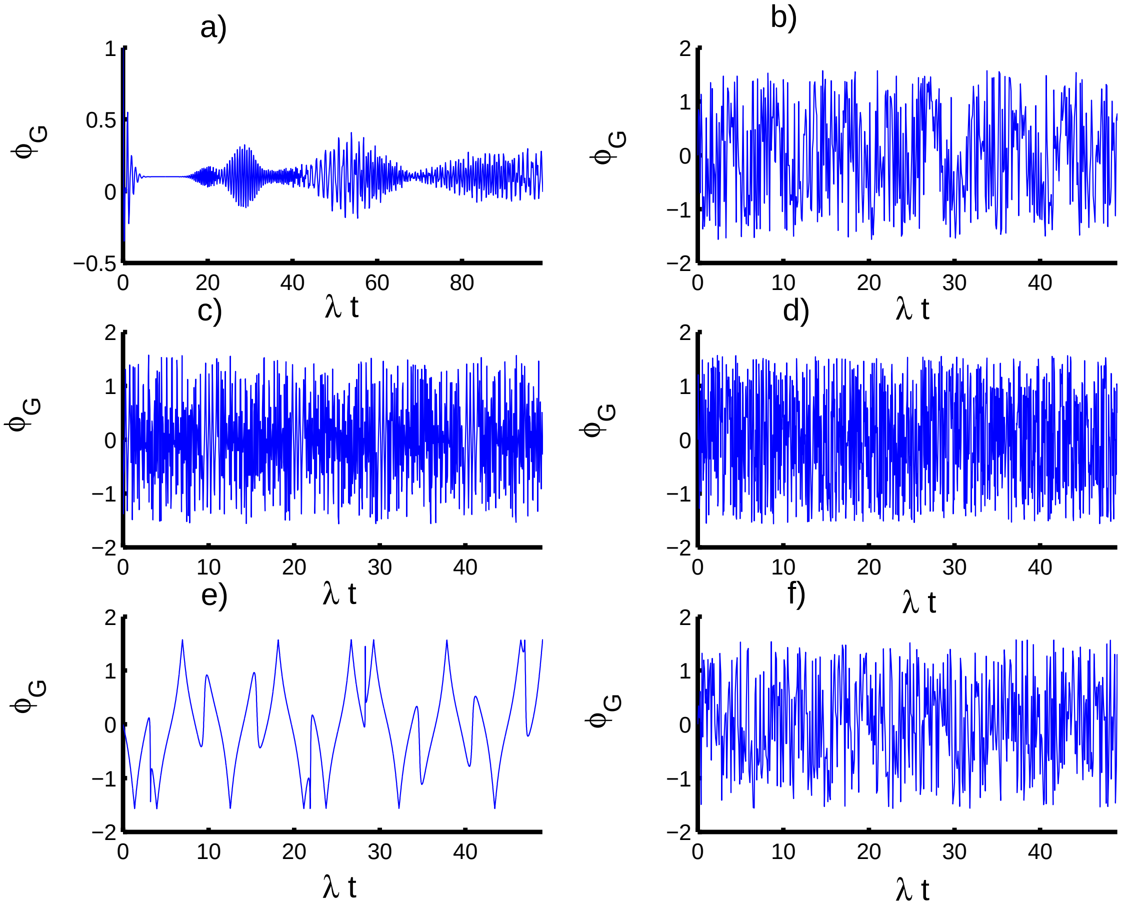

Figure 3: Time evolution of the *Geometric phase* $\Phi_G(t)$ with same conditions as Fig. 1

on the photon number).

7 Conclusion

In this article, some properties of a model describing N-type four-level atom interacting with a single-mode cavity field when both the Kerr medium and intensity dependent taken into accounts are studied. The feature wave function of the proposed system is obtained by solving the differential Schrödinger equation. The effect of two types of deformation on statistical quantities is also studied. The results showed that the phenomena of collapse and revival are achieved in the non-deformed case $f(\hat{m}) = \hat{I}$ and in absence of the Kerr medium term. This phenomenon disappears after adding the Kerr medium, and improves in the deformed case $f(\hat{m}) = \sqrt{\hat{m}}$ and does not appear at all in the second case $f(\hat{m}) = \frac{1}{\sqrt{\hat{m}}}$. A correlation appeared between the collapse, the revival periods and the entanglement function. The entanglement decreases after inclusion of the Kerr medium in the absence of deformation, while the entanglement increases in the presence of the $f(\hat{m}) = \frac{1}{\sqrt{\hat{m}}}$ case. The effect of the deformation and the Kerr medium on the geometric phase and its relation to the atomic population have been also studied.

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