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Anti- $\mathcal{PT}$ Transformations and Complex $\mathcal{PT}$ -Symmetric Superpartners

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ABSTRACT

A quantum mechanical system with unbroken super- and parity-time ($\mathcal{PT}$)-symmetry is derived and analyzed. Here, we propose a new formalism to construct the complex $\mathcal{PT}$ -symmetric superpartners by extending the additive shape invariant potentials to the complex domain. The probabilistic interpretation of a $\mathcal{PT}$ -symmetric quantum theory is correlated with the calculation of a new linear operator called the $\mathcal{C}$ operator, instead of complex conjugation in conventional quantum mechanics. At the present work, we introduce an anti- $\mathcal{PT}$ ($\mathcal{AP\mathcal{T}}$) conjugation to redefine a new version of the inner product without any additional considerations. This $\mathcal{PT}$ -supersymmetric quantum mechanics, satisfies essential requirements such as completeness, orthonormality as well as probabilistic interpretation.

1 Introduction

Space-time symmetries reflect underlying properties of any physical system and cause a deep insight into the physics of the system. Since revealing the importance of combined space-time transformations, extensive theoretical and experimental researches have been performed on classical and quantum systems described with non-Hermitian Hamiltonians [1-4]. The successes achieved have led to a new development of quantum mechanics called $\mathcal{PT}$ -symmetric quantum mechanics [5]. Based on $\mathcal{PT}$ -symmetric quantum mechanics, a dynamically balanced open system has an unbroken $\mathcal{PT}$ symmetry and therefore has real energy eigenvalues [6]. The applications of $\mathcal{PT}$ -symmetric Hamiltonians have been increasing in several areas of physics, such as quantum field theory, condensed matter, quantum optics, and non-equilibrium statistical physics since it has been introduced [7-9]. However, the conventional quantum mechanics is Hermitian, as a result, fundamental issues such as Hilbert space, unitary, inner product, and probabilistic interpretation for a non-Hermitian Hamiltonian are controversial.

On the other hand, the $\mathcal{AP\mathcal{T}}$ symmetry has also been the subject of theoretical and experimental researches in recent years [10-12]. As complex partners, such systems possess purely imaginary energy eigenvalues. Consequently, an $\mathcal{AP\mathcal{T}}$ -symmetric Hamiltonian $H^{(\mathcal{AP\mathcal{T}})}$ might be mathematically formed by a $\mathcal{PT}$ -symmetric Hamiltonian $H^{(\mathcal{PT})}$, via multiplying imaginary number 'i', such that $H^{(\mathcal{AP\mathcal{T}})} = \pm iH^{(\mathcal{PT})}$ [12]. Despite the theoretical elegant and empirical validations, these combined space-time transformations are hard to understand intuitively. In order to better understand as well as investigate the relationship between these transformations, here, we employ them for a quantum-mechanical solvable potential.

The solvability referred to a potential such that the energy eigenvalues, the bound-state eigenfunctions, and the scattering matrix can be determined in closed analytical form. Supersymmetric quantum mechanics (SUSY QM), as a formalism of quantum mechanics, suggests a mechanism to generate and classify solvable potentials [13]. On this basis, the shape invariant potentials (SIP) form the main class of exactly solvable potentials. Moreover, in the SUSY QM have been demonstrated superpartners are isospectral (except one level) by introducing the Hamiltonians hierarchy [14]. In the present study, we propose a new formalism to generate complex $\mathcal{PT}$ -symmetric superpartners and next analyze them under $\mathcal{AP\mathcal{T}}$ transformations.

2 $\mathcal{PT}$ -symmetric Square-Well Superpartners

The solutions of time independent of Schrödinger equation for ground state of the simplest one-dimensional problem in conventional quantum mechanics, i.e., particle in an infinite square-well, is known as,

$$\begin{cases} \psi_{0c}(x) = A \sin(kx) \\ \psi_{0r}(x) = B \cos(kx), \end{cases} \quad (1)$$

where k is the wave number ($\hbar = 2m = 1$). In SUSY QM, the n th-superpotential is defined as the logarithmic derivative of the n th ground state wave function [14],

$$W_n(x) = -\frac{d}{dx} \ln \psi_0^{(n)}(x), \quad (2)$$

where, $n = 1, 2, 3, \dots$. Therefore, the superpotentials for the wave functions of the Eq. (1) are the cotangent (denoted by subscript "c") and the tangent (denoted by subscript "t") functions. Now, to extend the superpotentials to the complex domain, we add an arbitrary imaginary functions, linearly,

$$\begin{cases} W_{1c}(x) = -k \cot(\alpha x) + i f_{1c}(x) \\ W_{1t}(x) = k \tan(\alpha x) + i f_{1t}(x). \end{cases} \quad (3)$$

According to the SUSY QM, the partner potentials are obtained by [14],

$$\begin{cases} V_n(x) = W_n^2(x) - W_n'(x) + E_0^{(n)} \\ V_{n+1}(x) = W_n^2(x) + W_n'(x) + E_0^{(n)}. \end{cases} \quad (4)$$

For $n = 1$ and due to unbroken SUSY, $E_0^{(1)} = 0$, by putting the superpotentials $W_{1c}(x)$ and $W_{1t}(x)$ to Eq. (4), we get for cotangent and tangent functions, respectively,

$$\begin{cases} V_{1c}(x) = k(k - \alpha) \csc^2(\alpha x) - k^2 - f_{1c}^2(x) + i[-f_{1c}'(x) - 2k \cot(\alpha x) f_{1c}(x)] \\ V_{2c}(x) = k(k + \alpha) \csc^2(\alpha x) - k^2 - f_{1c}^2(x) + i[f_{1c}'(x) - 2k \cot(\alpha x) f_{1c}(x)], \end{cases} \quad (5)$$

and,

$$\begin{cases} V_{1t}(x) = k(k - \alpha) \sec^2(\alpha x) - k^2 - f_{1t}^2(x) + i[-f_{1t}'(x) + 2k \tan(\alpha x) f_{1t}(x)] \\ V_{2t}(x) = k(k + \alpha) \sec^2(\alpha x) - k^2 - f_{1t}^2(x) + i[f_{1t}'(x) + 2k \tan(\alpha x) f_{1t}(x)], \end{cases} \quad (6)$$

If the V_1 and V_2 potentials are similar in shape and differ only in the parameters that appear in them, then they are said to shape invariant. The remainder that is defined as $R_1 = V_2(k, x) - V_1(k + \alpha, x)$, equals with,

$$\begin{cases} R_{1c} = \alpha(\alpha + 2k) + i2[f_{1c}'(x) + \alpha \cot(\alpha x) f_{1c}(x)] \\ R_{1t} = \alpha(\alpha + 2k) + i2[f_{1t}'(x) - \alpha \tan(\alpha x) f_{1t}(x)], \end{cases} \quad (7)$$

Accordingly, partner potentials are shape invariant only if the bracket terms of the remainders to be zero. As a result, the functions $f_{1c}(x)$ and $f_{1t}(x)$ are determined by this constraint as,

$$\begin{cases} f_{1c}(x) = q \csc(\alpha x) \\ f_{1t}(x) = q \sec(\alpha x), \end{cases} \quad (8)$$

where q is an arbitrary constant. By setting the $\alpha = k$, the superpotentials are gained as,

$$\begin{cases} W_{1c}(x) = -k \cot(kx) + iq \csc(kx) \\ W_{1t}(x) = k \tan(kx) + iq \sec(kx), \end{cases} \quad (9)$$

The real and imaginary terms W_{1cr} (W_{1tr}) and W_{1ci} (W_{1ti}) are plotted Fig. (1), for typically values $q = 2$ and $k = 1$. Proportionally, the complex partner potentials are,

$$\begin{cases} V_{1c}(x) = -q^2 \csc^2(kx) - k^2 - iqk \cot(kx) \csc(kx) \\ V_{2c}(x) = (2k^2 - q^2) \csc^2(kx) - k^2 - i3qk \cot(kx) \csc(kx), \end{cases} \quad (10)$$

and,

$$\begin{cases} V_{1t}(x) = -q^2 \sec^2(kx) - k^2 + iqk \tan(kx) \sec(kx) \\ V_{2t}(x) = (2k^2 - q^2) \sec^2(kx) - k^2 + i3qk \tan(kx) \sec(kx). \end{cases} \quad (11)$$

The potentials $V_1 = -1$ ($q = 0$ and $k = 1$) and real and imaginary parts V_{1cr} (V_{1tr}) and V_{1ci} (V_{1ti}) are depicted in Fig. (2), and the Fig. (3) illustrates the potentials V_{2c} (V_{2t}) ($q = 0$ and $k = 1$) and real and imaginary parts V_{2cr} (V_{2tr}) and V_{2ci} (V_{2ti}), for typically values $q = 2$ and $k = 1$. Finally, The wave functions are obtained by replacing superpotentials (9) in Eq. (2) as,

$$\begin{cases} \psi_{0c}^{(1)}(x) = A \sin(kx) \exp \left\{ -i \frac{q}{k} \ln [\csc(kx) - \cot(kx)] \right\} \\ \psi_{0t}^{(1)}(x) = B \cos(kx) \exp \left\{ -i \frac{q}{k} \ln [\sec(kx) + \tan(kx)] \right\}, \end{cases} \quad (12)$$

The coefficients A and B are determined by relevant boundary conditions and normalization.

3 Energy Eigenvalues and Eigenfunctions

Now consider an infinite square-well in one dimension with length $L = \pi$ in which a particle moves in the range $-\pi/2 \leq L \leq \pi/2$. The boundary conditions,

$$\psi(-\pi/2) = \psi(\pi/2) = 0, \quad (13)$$

imply that,

$$k_n = n + 1 \quad (14)$$

where $n = 0, 1, 2, \dots$. As a result, the remainders (7) are,

$$R_{nt} = R_{nc} = (2n + 1)k_n^2, \quad (15)$$

and according the unbroken SUSY, $E_0^{(1)} = 0$, the energy spectrum is,

$$E_n^{(1)} = k_n^2 - 1 = n(n + 2); \quad E_n^{(1)} = E_0^{(n+1)} \quad (16)$$

The Hamiltonian hierarchy yields nth superpotentials, potentials, and eigenfunctions, respectively as follow:

$$\begin{cases} W_{nc}(x) = -nk_n \cot(k_n x) + iq \csc(k_n x) \\ W_{nt}(x) = nk_n \tan(k_n x) + iq \sec(k_n x), \end{cases} \quad (17)$$

$$\begin{cases} V_{nc}(x) = [n(n-1)k_n^2 - q^2] \csc^2(k_n x) - k_n^2 - i(2n-1)qk_n \cot(k_n x) \csc(k_n x) \\ V_{nt}(x) = [n(n-1)k_n^2 - q^2] \sec^2(k_n x) - k_n^2 + i(2n-1)qk_n \tan(k_n x) \sec(k_n x). \end{cases} \quad (18)$$

and

$$\begin{cases} \psi_{0c}^{(n)}(x) = A \sin^n(k_n x) \exp \left\{ -i \frac{q}{k_n} \ln [\csc(k_n x) - \cot(k_n x)] \right\} \\ \psi_{0t}^{(n)}(x) = B \cos^n(k_n x) \exp \left\{ -i \frac{q}{k_n} \ln [\sec(k_n x) + \tan(k_n x)] \right\}, \end{cases} \quad (19)$$

If we set $q = 0$, these complex quantities become the known real form. So we can interpret that known real potentials are, in fact, particular cases of complex potentials, when the imaginary terms are zero.

4 $\mathcal{PT}$ and $\mathcal{AP}\mathcal{T}$ Transformations

The $\mathcal{PT}$ -symmetry requires that the potential under the following transformations be symmetric [1]:

$$\begin{cases} x \rightarrow -x \\ i \rightarrow -i \end{cases} \quad (20)$$

Consequently, the real $V_r(x)$ and imaginary $V_i(x)$ terms of the potential must be even and odd functions of x , respectively,

$$V^{\mathcal{PT}}(x) = V(x) = V_r^{\text{even}}(x) + iV_i^{\text{odd}}(x), \quad (21)$$

Fig. (2) shows that the potentials V_{1tr} and V_{1ti} and also Fig. (3) illustrates that V_{2tr} and V_{2ti} satisfy these conditions (preserve $\mathcal{PT}$ symmetry). These complex $\mathcal{PT}$ -symmetric superpartners are isospectrum with infinite square-well (except ground states). Therefore, only the tangent wave functions $\psi_{0t}^{(n)}(kx)$, corresponding to the bound states. If we shift the width of the well from the range $-\pi/2 \leq L \leq \pi/2$ to the range $0 \leq L \leq \pi$, we have to replace the tangent with the cotangent functions in order that invariance is preserved under the $\mathcal{PT}$ transformations.

However, the superpotential $W_{1t}(x)$ have anti-symmetric features, in the sense that the real $W_{1tr}(x)$ and imaginary $W_{1ti}(x)$ terms of the superpotential are odd and even functions, respectively (see Fig. (1)). Therefore, we can conclude that a $\mathcal{PT}$ -symmetric potential is produced by a $\mathcal{AP}\mathcal{T}$ -symmetric superpotential,

$$W_{1t}^{\mathcal{AP}\mathcal{T}}(x) = W_{1t}(x) = W_{1tr}^{\text{odd}}(x) + iW_{1ti}^{\text{even}}(x), \quad (22)$$

This is due to the change in the sign of the first spatial derivative, under $\mathcal{A}\mathcal{P}\mathcal{T}$ transformations. Moreover, to preserve the potential algebra in SUSY-QM formalism, it is necessary the $\mathcal{A}\mathcal{P}\mathcal{T}$ conjugation be similar to the complex conjugation on this operator,

$$\begin{cases} \left(\frac{d}{dx}\right)^{\mathcal{A}\mathcal{P}\mathcal{T}} \rightarrow -\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^\dagger \rightarrow -\frac{d}{dx} \end{cases} \quad (23)$$

The same argument also can be applied to integration. This is because each derivative or integration operation can exchange the odd and even properties of a function. Now, by examining the complex and $\mathcal{A}\mathcal{P}\mathcal{T}$ conjugations on the ladder operators, we have,

$$\begin{cases} A_{1l} = \frac{d}{dx} + W_{1l}(x) = \frac{d}{dx} + W_{1lr}(x) + iW_{1li}(x) \\ A_{1l}^{\mathcal{A}\mathcal{P}\mathcal{T}} = -\frac{d}{dx} + W_{1l}^{\mathcal{A}\mathcal{P}\mathcal{T}}(x) = -\frac{d}{dx} + W_{1lr}(x) + iW_{1li}(x) \\ A_{1l}^\dagger = -\frac{d}{dx} + W_{1l}^\dagger(x) = -\frac{d}{dx} + W_{1lr}(x) - iW_{1li}(x) \end{cases} \quad (24)$$

we see that the complex conjugation operator fails past section algebra because it changes the sign of superpotential imaginary term. Therefore, this mathematical operator is not longer applicable for a $\mathcal{P}\mathcal{T}$ -supersymmetric quantum theory.

5 Conclusion

In SUSY QM, a ground state wave function can be written in a general form [14],

$$\psi_0^{(n)}(x) = N \exp\{-f_n(x)\}, \quad (25)$$

where function $f_n(x)$ is obtained by integrating superpotential,

$$f_n(x) = \int W_n(x) dx, \quad (26)$$

According to past section arguments, in a $\mathcal{P}\mathcal{T}$ -supersymmetric system, $f_n(x)$ and $W_n(x)$ should be $\mathcal{P}\mathcal{T}$ - and $\mathcal{A}\mathcal{P}\mathcal{T}$ -symmetric, respectively,

$$\begin{cases} [f_n(x)]^{\mathcal{P}\mathcal{T}} = f_n(x) = f_{nr}^{even}(x) + i f_{ni}^{odd}(x) \\ [W_n(x)]^{\mathcal{A}\mathcal{P}\mathcal{T}} = W_n(x) = W_{nr}^{odd}(x) + i W_{ni}^{even}(x) \end{cases} \quad (27)$$

With respect to SUSY QM formalism, we know that supersymmetric eigenfunctions consist of a complete set and are orthonormal. Obviously, the $\mathcal{P}\mathcal{T}$ -supersymmetric eigenfunctions (20) posse also these properties. However, the coordinate-space inner product needs to redefine in such a way that the complex conjugation is replaced by the $\mathcal{A}\mathcal{P}\mathcal{T}$ conjugation as,

$$(\phi, \psi) = \int [\phi(x)]^{\mathcal{A}\mathcal{P}\mathcal{T}} \psi(x) dx, \quad (28)$$

As a consequence, orthonormality reads,

$$(\psi_0^{(m)}, \psi_0^{(n)}) = \int [\psi_0^{(m)}(x)]^{\mathcal{A}\mathcal{P}\mathcal{T}} \psi_0^{(n)}(x) dx = \delta_{m,n}, \quad (29)$$

and thus, the normalization constant is obtained by,

$$N = \frac{1}{\sqrt{\int \exp\{-2f_{nr}^{even}(x)\} dx}}, \quad (30)$$

The $\mathcal{A}\mathcal{P}\mathcal{T}$ transformations and symmetry are analyzed for a $\mathcal{P}\mathcal{T}$ -supersymmetric (both $\mathcal{P}\mathcal{T}$ and SUSY are unbroken) quantum system. However, the $\mathcal{A}\mathcal{P}\mathcal{T}$ transformations are less intuitive than $\mathcal{P}\mathcal{T}$ so we can not exhibit its effect on individual quantities similar to Eq. (21). In combined situations with imaginary number 'i', a typical function $ig(x)$ is $\mathcal{P}\mathcal{T}$ -symmetric if be odd, and is $\mathcal{A}\mathcal{P}\mathcal{T}$ -symmetric if be even function of x ,

$$\begin{cases} [ig^{odd}(x)]^{\mathcal{A}\mathcal{P}\mathcal{T}} = -ig^{odd}(x) \\ [ig^{even}(x)]^{\mathcal{A}\mathcal{P}\mathcal{T}} = ig^{even}(x) \end{cases} \quad (31)$$

Figure 1. The real and imaginary terms W_{1cr} (W_{1tr}) and W_{1ci} (W_{1ti}), for typically values $q = 2$ and $k = 1$.

From Eq. (24) we conclude that the first-order differential operators, named the ladder operators, should be redefined by $\mathcal{A}\mathcal{P}\mathcal{T}$ conjugation as,

$$\begin{cases} A = \frac{d}{dx} + W(x) = \frac{d}{dx} + W_r(x) + iW_i(x) \\ A^{\mathcal{A}\mathcal{P}\mathcal{T}} = -\frac{d}{dx} + W(x) = -\frac{d}{dx} + W_r(x) + iW_i(x) \end{cases} \quad (32)$$

accordingly Hamiltonian superpartners are defined as follows:

$$\begin{cases} H_1 = A^{\mathcal{A}\mathcal{P}\mathcal{T}} A \\ H_2 = A A^{\mathcal{A}\mathcal{P}\mathcal{T}} \end{cases} \quad (33)$$

so that $H_2^{\mathcal{A}\mathcal{P}\mathcal{T}} = H_1$. Therefore, the mathematical structure of $\mathcal{P}\mathcal{T}$ -SUSY QM is as same as SUSY QM, with difference that $\mathcal{A}\mathcal{P}\mathcal{T}$ conjugation exchange with complex conjugation.

Figure 2. The square-well $V_1 = -1$ and real and imaginary terms V_{1cr} (V_{1tr}) and V_{1ci} (V_{1ti}), for typically values $q = 2$ and $k = 1$

Figure 3. The potentials V_{2c} (V_{2t}) ($q = 0$ and $k = 1$) and real and imaginary terms V_{2cr} (V_{2tr}) and V_{2ci} (V_{2ti}).

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