Dynamic behaviors of soliton solutions for a three-coupled Lakshmanan-Porsezian-Daniel model

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Abstract

In this paper, we use the Riemann-Hilbert (RH) approach to examine the integrable three-coupled Lakshmanan-Porsezian-Daniel (LPD) model, which describe the dynamics of alpha helical protein with the interspine coupling at the fourth-order dispersion term. Through the spectral analysis of Lax pair, we construct the higher order matrix RH problem for the three-coupled LPD model, when the jump matrix of this particular RH problem is a 4×4 unit matrix, the exact N-soliton solutions of the three-coupled LPD model can be exhibited. As special examples, we also investigate the nonlinear dynamical behaviors of the single-soliton, two-soliton, three-soliton and breather soliton solutions. Finally, an integrable generalized N-component LPD model with its linear spectral problem is discussed.

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1 Introduction

Since the nonlinear evolution equations can be widely used to describe some of the physics physical phenomena, such as nonlinear optical, quantum mechanics, fluid physics, plasma physics, etc. The research on the method of solving nonlinear evolution equations becomes a challenging and vital task, and it has attracted more and more people's attention. With the development of soliton theory, a series of methods for solving nonlinear development equations are proposed, such as the inverse scattering method [1], the Hirota's bilinear method [2], the Bäcklund transformation method [3], the Darboux transformation (DT) method [4] and others [5, 6, 7, 8]. Based on these available methods, we have obtained a series of solutions of nonlinear evolution equations, including compaton solution, peakon solution, periodic sharp wave solution, Lump solution, breather solution, bright soliton, dark soliton, rogue waves, etc. These solutions can further help to understand natural phenomena and laws. In recent years, more and more mathematical physicists have begun to pay attention to Riemann-Hilbert (RH) approach [9, 10], which is a new powerful method for solving integrable linear and nonlinear evolution equations [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. The main idea of this method is to establish a corresponding matrix RH problem on the Lax pair of integrable equations. Furthermore, the RH approach is also an effective way to examine the initial boundary value problems [28, 29, 30, 31, 32] and the long-time asymptotic behavior [33, 34] of the integrable nonlinear evolution equations.

The Lakshmanan-Porsezian-Daniel (LPD) model [35] is one of the most paramount integrable systems in mathematics and physics which reads

$$iq_t + q_{xx} + 2q|q|^2 + \varepsilon(q_{xxxx} + 8|q|^2q_{xx} + 2q^2q_{xx}^* + 6q^*q_x^2 + 4qq_xq_x^* + 6q|q|^4) = 0. \quad (1.1)$$

This model can not only simulate the nonlinear transmission and interaction of ultrashort pulses in high-speed optical fiber transmission systems, but also describe the nonlinear spin excitation phenomenon of a one-dimensional Heisenberg ferromagnetic chain with octopole and dipole interactions [36]. When $\varepsilon = 0$, Eq.(1.1) can be reduced to the famous nonlinear schrödinger (NLS) equation. However, a slice of phenomena have been observed by experiment which cannot be explained by Eq.(1.1). For example, the dynamics of the alpha helical protein with the interspine coupling at the fourth-order dispersion term. In

order to explain these phenomena, the three-component case was introduced as follows [37, 38, 39]

$$\begin{aligned}
& i q_{\alpha,t} + q_{\alpha,xx} + 2 \sum_{k=1}^3 |q_k|^2 q_{\alpha} + \varepsilon [q_{\alpha,xxx} + 2 \sum_{k=1}^3 |q_{k,x}|^2 q_{\alpha} \\
& + 2 \sum_{k=1}^3 q_k q_{k,x}^* q_{\alpha,x} + 6 \sum_{k=1}^3 q_k^* q_{k,x} q_{\alpha,x} + 4 \sum_{k=1}^3 |q_k|^2 q_{\alpha,xx} \\
& + 4 \sum_{k=1}^3 q_k^* q_{k,xx} q_{\alpha} + 2 \sum_{k=1}^3 q_k q_{k,xx}^* q_{\alpha} + 6 (\sum_{k=1}^3 |q_k|^2)^2 q_{\alpha}] = 0, \tag{1.2}
\end{aligned}$$

where $q_{\alpha}(x, t)$, $(\alpha = 1, 2, 3)$ represent amplitude of molecular excitation in the α -th spine, the subscripts x and t represent the partial derivatives with respect to the scaled space variable and retarded time variable, respectively, the $*$ denotes the complex conjugate, and ε is a real parameter standing for the strength of higher-order linear and nonlinear effects. When $\varepsilon = 0$, Eq.(1.2) can be reduced to the three-component NLS equation [40, 41]. Indeed, the three-coupled LPD model (1.2) is a completely integrable model, and quite a few properties have been investigated, for instance, the Lax pair [37], the Hirota bilinear forms, the soliton solutions by the DT method and the binary Bell-polynomial approach [38], the semirational rogue waves by the generalized DT method [39]. In this paper, we aim to investigate the soliton solutions of three-coupled LPD model (1.2) via the RH approach, and discuss the dynamic behavior of the soliton solutions.

The organization of this paper is as follows. In section 2, we will construct a specific RH problem based on the inverse scattering transformation. In section 3, we compute N-soliton solutions of the three-coupled LPD model (1.2) from a specific RH problem, which possesses the identity jump matrix on the real axis. In section 4, the spatial structures and collision dynamics behaviors of soliton solutions are examined. In section 5, as a promotion, we briefly explain an integrable generalized multi-component LPD model.

2 The Riemann-Hilbert problem

The three-coupled LPD model (1.2) possesses the following Lax pair [39]

$$\Phi_x = U(x, t, \xi) \Phi = (i \xi \sigma_4 + M(x, t)) \Phi, \tag{2.1a}$$

$$\Phi_t = V(x, t, \xi)\Phi = (-8i\varepsilon\xi^4\sigma_4 + 2i\xi^2\sigma_4 + Q(x, t, \xi))\Phi, \quad (2.1b)$$

where ξ is a complex spectral parameter, $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T$ is a vector function, the 4×4 matrices $\sigma_4, M(x, t)$ are defined by

$$\sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad M(x, t) = \begin{pmatrix} 0 & q_1^* & q_2^* & q_3^* \\ q_1 & 0 & 0 & 0 \\ q_2 & 0 & 0 & 0 \\ q_3 & 0 & 0 & 0 \end{pmatrix}, \quad (2.2)$$

and $Q(x, t, \xi) = Q_{4 \times 4}$ is a matrix-value function defined by

$$\begin{aligned} Q(x, t, \xi) = & -8i\varepsilon M\xi^3 + 4\varepsilon(iM^2\sigma_4 - \sigma_4 M_x)\xi^2 + 2(i\varepsilon M_{xx} + 2i\varepsilon M^3 \\ & - \varepsilon(MM_x - M_x M) + iM)\xi + \varepsilon\sigma_4 M_{xxx} - i\varepsilon\sigma_4(MM_{xx} + M_{xx}M) \\ & - 3i\varepsilon\sigma_4 M^4 - i\sigma_4 M^2 + i\varepsilon M_x^2\sigma_4 + 3\varepsilon\sigma_4(M^2 M_x + M_x M^2) + \sigma_4 M_x. \end{aligned} \quad (2.3)$$

Eq.(2.1a)-(2.1b) is rewritten as

$$\Phi_x - i\xi\sigma_4\Phi = M(x, t)\Phi, \quad (2.4a)$$

$$\Phi_t + (8i\varepsilon\xi^4\sigma_4 - 2i\xi^2\sigma_4)\Phi = Q(x, t, \xi)\Phi, \quad (2.4b)$$

$\tilde{A}(x, t, \xi) = e^{(i\xi x - 8i\varepsilon\xi^4 t + 2i\xi^2 t)\sigma_4}$ is a solution for Eq.(2.4a)-(2.4b), by setting $\Phi(x, t, \xi) = \Psi(x, t, \xi)\tilde{A}(x, t, \xi)$, we have the following Lax pair forms

$$\Psi_x - i\xi[\sigma_4, \Psi] = M(x, t)\Psi, \quad (2.5a)$$

$$\Psi_t + (8i\varepsilon\xi^4 - 2i\xi^2)[\sigma_4, \Psi] = Q(x, t, \xi)\Psi. \quad (2.5b)$$

Now, we construct two Jost solutions $\Psi_{\pm} = \Psi_{\pm}(x, \xi)$ of Eq.(2.5a)

$$\Psi_+ = ([\Psi_+]_1, [\Psi_+]_2, [\Psi_+]_3, [\Psi_+]_4), \quad (2.6a)$$

$$\Psi_- = ([\Psi_-]_1, [\Psi_-]_2, [\Psi_-]_3, [\Psi_-]_4), \quad (2.6b)$$

with the boundary conditions

$$\Psi_+ \rightarrow I, \quad x \rightarrow +\infty, \quad (2.7a)$$

$$\Psi_- \rightarrow \mathbf{I}, \quad x \rightarrow -\infty, \quad (2.7b)$$

where $\mathbf{I} = \text{diag}\{1, 1, 1, 1\}$ is a 4×4 identity matrix, $[\Psi_\pm]_n (n = 1, 2, 3, 4)$ denote the n -th column vector of $\Psi_\pm$. Indeed, for $\xi \in \mathbb{R}$, the two Jost solutions $\Psi_\pm = \Psi_\pm(x, \xi)$ are determined by Volterra integral equations as follows

$$\Psi_+(x, \xi) = \mathbf{I} - \int_x^{+\infty} e^{i\xi\hat{\sigma}_4(x-\zeta)} M(\zeta) \Psi_+(\zeta, \xi) d\zeta, \quad (2.8a)$$

$$\Psi_-(x, \xi) = \mathbf{I} + \int_{-\infty}^x e^{i\xi\hat{\sigma}_4(x-\zeta)} M(\zeta) \Psi_-(\zeta, \xi) d\zeta, \quad (2.8b)$$

where $\hat{\sigma}_4$ represents a matrix operator (see [28]).

If $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the upper half ξ -plane and the lower half ξ -plane, respectively, then $[\Psi_-]_1, [\Psi_+]_2, [\Psi_+]_3$ and $[\Psi_+]_4$ admit analytic extensions to $\mathbb{C}_-$, $[\Psi_+]_1, [\Psi_-]_2, [\Psi_-]_3$ and $[\Psi_-]_4$ admit analytic extensions to the $\mathbb{C}_+$. Furthermore, the determinants of $\Psi_\pm$ are constants for all x because of the Abel's identity and $\text{Tr}(M) = 0$. It follows from the boundary conditions Eq.(2.8) that

$$\det \Psi_\pm = 1, \quad \xi \in \mathbb{R}. \quad (2.9)$$

Next, we introduce a new function $A(x, \xi) = e^{i\xi\sigma_4 x}$, then, Eq.(2.5a) exists two fundamental matrix solutions $\Psi_+ A$ and $\Psi_- A$ which satisfy the following relationship

$$\Psi_- A = \Psi_+ A \cdot S(\xi), \quad \xi \in \mathbb{R}, \quad (2.10)$$

where $S(\xi)$ is a 4×4 scattering matrix. From Eqs.(2.9)-(2.10), we know

$$\det S(\xi) = 1. \quad (2.11)$$

Let $x \rightarrow +\infty$, we have the 4×4 scattering matrix $S(\xi)$

$$S(\xi) = (s_{ij})_{4 \times 4} = \lim_{x \rightarrow +\infty} A^{-1} \Psi_+^{-1} \Psi_- A = \mathbf{I} + \int_{-\infty}^{+\infty} e^{-i\xi\hat{\sigma}_4\zeta} M \Psi_- d\zeta, \quad \xi \in \mathbb{R}. \quad (2.12)$$

It follows from the analytic property of Ψ_- that s_{11} allows analytic extensions to $\mathbb{C}_-$, the others $s_{ij}, i, j \neq 1$ allow analytic extensions to $\mathbb{C}_+$.

In order to discuss behavior of Jost solutions for large ξ , we assume that

$$\Psi = \Psi_0 + \frac{\Psi_1}{\xi} + \frac{\Psi_2}{\xi^2} + \frac{\Psi_3}{\xi^3} + \frac{\Psi_4}{\xi^4} + \cdots, \quad \xi \rightarrow \infty, \quad (2.13)$$

substituting the above expansion Eq.(2.13) into the Eq.(2.5a) and comparing the coefficients of the same order of ξ yields

$$O(\xi^1) : -i[\sigma_4, \Psi_0] = 0, \quad (2.14a)$$

$$O(\xi^0) : \Psi_{0,x} - i[\sigma_4, \Psi_1] - M\Psi_0 = 0, \quad (2.14b)$$

$$O(\xi^{-1}) : \Psi_{1,x} - i[\sigma_4, \Psi_2] - M\Psi_1 = 0, \quad (2.14c)$$

from $O(\xi^1)$ and $O(\xi^0)$, we have

$$i[\sigma_4, \Psi_1] = -M\Psi_0, \quad \Psi_{0,x} = 0. \quad (2.15)$$

On the one hand, we define another new Jost solution for Eq.(2.5a) by

$$H_+ = ([\Psi_-]_1, [\Psi_+]_2, [\Psi_+]_3, [\Psi_+]_4), \quad (2.16)$$

taking Eq.(2.6a)-(2.6b) into Eq.(2.10) gives rise to

$$\begin{aligned} ([\Psi_-]_1, [\Psi_-]_2, [\Psi_-]_3, [\Psi_-]_4) &= \Psi_+ AS(\xi) A^{-1} \\ &= ([\Psi_+]_1, [\Psi_+]_2, [\Psi_+]_3, [\Psi_+]_4) \begin{pmatrix} s_{11} & s_{12}e^{2i\xi x} & s_{13}e^{2i\xi x} & s_{14}e^{2i\xi x} \\ s_{21}e^{-2i\xi x} & s_{22} & s_{23} & s_{24} \\ s_{31}e^{-2i\xi x} & s_{32} & s_{33} & s_{34} \\ s_{41}e^{-2i\xi x} & s_{42} & s_{43} & s_{44} \end{pmatrix}, \end{aligned} \quad (2.17)$$

from Eq.(2.17), we have

$$[\Psi_-]_1 = s_{11}[\Psi_+]_1 + s_{21}e^{-2i\xi x}[\Psi_+]_2 + s_{31}e^{-2i\xi x}[\Psi_+]_3 + s_{41}e^{-2i\xi x}[\Psi_+]_4,$$

then, H_+ can be rewritten as the following matrix form

$$\begin{aligned} H_+ &= ([\Psi_-]_1, [\Psi_+]_2, [\Psi_+]_3, [\Psi_+]_4) = \Psi_+ S_+ \\ &= ([\Psi_+]_1, [\Psi_+]_2, [\Psi_+]_3, [\Psi_+]_4) \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ s_{21}e^{-2i\xi x} & 1 & 0 & 0 \\ s_{31}e^{-2i\xi x} & 0 & 1 & 0 \\ s_{41}e^{-2i\xi x} & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (2.18)$$

which is analytic for $\xi \in \mathbb{C}_-$ and admits asymptotic behavior for very large ξ as

$$H_+ \rightarrow \mathbf{I}, \xi \rightarrow +\infty, \xi \in \mathbb{C}_-. \quad (2.19)$$

On the other hand, in order to obtain the analytic counterpart of H_+ denoted by H_- in $\mathbb{C}_+$, we consider the adjoint scattering equation of Eq.(2.5a) as follows

$$G_x - i\xi[\sigma_4, G] = -GM, \quad (2.20)$$

obviously, the inverse matrices $\Psi_{\pm}^{-1}$ is defined as

$$[\Psi_+]^{-1} = \begin{pmatrix} [\Psi_+]^{-1}{}^1 \\ [\Psi_+]^{-1}{}^2 \\ [\Psi_+]^{-1}{}^3 \\ [\Psi_+]^{-1}{}^4 \end{pmatrix}, \quad [\Psi_-]^{-1} = \begin{pmatrix} [\Psi_-]^{-1}{}^1 \\ [\Psi_-]^{-1}{}^2 \\ [\Psi_-]^{-1}{}^3 \\ [\Psi_-]^{-1}{}^4 \end{pmatrix}, \quad (2.21)$$

which satisfy the adjoint equation (2.20), where $[\Psi_{\pm}^{-1}]^n (n = 1, 2, 3, 4)$ denote the n -th row vector of $\Psi_{\pm}^{-1}$, then, $[\Psi_+]^{-1}{}^1, [\Psi_+]^{-1}{}^2, [\Psi_+]^{-1}{}^3$ and $[\Psi_+]^{-1}{}^4$ admit analytic extensions to $\mathbb{C}_-$, $[\Psi_-]^{-1}{}^1, [\Psi_-]^{-1}{}^2, [\Psi_-]^{-1}{}^3$ and $[\Psi_-]^{-1}{}^4$ admit analytic extensions to the $\mathbb{C}_+$.

In addition, it is not difficult to find that the inverse matrices Ψ_+^{-1} and Ψ_-^{-1} satisfy the following boundary conditions

$$\Psi_+^{-1} \rightarrow \mathbf{I}, \quad x \rightarrow +\infty, \quad (2.22a)$$

$$\Psi_-^{-1} \rightarrow \mathbf{I}, \quad x \rightarrow -\infty, \quad (2.22b)$$

therefore, we can define

$$H_- = \begin{pmatrix} [\Psi_-]^{-1}{}^1 \\ [\Psi_+]^{-1}{}^2 \\ [\Psi_+]^{-1}{}^3 \\ [\Psi_+]^{-1}{}^4 \end{pmatrix}, \quad (2.23)$$

by similar techniques analysis, we can get that H_- is analytic in $\mathbb{C}_+$

$$H_- \rightarrow \mathbf{I}, \xi \rightarrow -\infty, \xi \in \mathbb{C}_+. \quad (2.24)$$

Assume that $R(\xi) = S^{-1}(\xi)$, we obtain

$$\Psi_-^{-1} = AR(\xi)A^{-1}\Psi_+^{-1}, \quad (2.25)$$

taking Eq.(2.21)-(2.6b) into Eq.(2.25) leads to

$$\begin{pmatrix} [\Psi_-^{-1}]^1 \\ [\Psi_-^{-1}]^2 \\ [\Psi_-^{-1}]^3 \\ [\Psi_-^{-1}]^4 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}e^{2i\xi x} & r_{13}e^{2i\xi x} & r_{14}e^{2i\xi x} \\ r_{21}e^{-2i\xi x} & r_{22} & r_{23} & r_{24} \\ r_{31}e^{-2i\xi x} & r_{32} & r_{33} & r_{34} \\ r_{41}e^{-2i\xi x} & r_{42} & r_{43} & r_{44} \end{pmatrix} \begin{pmatrix} [\Psi_+^{-1}]^1 \\ [\Psi_+^{-1}]^2 \\ [\Psi_+^{-1}]^3 \\ [\Psi_+^{-1}]^4 \end{pmatrix}, \quad (2.26)$$

from Eq.(2.26), we have

$$[\Psi_-^{-1}]^1 = r_{11}[\Psi_+^{-1}]^1 + r_{12}e^{2i\xi x}[\Psi_+^{-1}]^2 + r_{13}e^{2i\xi x}[\Psi_+^{-1}]^3 + r_{14}e^{2i\xi x}[\Psi_+^{-1}]^4,$$

then, H_+ can be rewritten as the following matrix form

$$H_- = \begin{pmatrix} [\Psi_-^{-1}]^1 \\ [\Psi_-^{-1}]^2 \\ [\Psi_-^{-1}]^3 \\ [\Psi_-^{-1}]^4 \end{pmatrix} = R_+ \Psi_+^{-1} = \begin{pmatrix} r_{11} & r_{12}e^{2i\xi x} & r_{13}e^{2i\xi x} & r_{14}e^{2i\xi x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} [\Psi_+^{-1}]^1 \\ [\Psi_+^{-1}]^2 \\ [\Psi_+^{-1}]^3 \\ [\Psi_+^{-1}]^4 \end{pmatrix}. \quad (2.27)$$

In fact, these two matrix functions $H_+(x, \xi)$ and $H_-(x, \xi)$ can be used to construct the following RH problem:

- $H_+(x, \xi)$ and $H_-(x, \xi)$ are analytic for ξ in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively.
- $H_+(x, \xi)$ and $H_-(x, \xi)$ satisfy the following jumping relationship on the real x -axis

$$H_-(x, \xi)H_+(x, \xi) = T(x, \xi), \quad \xi \in \mathbb{R}, \quad (2.28)$$

where

$$T(x, \xi) = \begin{pmatrix} 1 & r_{12}e^{2i\xi x} & r_{13}e^{2i\xi x} & r_{14}e^{2i\xi x} \\ s_{21}e^{-2i\xi x} & 1 & 0 & 0 \\ s_{31}e^{-2i\xi x} & 0 & 1 & 0 \\ s_{41}e^{-2i\xi x} & 0 & 0 & 1 \end{pmatrix}, \quad (2.29)$$

and $r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} + r_{14}s_{41} = 1$.

- $H_{\pm} \rightarrow I, \quad \xi \rightarrow \infty$.

Furthermore, since Ψ_- satisfies the temporal part of spectral equation

$$\Psi_{-,t} + (8i\varepsilon\xi^4 - 2i\xi^2)[\sigma_4, \Psi_-] = Q(x, t, \xi)\Psi_-, \quad (2.30)$$

from Eq.(2.10), we have

$$(\Psi_+AS)_t + (8i\varepsilon\xi^4 - 2i\xi^2)[\sigma_4, \Psi_+AS] = Q(x, t, \xi)\Psi_+AS, \quad (2.31)$$

for $x \rightarrow \infty$, assume that q_α , ($\alpha = 1, 2, 3$) are sufficient smoothness and fast decay, the matrix $Q(x, t, \xi) \rightarrow 0$ as $x \rightarrow \pm\infty$. Let $x \rightarrow +\infty$ in Eq.(2.31), we obtain

$$S_t = -(8i\varepsilon\xi^4 - 2i\xi^2)[\sigma_4, S], \quad (2.32)$$

which means that the scattering data $s_{11}, s_{22}, s_{33}, s_{23}, s_{32}$ are time independent

$$s_{11,t} = s_{22,t} = s_{33,t} = s_{44,t} = s_{23,t} = s_{24,t} = s_{32,t} = s_{34,t} = s_{42,t} = s_{43,t} = 0, \quad (2.33)$$

and the other scattering data satisfy

$$\begin{aligned} r_{12}(t, \xi) &= r_{12}(0, \xi)e^{-4i(4\varepsilon\xi^4 - \xi^2)t}, \quad r_{13}(t, \xi) = r_{13}(0, \xi)e^{-4i(4\varepsilon\xi^4 - \xi^2)t}, \\ r_{14}(t, \xi) &= r_{14}(0, \xi)e^{-4i(4\varepsilon\xi^4 - \xi^2)t}, \quad s_{21}(t, \xi) = s_{21}(0, \xi)e^{4i(4\varepsilon\xi^4 - \xi^2)t}, \\ s_{31}(t, \xi) &= s_{31}(0, \xi)e^{4i(4\varepsilon\xi^4 - \xi^2)t}, \quad s_{41}(t, \xi) = s_{41}(0, \xi)e^{4i(4\varepsilon\xi^4 - \xi^2)t}. \end{aligned} \quad (2.34)$$

3 The soliton solutions for three-coupled LPD model

In this section, based on the RH problem constructed in Sec. 2, we would like to formulate the N-soliton solutions of three-coupled LPD model (1.2). In fact, the solution to this RH problem will not be unique unless the zeros of $\det H_+$ and $\det H_-$ in the upper and lower half of the ξ -plane are also specified, and the kernel structures of $H_\pm$ at these zeros are provided. It follows from the definitions of H_+ and H_- that

$$\det H_+(x, \xi) = s_{11}(\xi), \quad \det H_-(x, \xi) = r_{11}(\xi), \quad (3.1)$$

which means that the zeros of $\det H_+$ and $\det H_-$ are the same as $s_{11}(\xi)$ and $r_{11}(\xi)$, respectively. Furthermore, the scattering data s_{11} and r_{11} are time independent (2.33), we can get that the roots of $s_{11} = 0$ and $r_{11} = 0$ are also time independent, since

$$\sigma_4 M(x, t) \sigma_4 = -M(x, t), \quad M^\dagger(x, t) = M(x, t),$$

where $\dagger$ represents the Hermitian of a matrix. It is easy to see that

$$\Psi_{\pm}(x, t, -\xi) = \sigma_4 \Psi_{\pm}(x, t, \xi) \sigma_4, \quad \Psi_{\pm}^{-1}(x, t, \xi) = \Psi_{\pm}^{\dagger}(x, t, \xi^*), \quad (3.2)$$

according to reduction conditions given by Eqs.(2.6a)-(2.6b), we obtain

$$S(-\xi) = \sigma_4 S(\xi) \sigma_4, \quad R(\xi) = S^{-1}(\xi) = S^{\dagger}(\xi^*), \quad H_{-}(x, \xi) = H_{+}^{\dagger}(x, \xi^*). \quad (3.3)$$

Assume that s_{11} enjoys $N \geq 0$ possible zeros denoted as $\{\xi_m, 1 \leq m \leq N\}$ in $\mathbb{C}_{-}$, r_{11} enjoys $N \geq 0$ possible zeros denoted as $\{\tilde{\xi}_m, 1 \leq m \leq N\}$ in $\mathbb{C}_{+}$. For the convenience of later discussion, without loss of generality, we suppose that all zeros $\{(\xi_m, \tilde{\xi}_m), m = 1, 2, \dots, N\}$ of s_{11} and r_{11} are simple zeros. In this case, each of $\ker H_{+}(\xi_m)$ and $\ker H_{-}(\tilde{\xi}_m)$ includes only a single column vector v_m and row vector $\tilde{v}_m$, respectively, i.e.

$$H_{+}(\xi_m)v_m = 0, \quad \tilde{v}_m H_{-}(\tilde{\xi}_m) = 0, \quad (3.4)$$

because $H_{+}(\xi)$ is the solution of Eq.(2.5a), for large ξ , we suppose that $H_{+}(\xi)$ possesses the following asymptotic expansion

$$H_{+} = I + \frac{H_{+}^{(1)}}{\xi} + O(\xi^{-2}), \quad \xi \rightarrow \infty, \quad (3.5)$$

taking Eq.(3.5) into Eq.(2.5a) and comparing $O(1)$ terms gets

$$M = -i[\sigma_4, H_{+}^{(1)}] = \begin{pmatrix} 0 & -2i(H_{+}^{(1)})_{12} & -2i(H_{+}^{(1)})_{13} & -2i(H_{+}^{(1)})_{14} \\ 2i(H_{+}^{(1)})_{21} & 0 & 0 & 0 \\ 2i(H_{+}^{(1)})_{31} & 0 & 0 & 0 \\ 2i(H_{+}^{(1)})_{41} & 0 & 0 & 0 \end{pmatrix}, \quad (3.6)$$

then, the potential functions solutions $q_1(x, t)$, $q_2(x, t)$ and $q_3(x, t)$ of the three-coupled LPD model (1.2) can be reconstructed by

$$q_1(x, t) = 2i(H_{+}^{(1)})_{21}, \quad q_2(x, t) = 2i(H_{+}^{(1)})_{31}, \quad q_3(x, t) = 2i(H_{+}^{(1)})_{41}, \quad (3.7)$$

where $H_{+}^{(1)} = (H_{+}^{(1)})_{4 \times 4}$ and $(H_{+}^{(1)})_{ij}$ are the (i, j) -entry of $H_{+}^{(1)}$, $i, j = 1, 2, 3, 4$.

In order to get the spatial evolutions for vectors $v_m(x, t)$, on the one hand, taking the derivative of equation $H_{+}v_m = 0$ with respect to x and with Eq.(2.5a), we obtain

$$H_{+}v_{m,x} + i\xi_m H_{+} \sigma_4 v_m = 0, \quad (3.8)$$

thus

$$v_{m,x} = -i\xi_m \sigma_4 v_m, \quad (3.9)$$

on the other hand, taking the derivative of equation $H_+ v_m = 0$ with respect to t and with Eq.(2.5b), we get

$$H_+ v_{m,t} - (8i\varepsilon \xi_m^4 - 2i\xi_m^2) H_+ \sigma_4 v_m = 0, \quad (3.10)$$

thus

$$v_{m,t} = (8i\varepsilon \xi_m^4 - 2i\xi_m^2) \sigma_4 v_m, \quad (3.11)$$

according to Eq.(3.9) and Eq.(3.11), we have

$$v_m(x, t) = e^{-i\xi_m \sigma_4 x + (8i\varepsilon \xi_m^4 - 2i\xi_m^2) \sigma_4 t} v_{m0} e^{\int_{x_0}^x \rho_m(y) dy + \int_{t_0}^t \omega_m(\tau) d\tau}, \quad (3.12)$$

where v_{m0} is constant, $\rho_m(y)$ and $\omega_m(\tau)$ are two scalar functions, at the same time, we have

$$\tilde{v}_m(x, t) = v_m^\dagger(x, t). \quad (3.13)$$

In order to obtain multi-soliton solutions for the three-coupled LPD model (1.2), we assume that the jump matrix $T(x, \xi) = I$ is a 4×4 unit matrix in Eq.(2.28). In other words, the discrete scattering data $r_{12} = r_{13} = r_{14} = s_{21} = s_{31} = s_{41} = 0$, consequently, the unique solution to this special RH problem can be described as

$$H_+(\xi) = I + \sum_{m=1}^N \sum_{n=1}^N \frac{v_m \tilde{v}_n (P^{-1})_{mn}}{\xi - \tilde{\xi}_n}, \quad (3.14a)$$

$$H_-(\xi) = I - \sum_{m=1}^N \sum_{n=1}^N \frac{v_m \tilde{v}_n (P^{-1})_{mn}}{\xi - \tilde{\xi}_n}, \quad (3.14b)$$

where $P = (P_{mn})_{N \times N}$ is a $N \times N$ matrix defined by

$$P_{mn} = \frac{\tilde{v}_m v_n}{\xi_m - \xi_n}, \quad 1 \leq m, n \leq N, \quad (3.15)$$

therefore, from (3.14a) and (3.14b), we obtain

$$H_+^{(1)} = \sum_{m=1}^N \sum_{n=1}^N v_m (P^{-1})_{mn} \tilde{v}_n, \quad (3.16)$$

choosing $v_{m0} = (a_m, b_m, c_m, d_m)^T$ and $\rho_m = \omega_m = 0$, from (3.16), we get that the general N-soliton solution for the three-coupled LPD model (1.2) reads

$$q_1 = 2i \sum_{m=1}^N \sum_{n=1}^N b_m^* a_n e^{\theta_n^* - \theta_m} (P^{-1})_{mn}, \quad (3.17a)$$

$$q_2 = 2i \sum_{m=1}^N \sum_{n=1}^N c_m^* a_n e^{\theta_n^* - \theta_m} (P^{-1})_{mn}, \quad (3.17b)$$

$$q_c = 2i \sum_{m=1}^N \sum_{n=1}^N d_m^* a_n e^{\theta_n^* - \theta_m} (P^{-1})_{mn}, \quad (3.17c)$$

where $\theta_n = i\xi_n x - 8i\varepsilon \xi_n^4 t + 2i\xi_n^2 t$, $P = (P_{mn})_{N \times N}$ is a $N \times N$ matrix with elements given by

$$P_{mn} = \frac{a_m^* a_n e^{\theta_m^* + \theta_n} + (b_m^* b_n + c_m^* c_n + d_m^* d_n) e^{-(\theta_m^* + \theta_n)}}{\xi_m^* - \xi_n}, \quad 1 \leq m, n \leq N. \quad (3.18)$$

Furthermore, the N-soliton solutions Eq.(3.17a)-(3.17c) can be rewritten as the following determinant ratio form

$$q_1(x, t) = -2i \frac{\det D_1}{\det P}, \quad q_2(x, t) = -2i \frac{\det D_2}{\det P}, \quad q_3(x, t) = -2i \frac{\det D_3}{\det P}, \quad (3.19)$$

where the $N \times N$ matrix P is defined by Eq.(3.18), and D_1, D_2, D_3 are $(N+1) \times (N+1)$ matrices given by

$$\begin{aligned} D_1 &= \begin{pmatrix} P_{11} & \cdots & P_{1N} & a_1 e^{-\theta_1} \\ \vdots & \vdots & \vdots & \vdots \\ P_{N1} & \cdots & P_{NN} & a_N e^{-\theta_N} \\ b_1^* e^{\theta_1^*} & \cdots & b_N^* e^{\theta_N^*} & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} P_{11} & \cdots & P_{1N} & a_1 e^{-\theta_1} \\ \vdots & \vdots & \vdots & \vdots \\ P_{N1} & \cdots & P_{NN} & a_N e^{-\theta_N} \\ c_1^* e^{\theta_1^*} & \cdots & c_N^* e^{\theta_N^*} & 0 \end{pmatrix}, \\ D_3 &= \begin{pmatrix} P_{11} & \cdots & P_{1N} & a_1 e^{-\theta_1} \\ \vdots & \vdots & \vdots & \vdots \\ P_{N1} & \cdots & P_{NN} & a_N e^{-\theta_N} \\ d_1^* e^{\theta_1^*} & \cdots & d_N^* e^{\theta_N^*} & 0 \end{pmatrix}. \end{aligned} \quad (3.20)$$

4 Dynamic behaviors of soliton solutions

In what follows, we can examine the nonlinear dynamical behaviors of the breather and soliton solutions for the three-coupled LPD model (1.2).

Firstly, as a simple example, choosing $N = 1$ in Eqs.(3.17a)-(3.17c) and with Eq.(3.15) gives rise to the following single-soliton solution of the three-coupled LPD model (1.2)

$$q_1(x, t) = \frac{2ib_1^*a_1e^{\theta_1^*-\theta_1}(\xi_1^*-\xi_1)}{|a_1|^2e^{\theta_1^*+\theta_1} + (|b_1|^2 + |c_1|^2 + |d_1|^2)e^{-(\theta_1^*+\theta_1)}}, \quad (4.1a)$$

$$q_2(x, t) = \frac{2ic_1^*a_1e^{\theta_1^*-\theta_1}(\xi_1^*-\xi_1)}{|a_1|^2e^{\theta_1^*+\theta_1} + (|b_1|^2 + |c_1|^2 + |d_1|^2)e^{-(\theta_1^*+\theta_1)}}, \quad (4.1b)$$

$$q_3(x, t) = \frac{2id_1^*a_1e^{\theta_1^*-\theta_1}(\xi_1^*-\xi_1)}{|a_1|^2e^{\theta_1^*+\theta_1} + (|b_1|^2 + |c_1|^2 + |d_1|^2)e^{-(\theta_1^*+\theta_1)}}, \quad (4.1c)$$

where $\theta_1 = i\xi_1x - 8i\varepsilon\xi_1^4t + 2i\xi_1^2t$. Let $\xi_1 = \beta_1 + i\gamma_1$, $\varepsilon = \frac{1}{2}$, $a_1 = 1$, $|b_1|^2 + |c_1|^2 + |d_1|^2 = e^{-2\eta_1}$, the single-soliton solution Eqs.(4.1a)-(4.1c) becomes to

$$q_1(x, t) = 2b_1^*\gamma_1e^{-2i[\beta_1x-4(\beta_1^4-6\beta_1^2\gamma_1^2+\gamma_1^4)t+2(\beta_1^2-\gamma_1^2)t+\eta_1]}\text{sech}\Delta(x, t), \quad (4.2a)$$

$$q_2(x, t) = 2c_1^*\gamma_1e^{-2i[\beta_1x-4(\beta_1^4-6\beta_1^2\gamma_1^2+\gamma_1^4)t+2(\beta_1^2-\gamma_1^2)t+\eta_1]}\text{sech}\Delta(x, t), \quad (4.2b)$$

$$q_3(x, t) = 2d_1^*\gamma_1e^{-2i[\beta_1x-4(\beta_1^4-6\beta_1^2\gamma_1^2+\gamma_1^4)t+2(\beta_1^2-\gamma_1^2)t+\eta_1]}\text{sech}\Delta(x, t), \quad (4.2c)$$

where $\Delta(x, t) = -2\gamma_1x + 32\beta_1\gamma_1(\beta_1^2 - \gamma_1^2)t - 8\beta_1\gamma_1t + \eta_1$.

By choosing different parameters, we can obtain some plots which are displayed to particularly describe the dynamic behaviors for single-soliton solution Eqs.(4.2a)-(4.2c) in the following. Figure 1 illustrates that the single-soliton solution $q_1(x, t)$ is a line-soliton, and its widths, amplitudes and velocities remain invariant during the propagation.

Secondly, as a special example, choosing $N = 2$ in Eqs.(3.17a)-(3.17c) and with Eq.(3.15) arrives at the following two-soliton solution of the three-coupled LPD model (1.2)

$$q_1(x, t) = 2ib_1^*a_1e^{\theta_1^*-\theta_1}(P^{-1})_{11} + 2ib_1^*a_2e^{\theta_1^*-\theta_2}(P^{-1})_{12} \\ + 2ib_2^*a_1e^{\theta_2^*-\theta_1}(P^{-1})_{21} + 2ib_2^*a_2e^{\theta_2^*-\theta_2}(P^{-1})_{22}, \quad (4.3a)$$

$$q_2(x, t) = 2ic_1^*a_1e^{\theta_1^*-\theta_1}(P^{-1})_{11} + 2ic_1^*a_2e^{\theta_1^*-\theta_2}(P^{-1})_{12} \\ + 2ic_2^*a_1e^{\theta_2^*-\theta_1}(P^{-1})_{21} + 2ic_2^*a_2e^{\theta_2^*-\theta_2}(P^{-1})_{22}, \quad (4.3b)$$

$$q_3(x, t) = 2id_1^*a_1e^{\theta_1^*-\theta_1}(P^{-1})_{11} + 2id_1^*a_2e^{\theta_1^*-\theta_2}(P^{-1})_{12}$$

Figure 1: The single-soliton solution (4.2a) $q_1(x,t)$ for three-coupled LPD model (1.2): $\eta_1 = 0, b_1 = 1, \beta_1 = \frac{1}{5}, \gamma_1 = -\frac{1}{5}$. (a) Perspective view of modulus of $q_1(x,t)$. (b) Density plot of modulus of $q_1(x,t)$. (c) The wave propagation along the x -axis with $t = 0$.

$$+ 2id_2^* a_1 e^{\theta_2^* - \theta_1} (P^{-1})_{21} + 2id_2^* a_2 e^{\theta_2^* - \theta_2} (P^{-1})_{22}, \quad (4.3c)$$

where $\theta_l = i\xi_l x - 8i\varepsilon\xi_l^4 t + 2i\xi_l^2 t$, ($l = 1, 2$), and $P = (P_{mn})_{2 \times 2}$ is a 2×2 matrix with the following elements

$$\begin{aligned} P_{11} &= \frac{|a_1|^2 e^{\theta_1 + \theta_1^*} + (|b_1|^2 + |c_1|^2 + |d_1|^2) e^{-(\theta_1 + \theta_1^*)}}{\xi_1^* - \xi_1}, \\ P_{12} &= \frac{a_1^* a_2 e^{\theta_2 + \theta_1^*} + (b_1^* b_2 + c_1^* c_2 + d_1^* d_2) e^{-(\theta_2 + \theta_1^*)}}{\xi_1^* - \xi_2}, \\ P_{21} &= \frac{a_2^* a_1 e^{\theta_1 + \theta_2^*} + (b_2^* b_1 + c_2^* c_1 + d_2^* d_1) e^{-(\theta_1 + \theta_2^*)}}{\xi_2^* - \xi_1}, \\ P_{22} &= \frac{|a_2|^2 e^{\theta_2 + \theta_2^*} + (|b_2|^2 + |c_2|^2 + |d_2|^2) e^{-(\theta_2 + \theta_2^*)}}{\xi_2^* - \xi_2}. \end{aligned}$$

Let $\xi_l = \beta_l + i\gamma_l$, ($l = 1, 2$), $\varepsilon = \frac{1}{2}$, $a_1 = a_2 = 1, b_1 = b_2, c_1 = c_2, d_1 = d_2$, $|b_l|^2 + |c_l|^2 + |d_l|^2 = e^{-2\eta_2}$, the two-soliton solution Eqs.(4.3a)-(4.3c) becomes to

$$q_1(x,t) = \frac{2ib_1^* [e^{\theta_1^* - \theta_1} P_{22} - e^{\theta_2^* - \theta_1} P_{21} - e^{\theta_1^* - \theta_2} P_{12} + e^{\theta_2^* - \theta_2} P_{11}]}{P_{11}P_{22} - P_{12}P_{21}}, \quad (4.4a)$$

$$q_2(x,t) = \frac{2ic_1^* [e^{\theta_1^* - \theta_1} P_{22} - e^{\theta_2^* - \theta_1} P_{21} - e^{\theta_1^* - \theta_2} P_{12} + e^{\theta_2^* - \theta_2} P_{11}]}{P_{11}P_{22} - P_{12}P_{21}}, \quad (4.4b)$$

$$q_3(x, t) = \frac{2id_1^*[e^{\theta_1^* - \theta_1} P_{22} - e^{\theta_2^* - \theta_1} P_{21} - e^{\theta_1^* - \theta_2} P_{12} + e^{\theta_2^* - \theta_2} P_{11}]}{P_{11}P_{22} - P_{12}P_{21}}, \quad (4.4c)$$

where

$$P_{11} = \frac{ie^{-\eta_2}}{\gamma_1} \cosh(\theta_1 + \theta_1^* + \eta_2), \quad P_{12} = \frac{2e^{-\eta_2}}{(\beta_1 - \beta_2) - i(\gamma_1 + \gamma_2)} \cosh(\theta_2 + \theta_1^* + \eta_2),$$

$$P_{22} = \frac{ie^{-\eta_2}}{\gamma_2} \cosh(\theta_2 + \theta_2^* + \eta_2), \quad P_{21} = \frac{2e^{-\eta_2}}{(\beta_2 - \beta_1) - i(\gamma_2 + \gamma_1)} \cosh(\theta_1 + \theta_2^* + \eta_2).$$

By choosing different parameters, we can obtain the interaction solutions Eqs.(4.4a)-(4.4c) which are displayed in Figure 2 and Figure 3. Figure 2 illustrates that soliton transmission, and the two solitons moving over each other, and their polarizations remaining unchanged. Figure 3 illustrates that soliton reflection produces the breather soliton solution.

Figure 2: The two-soliton solution (4.4a) $q_1(x, t)$ for three-coupled LPD model (1.2) by choosing suitable parameters: (a) $\eta_2 = 0, b_1 = 1, \beta_1 = -0.3, \beta_2 = 0.4, \gamma_1 = -0.3, \gamma_2 = 0.3$. (b) $\eta_2 = 0, b_1 = 1, \beta_1 = 0.7, \beta_2 = -0.5, \gamma_1 = 0.2, \gamma_2 = -0.3$.

Finally, as another special example, choosing $N = 3$ in Eqs.(3.17a)-(3.17c) and with Eq.(3.15) arrives at the following three-soliton solution of the three-coupled LPD model (1.2)

$$q_1(x, t) = 2ib_1^*a_1e^{\theta_1^* - \theta_1}(P^{-1})_{11} + 2ib_1^*a_2e^{\theta_1^* - \theta_2}(P^{-1})_{12} + 2ib_1^*a_3e^{\theta_1^* - \theta_3}(P^{-1})_{13}$$

$$+ 2ib_2^*a_1e^{\theta_2^* - \theta_1}(P^{-1})_{21} + 2ib_2^*a_2e^{\theta_2^* - \theta_2}(P^{-1})_{22} + 2ib_2^*a_3e^{\theta_2^* - \theta_3}(P^{-1})_{23}$$

Figure 3: The breather soliton solution (4.4a) $q_1(x, t)$ for three-coupled LPD model (1.2) by choosing suitable parameters: $\eta_2 = 0, b_1 = 0.5 - 0.5i, \beta_1 = 0.4, \beta_2 = -0.7, \gamma_1 = 0.5, \gamma_2 = -0.2$. (a) Perspective view of modulus of breather soliton $q_1(x, t)$. (b) Density plot of modulus of breather soliton $q_1(x, t)$.

$$+ 2ib_3^*a_1e^{\theta_3^*-\theta_1}(P^{-1})_{31} + 2ib_3^*a_2e^{\theta_3^*-\theta_2}(P^{-1})_{32} + 2ib_3^*a_3e^{\theta_3^*-\theta_3}(P^{-1})_{33}, \quad (4.5a)$$

$$\begin{aligned} q_2(x, t) = & 2ic_1^*a_1e^{\theta_1^*-\theta_1}(P^{-1})_{11} + 2ic_1^*a_2e^{\theta_1^*-\theta_2}(P^{-1})_{12} + 2ic_1^*a_3e^{\theta_1^*-\theta_3}(P^{-1})_{13} \\ & + 2ic_2^*a_1e^{\theta_2^*-\theta_1}(P^{-1})_{21} + 2ic_2^*a_2e^{\theta_2^*-\theta_2}(P^{-1})_{22} + 2ic_2^*a_3e^{\theta_2^*-\theta_3}(P^{-1})_{23} \\ & + 2ic_3^*a_1e^{\theta_3^*-\theta_1}(P^{-1})_{31} + 2ic_3^*a_2e^{\theta_3^*-\theta_2}(P^{-1})_{32} + 2ic_3^*a_3e^{\theta_3^*-\theta_3}(P^{-1})_{33}, \end{aligned} \quad (4.5b)$$

$$\begin{aligned} q_3(x, t) = & 2id_1^*a_1e^{\theta_1^*-\theta_1}(P^{-1})_{11} + 2id_1^*a_2e^{\theta_1^*-\theta_2}(P^{-1})_{12} + 2id_1^*a_3e^{\theta_1^*-\theta_3}(P^{-1})_{13} \\ & + 2id_2^*a_1e^{\theta_2^*-\theta_1}(P^{-1})_{21} + 2id_2^*a_2e^{\theta_2^*-\theta_2}(P^{-1})_{22} + 2id_2^*a_3e^{\theta_2^*-\theta_3}(P^{-1})_{23} \\ & + 2id_3^*a_1e^{\theta_3^*-\theta_1}(P^{-1})_{31} + 2id_3^*a_2e^{\theta_3^*-\theta_2}(P^{-1})_{32} + 2id_3^*a_3e^{\theta_3^*-\theta_3}(P^{-1})_{33}, \end{aligned} \quad (4.5c)$$

where $\theta_j = i\xi_j x - 8i\varepsilon\xi_j^4 t + 2i\xi_j^2 t$, ($j = 1, 2, 3$), and $M = (P_{mn})_{3 \times 3}$ is a 3×3 matrix with following elements

$$\begin{aligned} P_{11} &= \frac{|a_1|^2 e^{\theta_1 + \theta_1^*} + (|b_1|^2 + |c_1|^2 + |d_1|^2) e^{-(\theta_1 + \theta_1^*)}}{\xi_1^* - \xi_1}, \\ P_{12} &= \frac{a_1^* a_2 e^{\theta_2 + \theta_1^*} + (b_1^* b_2 + c_1^* c_2 + d_1^* d_2) e^{-(\theta_2 + \theta_1^*)}}{\xi_1^* - \xi_2}, \\ P_{13} &= \frac{a_1^* a_3 e^{\theta_3 + \theta_1^*} + (b_1^* b_3 + c_1^* c_3 + d_1^* d_3) e^{-(\theta_3 + \theta_1^*)}}{\xi_1^* - \xi_3}, \end{aligned}$$

$$\begin{aligned}
P_{21} &= \frac{a_2^* a_1 e^{\theta_1 + \theta_2^*} + (b_2^* b_1 + c_2^* c_1 + d_2^* d_1) e^{-(\theta_1 + \theta_2^*)}}{\xi_2^* - \xi_1}, \\
P_{22} &= \frac{|a_2|^2 e^{\theta_2 + \theta_2^*} + (|b_2|^2 + |c_2|^2 + |d_2|^2) e^{-(\theta_2 + \theta_2^*)}}{\xi_2^* - \xi_2}, \\
P_{23} &= \frac{a_2^* a_3 e^{\theta_3 + \theta_2^*} + (b_2^* b_3 + c_2^* c_3 + d_2^* d_3) e^{-(\theta_3 + \theta_2^*)}}{\xi_2^* - \xi_3}, \\
P_{31} &= \frac{a_3^* a_1 e^{\theta_1 + \theta_3^*} + (b_3^* b_1 + c_3^* c_1 + d_3^* d_1) e^{-(\theta_1 + \theta_3^*)}}{\xi_3^* - \xi_1}, \\
P_{32} &= \frac{a_3^* a_2 e^{\theta_2 + \theta_3^*} + (b_3^* b_2 + c_3^* c_2 + d_3^* d_2) e^{-(\theta_2 + \theta_3^*)}}{\xi_3^* - \xi_2}, \\
P_{33} &= \frac{|a_3|^2 e^{\theta_3 + \theta_3^*} + (|b_3|^2 + |c_3|^2 + |d_3|^2) e^{-(\theta_3 + \theta_3^*)}}{\xi_3^* - \xi_3}.
\end{aligned}$$

Let $\xi_j = \beta_j + i\gamma_j$, ($j = 1, 2, 3$), $\varepsilon = \frac{1}{2}$, $a_1 = a_2 = a_3 = 1$, $b_1 = b_2 = b_3$, $c_1 = c_2 = c_3$, $d_1 = d_2 = d_3$, $|b_j|^2 + |c_j|^2 + |d_j|^2 = e^{-2\eta_3}$, by choosing different parameters, we can obtain that the interaction solutions Eqs.(4.5a)-(4.5c) are displayed in Figure 4 and Figure 5. Figure 4 illustrates that soliton transmission, and the three solitons interact each other. Figure 5 illustrates that soliton reflection produces the breather and soliton solution.

5 Discussions and conclusions

Indeed, as a promotion, the integrable three-coupled LPD model (1.2) can be extended to the integrable generalized N-component LPD model as follows:

$$\begin{aligned}
& i q_{\alpha,t} + q_{\alpha,xx} + 2 \sum_{k=1}^N |q_k|^2 q_{\alpha} + \varepsilon [q_{\alpha,xxxx} + 2 \sum_{k=1}^N |q_{k,x}|^2 q_{\alpha} \\
& + 2 \sum_{k=1}^N q_k q_{k,x}^* q_{\alpha,x} + 6 \sum_{k=1}^N q_k^* q_{k,x} q_{\alpha,x} + 4 \sum_{k=1}^N |q_k|^2 q_{\alpha,xx} \\
& + 4 \sum_{k=1}^N q_k^* q_{k,xx} q_{\alpha} + 2 \sum_{k=1}^N q_k q_{k,xx}^* q_{\alpha} + 6 \left(\sum_{k=1}^N |q_k|^2 \right)^2 q_{\alpha}] = 0, \tag{5.1}
\end{aligned}$$

where $q_{\alpha}(x, t)$, ($\alpha = 1, 2, \dots, N$) represents amplitude of molecular excitation in the α -th spine, ε is a real parameter which stands for the strength of higher-order linear and

Figure 4: The three-soliton solution (4.5a) $q_1(x, t)$ for three-coupled LPD model (1.2) by choosing suitable parameters: $\eta_3 = 0, b_1 = b_2 = b_3 = 1, \beta_1 = 0.65, \beta_2 = -0.5, \beta_3 = 0.3, \gamma_1 = -0.05, \gamma_2 = 0.25, \gamma_3 = 0.35$. (a) Perspective view of modulus of three-soliton $q_1(x, t)$. (b) Density plot of modulus of breather soliton $q_1(x, t)$.

Figure 5: The breather soliton solution (4.5a) $q_1(x, t)$ for three-coupled LPD model (1.2) by choosing suitable parameters: $\eta_3 = 0, b_1 = b_2 = b_3 = 1, \beta_1 = 0.4, \beta_2 = -0.1, \beta_3 = 0.1, \gamma_1 = -0.5, \gamma_2 = 0.5, \gamma_3 = 0.01$. (a) Perspective view of modulus of three-soliton $q_1(x, t)$. (b) Density plot of modulus of breather soliton $q_1(x, t)$.

nonlinear effects. Let $\mathbf{q} = (q_1, q_2, \dots, q_N)^T$, Eq.(5.1) has the following vector form

$$\begin{aligned} i\mathbf{q}_t + \mathbf{q}_{xx} + 2\mathbf{q}\mathbf{q}^\dagger\mathbf{q} + \varepsilon[\mathbf{q}_{xxxx} + 2\mathbf{q}_x\mathbf{q}_x^\dagger\mathbf{q} + 2\mathbf{q}\mathbf{q}_x^\dagger\mathbf{q}_x \\ + 6\mathbf{q}_x\mathbf{q}^\dagger\mathbf{q}_x + 4\mathbf{q}\mathbf{q}^\dagger\mathbf{q}_{xx} + 4\mathbf{q}_{xx}\mathbf{q}^\dagger\mathbf{q} + 2\mathbf{q}\mathbf{q}_{xx}^\dagger\mathbf{q} + 6(\mathbf{q}\mathbf{q}^\dagger)^2\mathbf{q}] = 0, \end{aligned} \quad (5.2)$$

Eq.(5.1) possesses the following $(N+1) \times (N+1)$ matrix spectral problem

$$\Phi_x = [i\xi\Lambda + \widetilde{M}(x, t)]\Phi, \quad (5.3a)$$

$$\Phi_t = [-8i\varepsilon\xi^4\Lambda + 2i\xi^2\Lambda + \widetilde{Q}(x, t, \xi)]\Phi, \quad (5.3b)$$

where ξ is a complex spectral parameter, $\Lambda, \widetilde{M}(x, t)$ are $(N+1) \times (N+1)$ matrices defined by

$$\Lambda = \begin{pmatrix} 1 & \mathbf{0}_{1 \times N} \\ \mathbf{0}_{1 \times N} & -\mathbf{I}_{1 \times N} \end{pmatrix}, \quad \widetilde{M}(x, t) = \begin{pmatrix} 0 & \mathbf{q}^\dagger \\ \mathbf{q} & \mathbf{0}_{N \times N} \end{pmatrix}, \quad (5.4)$$

and $\widetilde{Q}(x, t, \xi)$ is a $(N+1) \times (N+1)$ matrix as follows

$$\begin{aligned} \widetilde{Q}(x, t, \xi) = & -8i\varepsilon\widetilde{M}\xi^3 + 4\varepsilon(i\widetilde{M}^2\Lambda - \Lambda\widetilde{M}_x)\xi^2 \\ & + 2(i\varepsilon\widetilde{M}_{xx} + 2i\varepsilon\widetilde{M}^3 - \varepsilon(\widetilde{M}\widetilde{M}_x - \widetilde{M}_x\widetilde{M}) + i\widetilde{M})\xi \\ & + \varepsilon\Lambda\widetilde{M}_{xxx} - i\varepsilon\Lambda(\widetilde{M}\widetilde{M}_{xx} + \widetilde{M}_{xx}\widetilde{M}) - 3i\varepsilon\Lambda\widetilde{M}^4 - i\Lambda\widetilde{M}^2 \\ & + i\varepsilon\widetilde{M}_x^2\Lambda + 3\varepsilon\Lambda(\widetilde{M}^2\widetilde{M}_x + \widetilde{M}_x\widetilde{M}^2) + \Lambda\widetilde{M}_x. \end{aligned} \quad (5.5)$$

Accordingly, we can also examine the N-soliton solutions to the integrable generalized N-component LPD model (5.1) by the same way in Section 3. However, we don't report them here since the procedure is mechanical.

In this work, we utilize the RH approach to study the three-coupled LPD model (1.2). By constructing a special matrix RH problem, we have obtained the multi-soliton solution of the LPD model (1.2). In addition, some graphical analysis gives the dynamic characteristics of the soliton solution, including the interaction of single soliton, two-solitons, three-soliton and breather soliton solutions. It is hoped that our results can help enrich and explain other nonlinear integrable models.

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