An Investigation on M-Polar Fuzzy Saturation Graph and Its Application

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An investigation on m -polar fuzzy saturation graph and its application

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Abstract

The saturation graph is a well-defined topic. But, saturation in a fuzzy graph was defined recently and investigated many properties. In a fuzzy saturation graph, only one saturation is considered for every vertex. In a m -polar fuzzy graph (m PFG), each vertex and edge has a m number of membership values. So, defining saturation for m PFG is not easy and needs some new ideas. By considering m saturation for each component out of m components of membership values of a vertex, we defined saturation in m PFG. α -saturation as well as β -saturation in m PFG is introduced here. Many interesting properties of it are also presented. α -vertex count and β -vertex count in m PFG are also studied and the upper bound on some well known m PFG is also found here. Finally, a real-life application using saturation in m PFG is also presented.

Keywords: m -polar fuzzy graph, α -vertex count of m -polar fuzzy graph, β -vertex count of m -polar fuzzy graph, saturation in m -polar fuzzy graph.

1 Introduction

1.1 Research background and Related works

Real-world conditions can suitably be stated through a model which consists of nodes set along with lines connecting particular pairs of nodes. Due to graph theory, we present the current interlink between the network or system. It has become conventional to preserve graph theory in different situations such as computer networks, electric networks, etc.

Zadeh [24], in 1965, identified the phenomena of doubtfulness as well as the ambiguity of the real-life situation and brought in a fuzzy set which shifted faces of technology and science. Zhang [22, 23] explained the fuzzy set idea as well as presented the possibility of bipolar fuzzy

sets. Next, Chen et al. [4] presented the thought of fuzzy sets in m -dimension as speculation of bipolar fuzzy sets. First, Kafmann [10] presented the fuzzy graph concept utilizing Zadeh's fuzzy relation. After that Rosenfeld [20] supplied the possibility of nodes, edges along with several hypothetical ideas like paths, connectedness, cycle, etc., in fuzziness. Different concepts and definitions are presented thereafter on fuzzy graphs [15, 17, 21]. Nair and Cheng presented fuzzy cliques in fuzzy graphs [18]. Mathew et al. also worked on different properties on fuzzy graphs [15, 17, 21]. Chen et al. [4] first presented a m -polar fuzzy graph (m PFG). Later on, Ghorai and Pal discussed several properties of m PFG [6, 7, 8, 9, 19]. Next, Mandal et al. studied different types of arcs on m PFG [13]. Akram and Adeel have also deeded on m PFGs and line graphs [1]. Akram et al. concentrated a few properties of edge on m PFG [2]. Next, Mahapatra and Pal presented fuzzy colouring of m PFG [11] and recently, Mahapatra et al. studied fuzzy fractional colouring on fuzzy graph [12].

1.2 Framework of this study

This paper is structured as follows: Section 2 describe some definitions which are useful in these manuscripts. In section 3, we have discussed the definitions of the strong vertex as well as SE count, α -vertex as well as α -edge count, β -vertex as well as β -edge count of m PFG and give the lower and upper bound of them in an m PFG. In section 3.1, we investigate vertex as well as edge counts of some well-known m PFG. In section 4, we introduced saturation in m PFG with the help of α -saturation and β -saturation. Section 5 describe algorithms to find α -saturated as well as β -saturated and saturation in m PFG. A real-life application based on the allocation problem has been solved using saturation in m PFG, given in Section 6. Lastly, the conclusion has been given in Sections 7.

Notations and symbols

In this portion, we reform some of the significant as well as useful notations which are used in the whole paper for development of the theories. The abbreviation form and their meanings are given in Table 1.2.

Full name	Abbreviation Form
Fuzzy graph	FG
m -polar fuzzy graph	m PFG
Underlying crisp graph	UCG
Membership value	MV
m -polar fuzzy set	m PFS
Strength of connectedness	SC
Strong edge	SE
Maximal spanning tree	MST

Table 1.2: Abbreviation form of some terms

2 Preliminaries

Here, we briefly call again some basic as well as useful definitions of graphs, $mPFG$ and related terms connecting with it. Suppose $G = (V, E)$ be a graph, where V (non-null set) is indicated as a node-set as well as E is indicated as edge-set. If a node is separate from all edges, then the node is said to be an isolated node. Otherwise, it is called a non-isolated node.

Definition 2.1. [20] A FG $G = (V, \sigma, \mu)$ having UCG G^* in which $\sigma : V \rightarrow [0, 1]$ as well as $\mu : (\widetilde{V^2}) \rightarrow [0, 1]$ is a fuzzy set in V as well as $\widetilde{V^2}$ respectively and which obey the following rule $\mu(a, d) \leq \{\sigma(a) \wedge \sigma(d)\}$, $\forall(a, d) \in \widetilde{V^2}$ as well as $\mu(a, d) = 0$, $\forall(a, d) \in (\widetilde{V^2} - E)$. $\sigma(a)$ as well as $\mu(a, d)$ indicates the node a and edge (a, d) MV.

Throughout this article $[0, 1]^m$ will be a partial order set having component wise order $\leq$, $m \in N$ and “ $\leq$ ” stands for $x \leq z \Leftrightarrow \forall i = 1, 2, \dots, m, p_i(x) \leq p_i(z)$, x, z are taken from $[0, 1]^m$ as well as $p_i : [0, 1]^m \rightarrow [0, 1]$ represents of i^{th} components of projection mapping.

Definition 2.2. [4] An $mPFS$ A on X is a mapping $A : X \rightarrow [0, 1]^m$.

Definition 2.3. [4] Let A be an $mPFS$. Then $h(A)$ denotes height of A and it is given by

$$(\sup_{x \in A} p_1 \circ A(x), \sup_{x \in A} p_2 \circ A(x), \dots, \sup_{x \in A} p_m \circ A(x))$$

Definition 2.4. [15] The support of an $mPFS$ A is given as $supp(A) = \{c \in A : p_i \circ B(c) > 0\}$, $i = 1, 2, \dots, m$, indicated by $supp(A)$, where $B : A \rightarrow [0, 1]^m$ is a mapping.

Clearly, $supp(A) = \phi$ iff $A = \phi$ and $supp(A) \neq \phi$ iff $A \neq \phi$. Therefore, A and $supp(A)$ are equivalent in $mPFS$.

Definition 2.5. [7] An $mPFG$ $G = (V, A, B)$ having UCG $G^* = (V, E)$, where $A : V \rightarrow [0, 1]^m$ as well as $B : V \times V \rightarrow [0, 1]^m$ represents an $mPFS$ of V as well as $\widetilde{V \times V}$ respectively and which obey the rule such that $\forall i = 1, 2, 3, \dots, m, p_i \circ B(a, c) \leq \{p_i \circ A(a) \wedge p_i \circ A(c)\}$ $\forall(a, c) \in \widetilde{V \times V}$ as well as $B(a, c) = \mathbf{0}$ $\forall(a, c) \in (V \times V - E)$. $p_i \circ A(a)$ and $p_i \circ B(a, c)$ indicates i^{th} component of membership function of node a and edge (a, c) of $mPFG$.

Definition 2.6. [6] An $mPFG$ $G = (V, \sigma, \mu)$ is entitled as complete whenever $p_i \circ \mu(a, c) = \{p_i \circ \sigma(a) \wedge p_i \circ \sigma(c)\}$, $\forall a, c \in V$, $i = 1, 2, \dots, m$.

Definition 2.7. [7] An $mPFG$ $G = (V, \sigma, \mu)$ is entitled as strong whenever

$$p_i \circ \mu(a, c) = \{p_i \circ \sigma(a) \wedge p_i \circ \sigma(c)\},$$

$\forall (a, c) \in E, i = 1, 2, \dots, m$.

Definition 2.8. [9] Suppose $G = (V, \sigma, \mu)$ as well as $G' = (V', \sigma', \mu')$ be two $mPFG$ s. If there exists a mapping $\phi : G \rightarrow G'$ such that for each $i = 1, 2, \dots, m$

$$(i) \ p_i \circ \sigma(a) = p_i \circ \sigma'(\phi(a)), \forall a \in V.$$

$$(ii) \ p_i \circ \mu(a, c) = p_i \circ \mu'(\phi(a), \phi(c)), \forall (a, c) \in \widetilde{V \times V}.$$

Then G as well as G' are called isomorphic. We write it as $G \cong G'$.

Definition 2.9. [3] Let $G = (V, \sigma, \mu)$ be an mPFG having UCG $G^* = (V, E)$. Let $H = (V', \sigma', \mu')$ be a subgraph of G having UCG $H^* = (V', E')$. Then H is called mPF cycle if $(\text{supp}(V'), \text{supp}(E'))$ is a cycle and there does not exist unique $(x, y) \in E'$ (edge set of H) such that $p_i \circ \mu(x, z) = \inf\{p_i \circ \mu(a, c) : (a, c) \in E'\}$, for $i = 1, 2, \dots, m$.

Definition 2.10. [13] Suppose $G = (V, \sigma, \mu)$ be an mPFG as well as $P : a_1, a_2, \dots, a_k$ be a path in G . Then $S(P)$ denotes the strength of the path P which is given by $S(P) = (\min_{1 \leq i < j \leq k} p_1 \circ \mu(a_i, a_j), \min_{1 \leq i < j \leq k} p_2 \circ \mu(a_i, a_j), \dots, \min_{1 \leq i < j \leq k} p_m \circ \mu(a_i, a_j)) = (\mu_1^n(a_i, a_j), \mu_2^n(a_i, a_j), \dots, \mu_m^n(a_i, a_j))$.

The SC of the path in between a_1 and a_k is given in the following way: $\text{CONN}_G(a_1, a_k) = (p_1 \circ \mu(a_i, a_j)^\infty, p_2 \circ \mu(a_i, a_j)^\infty, \dots, p_m \circ \mu(a_i, a_j)^\infty)$, where $(p_i \circ \mu(a, b)^\infty) = \max_{n \in \mathbb{N}} (\mu_i^n(a, b))$.

Definition 2.11. [13] Suppose $G = (V, \alpha, \mu)$ is an mPFG as well as (x, z) be an edge in G . If $\forall i = 1, 2, \dots, m$, $p_i \circ \mu(x, z) > p_i \circ \text{CONN}_{G-(x,z)}(x, z)$, $p_i \circ \mu(x, z) = p_i \circ \text{CONN}_{G-(x,z)}(x, z)$, $p_i \circ \mu(x, z) < p_i \circ \text{CONN}_{G-(x,z)}(x, z)$ then (x, z) is called α -strong, β -strong, δ -SE respectively.

Definition 2.12. [14] An mPFG $G = (V, \sigma, \mu)$ is said to be mPF tree if there exists a spanning mPF subgraph $H = (V, \sigma', \mu')$ which is an mPF tree and $p_i \circ \mu'(a, c) = 0$ means $p_i \circ \text{CONN}_H(a, c) > p_i \circ \mu'(a, c)$, for $i = 1, 2, \dots, m$.

Definition 2.13. [14] Suppose $G = (V, \alpha, \mu)$ is an mPFG. An arc (a, c) is said to be mPF bridge if deletion of it decreases the SC between some other pair of vertices of G .

In this article $G^* = (V, E)$ stands for the UCG of an mPFG G .

3 Vertex as well as edge saturation counts of mPFG

In this section, we introduced vertex as well as edge saturation counts in mPFG and discussed various useful properties of them. Vertex saturation count of an mPFG gives the dimension of the mean strong degree of mPFG and edge saturation count indicates the portion of SEs of mPFG. Here, we consider $\sigma(u) = \mathbf{1} = (1, 1, \dots, 1)$, for all $u \in V$, where $G = (V, \sigma, \mu)$.

Definition 3.1. Suppose $G = (V, \sigma, \mu)$ be an mPFG having UCG $G^* = (V, E)$. Then strong vertex count of G is indicated by $S_V(G)$ and given by

$$p_i \circ S_V(G) = \frac{\text{number of SE of } G}{|V|} = \frac{\text{number of } \alpha \text{ or } \beta\text{-SE of } G}{|V|}$$

$\forall i = 1, 2, \dots, m$ and SE count of G is indicated by $S_E(G)$ as well as given by

$$p_i \circ S_V(G) = \frac{\text{number of SE of } G}{|E|} = \frac{\text{number of } \alpha \text{ or } \beta\text{-SE of } G}{|E|}$$

$\forall i = 1, 2, \dots, m.$

Definition 3.2. Suppose $G = (V, \sigma, \mu)$ be an mPFG having UCG $G^* = (V, E)$. Then α -vertex count of G is indicated by $\alpha_V(G)$ as well as given by

$$p_i \circ \alpha_V(G) = \frac{\text{number of } \alpha\text{-SE of } G}{|V|}$$

$\forall i = 1, 2, \dots, m$ and α -SE count of G is indicated by $\alpha_E(G)$ as well as given by

$$p_i \circ \alpha_E(G) = \frac{\text{number of } \alpha\text{-SE of } G}{|E|}$$

$\forall i = 1, 2, \dots, m.$

Definition 3.3. Suppose $G = (V, \sigma, \mu)$ is an mPFG having UCG $G^* = (V, E)$. Then β -vertex count of G is indicated by $\beta_V(G)$ as well as given by

$$p_i \circ \beta_V(G) = \frac{\text{number of } \beta\text{-SE of } G}{|V|}$$

$\forall i = 1, 2, \dots, m$ and the β -SE count of G is indicated by $\beta_E(G)$ and given by

$$p_i \circ \beta_E(G) = \frac{\text{number of } \beta\text{-SE of } G}{|E|}$$

$\forall i = 1, 2, \dots, m.$

Example 1. Here, we consider an 3PFG to depicted the above definitions. Here, we consider $\sigma(x) = (1, 1, 1)$, for all $x \in V$.

Figure 1: 3PFG G

Here, the classified edges are given in tabular form.

Edge	Classification
(a, b)	β -strong
(b, c)	α -strong
(a, d)	β -strong
(b, d)	α -strong
(a, c)	δ -strong
(d, c)	δ -strong

α -vertex count of G is

$$p_i \circ \alpha_V(G) = \frac{2}{4} = \frac{1}{2}, \text{ for } i = 1, 2, 3$$

and α -edge count of G is

$$p_i \circ \alpha_E(G) = \frac{2}{6} = \frac{1}{3}, \text{ for } i = 1, 2, 3.$$

β -vertex count of G is

$$p_i \circ \beta_V(G) = \frac{2}{4} = \frac{1}{2}, \text{ for } i = 1, 2, 3$$

and β -edge count of G is

$$p_i \circ \beta_E(G) = \frac{2}{6} = \frac{1}{3}, \text{ for } i = 1, 2, 3$$

Strong-vertex count of G is

$$p_i \circ S_V(G) = \frac{4}{4} = 1, \text{ for } i = 1, 2, 3$$

and Strong-edge count of G is

$$p_i \circ S_E(G) = \frac{4}{6} = \frac{2}{3}, \text{ for } i = 1, 2, 3.$$

Since every edges in an m PF tree is α -strong by the Theorem 3.18 of [13] therefore $p_i \circ \alpha_V(G) = \frac{n-1}{n}$ and $p_i \circ \alpha_E(G) = \frac{n-1}{n-1} = 1$, where n = total number of vertex in an m PF tree $G = (V, \sigma, \mu)$.

Any other m PFG except the m PF tree, the count of α -strong vertex never exceeds the count of nodes. For a complete m PFG, all possible edges can be made β -strong by allotting the same MV to the nodes. Then $p_i \circ \beta_V(G) = \frac{\binom{n}{2}}{2}$ and $p_i \circ \beta_E(G) = \frac{\binom{n}{2}}{\binom{n}{2}} = 1$, for $i = 1, 2, \dots, m$.

Depending on the above observation, we can say the following:

Proposition 1. Suppose $G = (V, \sigma, \mu)$ is an m PFG where $|V| = n$. Then

$$(i) \ 0 \leq p_i \circ \alpha_V(G) \leq \frac{n-1}{n}.$$

$$(ii) \ 0 \leq p_i \circ \alpha_E(G) \leq 1.$$

$$(iii) \ 0 \leq p_i \circ \beta_V(G) \leq \frac{\binom{n}{2}}{2}.$$

$$(iv) \ 0 \leq p_i \circ \beta_E(G) \leq 1.$$

$$(v) \ 0 \leq p_i \circ S_V(G) \leq \frac{\binom{n}{2}}{2}.$$

$$(vi) \ 0 \leq p_i \circ S_E(G) \leq 1.$$

for each $i = 1, 2, \dots, m$.

Proposition 2. Suppose $G = (V, \sigma, \mu)$ is an mPF tree. Then $0 \leq p_i \circ \alpha_V(G) \leq p_i \circ \alpha_E(G)$, $\forall i = 1, 2, \dots, m$.

Proof. As G is an mPF tree therefore $p_i \circ \alpha_V(G) = \frac{n-1}{n}$, $\forall i = 1, 2, \dots, m$ and $p_i \circ \alpha_E(G) = \frac{n-1}{n-1} = 1$, $\forall i = 1, 2, \dots, m$. Hence, $0 \leq p_i \circ \alpha_V(G) \leq p_i \circ \alpha_E(G)$, $\forall i = 1, 2, \dots, m$.

3.1 Vertex and edge counts of some well-known $mPFG$

In this portion, we talk over saturation counts of $mPFG$ structures like mPF cycles, trees, and blocks in $mPFG$. Some necessary parts for these structures are also obtained.

Theorem 3.1. Suppose $G = (V, \sigma, \mu)$ be an $mPFG$ having UCG $G^* = (V, E)$ where $|V| = n$. Then, the following condition are identical:

- (i) G be an mPF tree.
- (ii) $p_i \circ \alpha_V(G) = \frac{n-1}{n}$ as well as $p_i \circ \alpha_E(G) = 1$, $\forall i = 1, 2, \dots, m$.
- (iii) $n \times p_i \circ \alpha_V(G) = (n-1) \times p_i \circ \alpha_E(G)$, $\forall i = 1, 2, \dots, m$.

Proof. (i) $\Rightarrow$ (ii) done previously.

(ii) $\Rightarrow$ (iii)

Suppose that $p_i \circ \alpha_V(G) = \frac{n-1}{n}$ and $p_i \circ \alpha_E(G) = 1$, $\forall i = 1, 2, \dots, m$.

$$\begin{aligned}
 p_i \circ \alpha_V(G) &= \frac{n-1}{n} \\
 \Rightarrow n \times p_i \circ \alpha_V(G) &= (n-1) \\
 \Rightarrow n \times p_i \circ \alpha_V(G) &= (n-1) \times 1 \\
 \Rightarrow n \times p_i \circ \alpha_V(G) &= (n-1) \times p_i \circ \alpha_E(G) \quad [As p_i \circ \alpha_E(G) = 1]
 \end{aligned}$$

Hence, $n \times p_i \circ \alpha_V(G) = (n-1) \times p_i \circ \alpha_E(G)$, $\forall i = 1, 2, \dots, m$.

(iii) $\Rightarrow$ (i)

Suppose that $n \times p_i \circ \alpha_V(G) = (n-1) \times p_i \circ \alpha_E(G)$, $\forall i = 1, 2, \dots, m$.

Since,

$$\begin{aligned}
 n \times p_i \circ \alpha_V(G) &= (n-1) \times p_i \circ \alpha_E(G) \\
 \Rightarrow \frac{p_i \circ \alpha_V(G)}{p_i \circ \alpha_E(G)} &= \frac{(n-1)}{n} \\
 \Rightarrow \frac{p_i \circ \alpha_V(G)}{p_i \circ \alpha_E(G)} &= \frac{(n-1)}{n} = p_i \circ \alpha_V(G)
 \end{aligned}$$

This shows that $p_i \circ \alpha_E(G) = 1$, for each $i = 1, 2, \dots, m$.

Hence, G is connected and acyclic only when all edges are α -strong and therefore, G is a tree.

Theorem 3.2. Suppose $G = (V, \sigma, \mu)$ is a connected $mPFG$. G is an mPF tree iff $p_i \circ \alpha_V(G) = p_i \circ S_V(G)$ as well as $p_i \circ \alpha_E(G) = p_i \circ S_E(G)$, $\forall i = 1, 2, \dots, m$.

Proof. Suppose G be a connected $mPFG$ as well as mPF tree. Now from Theorem 3.19 of [13], we know that G is free from β -SEs. Therefore,

$$p_i \circ \beta_V(G) = \frac{0}{|V|} = 0, \forall i = 1, 2, \dots, m$$

and

$$p_i \circ \beta_E(G) = \frac{0}{|E|} = 0, \forall i = 1, 2, \dots, m.$$

Therefore,

$$p_i \circ \alpha_V(G) = p_i \circ S_V(G), \forall i = 1, 2, \dots, m$$

and

$$p_i \circ \alpha_E(G) = p_i \circ S_E(G), \forall i = 1, 2, \dots, m.$$

Conversely, let $p_i \circ \alpha_V(G) = p_i \circ S_V(G)$ and $p_i \circ \alpha_E(G) = p_i \circ S_E(G)$, for $i = 1, 2, \dots, m$. If G is acyclic, then G is an mPF tree. Let C be a cycle in G . Hence C must have only α -strong as well as δ -SEs only. Again, consider G does not have all α -SEs. Therefore, G contains at least one δ -SE. Suppose e is an δ -SE. Then, we remove it from C . If a unique MST is found then the condition is done. Otherwise, removing δ -SEs one by one from C until we get a unique MST of G .

Theorem 3.3. A connected $mPFG$ $G = (V, \sigma, \mu)$ be an mPF tree iff $p_i \circ \alpha_V(G) = p_i \circ \alpha_V(F)$, for each $i = 1, 2, \dots, m$, where F is MST of G .

Proof. Suppose G is an mPF tree. Then G and F are isomorphic. Therefore,

$$\begin{aligned} p_i \circ \alpha_V(G) &= \frac{\text{count of } \alpha - SE \text{ of } G}{\text{number of nodes}} \\ &= \frac{\text{count of } \alpha - SE \text{ of } F}{\text{number of nodes}} \\ &= p_i \circ \alpha_V(F) \end{aligned}$$

for $i=1, 2, \dots, m$.

Now, we consider another case. Suppose, G contains a cycle, say C . Then, it is not free from δ -SE. Let e be a δ -SE. If $G - e$ is a tree therefore $G - e$ as well as F are isomorphic. Therefore,

$$p_i \circ \alpha_V(G) = p_i \circ \alpha_V(F), \forall i = 1, 2, \dots, m.$$

If $G - e$ is not a tree, deleting the δ -SEs in $G - e$ in similar manner to obtain a MST F of G such that $p_i \circ \alpha_V(G) = p_i \circ \alpha_V(F)$, for each $i = 1, 2, \dots, m$.

Conversely, let $p_i \circ \alpha_V(G) = p_i \circ \alpha_V(F)$, $\forall i = 1, 2, \dots, m$, where F is the corresponding MST of G . We have to show that G is an mPF tree. Suppose G is not an mPF tree, then it must have one β -SE, say (a, b) . Let $c - d$ be another path P in G for which $p_i \circ \mu(c, d) \geq p_i \circ \mu(a, b)$,

$\forall i = 1, 2, \dots, m$ and $\forall (c, d) \in P$. Now, the join of P as well as (a, b) creates a cycle in G . Let k be the count of α -SEs which are incident at a . To find F , remove (a, b) from G , which has minimum weight in C . Then the count of α -SEs connected to c in F is $k + 1$. Suppose the remaining counts of α -SEs is k_1 . Hence, $p_i \circ \alpha_V(G) = \frac{k+1}{|V|}$ as well as $p_i \circ \alpha_V(F) = \frac{k+k_1+1}{|V|}$, $\forall i = 1, 2, \dots, m$, a contrast. Therefore, the theorem.

4 Saturation in m -polar fuzzy graph

Here, saturation in terms of a node as well as edge counts is presented. In this section, we also studied some of the interesting facts of it. We also studied saturated blocks in m PFG.

Definition 4.1. Suppose G is an m PFG. Then G is called α -saturate if it must have one α -SEs incident with each node of G . G is said to be β -strong saturate if it must have one β -SE incident with each nodes of G .

Definition 4.2. Suppose G is an m PFG. G is called a saturate graph if at least one α -SE as well as β -SEs incident with every vertex of G . Otherwise G is called an unsaturated m PFG.

Example 2. To illustrate the above definition we consider a 3PFG G displayed in Fig. 2 whose all vertices having membership value $(1, 1, 1)$.

Figure 2: 3PFG G having four vertices

Here, we see that the edges $(a, b), (c, d)$ are α -SEs and $(c, b), (a, d)$ are β -SEs. The edges (a, c) are δ -SE. Each vertex is connected with α -SE as well as β -SE. Therefore, G be a saturated 3PFG.

Theorem 4.1. Suppose $G = (V, \sigma, \mu)$ as well as $G' = (V', \sigma', \mu')$ are two isomorphic m PFGs. If G is saturated, G' is also saturated.

Proof. Let $\phi : G \rightarrow G'$ be the isomorphism between two m PFGs. Therefore, we have $\forall i = 1, 2, \dots, m$

- (i) $p_i \circ \sigma(a) = p_i \circ \sigma'(\phi(a)), \forall a \in V$.

$$(ii) \quad p_i \circ \mu(a, c) = p_i \circ \mu'(\phi(a), \phi(c)), \forall (a, c) \in \widetilde{V \times V}.$$

To show G' is saturated, we have to show that each node is connected with at least one α -SE as well as β -SEs. Let $w' \in V'$. Then there must be a node, say w , in G for which $\phi(w) = w'$. Since G is saturated, therefore w is an incident with at least one α -SEs as well as β -SEs. Since, G and G' are isomorphic to each other. Therefore, w' is also incident to at least one α -SEs and β -SEs. Hence, G' is also saturated.

Let G be an $mPFG$ having UCG $G^* = (V, E)$ where $|V| = k$. We define a finite sequence $\alpha_S(G) = (n_1, n_2, \dots, n_k)$ called α -strong sequence where, n_j = count of α -SEs connect at node v_j . We define a finite sequence $\beta_S(G) = (n_1, n_2, \dots, n_k)$ called β -strong sequence where, n_j = count of β -SEs connect at node v_j . Since, the count of SEs of G = (the count of α -SEs of G + the count of β -SEs of G) therefore,

$$\sum_{n_j \in \alpha_S(G)} n_j + \sum_{n_j \in \beta_S(G)} n_j = \sum_{n_j \in S_S(G)} n_j$$

Theorem 4.2. Suppose $G = (V, \sigma, \mu)$ is an $mPFG$ having UCG $G^* = (V, E)$ where $|V| = k$. Then G is α -saturated iff $\sum_{n_j \in \alpha_S(G)} n_j \geq k$.

Proof. Suppose G be an α -saturated $mPFG$. Therefore, at least one α -SEs incident with each vertex of G . Thus

$$\begin{aligned} \sum_{n_j \in \alpha_S(G)} n_j &\geq 1 + 1 + \dots + 1 \\ \Rightarrow \sum_{n_j \in \alpha_S(G)} n_j &\geq k \end{aligned}$$

Conversely, let $\sum_{n_j \in \alpha_S(G)} n_j \geq k$. Then, all k nodes of G are connected with at least one α -SEs.

Therefore, G is α -saturated $mPFG$.

Theorem 4.3. Suppose $G = (V, \sigma, \mu)$ is an $mPFG$ having UCG $G^* = (V, E)$ where $|V| = k$. Then G is β -saturated iff $\sum_{n_j \in \beta_S(G)} n_j \geq k$.

Proof. Similar to the above theorem.

Theorem 4.4. Suppose $G = (V, \sigma, \mu)$ is an $mPFG$ having UCG $G^* = (V, E)$ where $|V| = k$. If G is β -saturated then $\sum_{n_j \in S_S(G)} n_j \geq 2k$.

Proof. Let G be saturated. Therefore, each node of G is connected with at least one α -SE as well as one β -SE. Thus, $\sum_{n_j \in S_S(G)} n_j \geq 2k$.

Theorem 4.5. Suppose $G = (V, \sigma, \mu)$ is an mPFG having UCG $G^* = (V, E)$ where $|V| = n$. If,

(i) $p_i \circ \alpha_V(G) \geq 0.5$ if α -saturated.

(ii) $p_i \circ \beta_V(G) \geq 0.5$ if β -saturated.

(iii) $p_i \circ S_V(G) \geq 1$ if saturated.

$\forall i = 1, 2, \dots, m$.

Proof. i) Let G is α -saturated then every vertex of G is incident with at least one α -SEs. Therefore, G must have $\frac{n}{2}$, α -SEs. Therefore, $p_i \circ \alpha_V(G) \geq \frac{\frac{n}{2}}{n} = 0.5$.

ii) Similar to the above.

iii) Let G be saturated. Therefore, each node of G is connected with at least one α -SE as well as β -SEs. Since, the count of SEs of $G = (\text{the count of } \alpha\text{-SEs of } G + \text{the count of } \beta\text{-SEs of } G) \geq \frac{n}{2} + \frac{n}{2} = n$. Hence, $p_i \circ S_V(G) \geq \frac{n}{n} = 1$.

Figure 3: 3PFG G having even number of vertices

In the Fig. 3, all the vertices have membership value $(1, 1, 1)$, that is $\sigma(a_j) = (1, 1, 1)$, for $j = 1, 2, \dots, 12$. The edges membership value is $\mu(a_j, a_k) = (0.6, 0.6, 0.6)$, where $1 \leq j < k \leq 12$ and j is odd and k is even. The edges membership value is $\mu(a_j, a_k) = (0.4, 0.4, 0.4)$, where $1 < j < k < 12$ and j is even and k is odd. The edge membership value between a_1 and a_{12} is $(0.4, 0.4, 0.4)$.

In the Fig. 3, we see that all the edges having membership value $(0.6, 0.6, 0.6)$ are α -strong and the edges having membership value $(0.4, 0.4, 0.4)$ are β -strong. Therefore, Fig. 3 is saturated.

Figure 4: 3PF G having odd number of vertices

In the Fig. 4, all the vertices have membership value $(1, 1, 1)$, that is $\sigma(a_j) = (1, 1, 1)$, for $j = 1, 2, \dots, 9$. The edges membership value is $\mu(a_j, a_k) = (0.5, 0.5, 0.5)$, where $1 \leq j < k \leq 9$ and j is odd and k is even. The edges membership value is $\mu(a_j, a_k) = (0.7, 0.7, 0.7)$, where $1 < j < k < 9$ and j is even and k is odd. The edge membership value between a_1 and a_9 is $(0.5, 0.5, 0.5)$.

In the Fig. 4, we see that all the edges having membership value $(0.7, 0.7, 0.7)$ are α -strong and the edges having membership value $(0.5, 0.5, 0.5)$ are β -strong. Therefore, Fig. 4 is unsaturated as the vertex a_1 connected with both the β -SEs.

One simple observation of the above discussion is that Fig. 3 has an even number of vertices while in Fig. 4 has an odd number of vertices. Thus we have the following theorem.

Theorem 4.6. Suppose C_n be an mPF cycle. It is saturated iff the following two hold:

- (i) $n = 2t$, t is a positive integer.
- (ii) α -SE as well as β -SEs occur alternatively on C_n .

Proof. Suppose C_n is an mPF cycle. Therefore, it is free from δ -SEs. All arcs that occur on C_n are α -SE or β -SE. Let us assume that C_n be saturated. Therefore, each node is connected with at least one α -SE and one β -SEs. Hence, count of α -SEs $= t =$ count of β -SEs. Therefore, $n = 2t$. Again, every node connected with both α -SE as well as β -SEs happen if they occur alternatively on C_n .

Conversely, let C_n is a fuzzy cycle with an even number of nodes in which each node is connected with both α -SE as well as β -SEs alternatively. Therefore, each node is connected with precisely one α -SE as well as β -SEs. Hence, C_n be a saturated fuzzy cycle.

Theorem 4.7. Suppose $G = (V, \sigma, \mu)$ be an mPF cycle. If G is saturated, it must be a block.

Proof. Since G is saturated, each node is connected with at least one α -SE as well as β -SEs. Again, since G is an mPF cycle; therefore every node is connected with just two nodes.

Therefore, every node is incident with precisely one α -SE and β -SEs. Hence, removing any node from G may not decrease SC between other nodes. This shows that G is free from the mPF cut node; therefore, G is a block.

Theorem 4.8. *Let $G = (V, \sigma, \mu)$ is mPF cycle. If G be an mPF blocks, then it is either saturated or β -saturated.*

Proof. Let a block be G . We demand that G is free from δ -SEs. If possible, let e be a δ -SE. Then the remaining edges must be α -SE, and therefore G contains $n - 2$ fuzzy cut nodes, an irrelevance. So, G has no δ -SEs. Thus, G is free from δ -SEs.

If G has only α -SE as well as β -SEs, they appear alternatively; else, the block shape will not be found. If count of α -SE = count of β -SEs = $\frac{n}{2}$, then G be α -saturated as well as β -saturated and therefore it is saturated. If the count of α -SEs is less than the count of β -SEs, then G must be only β -saturated. For another case, when the count of α -SEs is greater than the count of β -SEs, it will not be true as it does not form a block. If every arc is β -strong of G , it must be β -saturated. Therefore, the theorem.

Theorem 4.9. *A complete $mPFG$ has no δ -arcs.*

Proof. Suppose $G = (V, \sigma, \mu)$ is a complete $mPFG$. Let G has a δ -arcs. Let (p, q) be the δ -arcs. Then we have,

$$p_i \circ \mu(p, q) < p_i \circ CONN_{G-(p,q)} \mu(p, q), \forall i = 1, 2, \dots, m.$$

A stronger path P except the arc (p, q) in G must exist there. Suppose $p_i \circ \mu(p, q) = t_i$, $i = 1, 2, \dots, m$ and the strength of P be $(u_1, u_2, \dots, u_m)$. Therefore, we have $t_i < u_i$, $\forall i = 1, 2, \dots, m$. Suppose r be the first vertex after u in the path P . Then, we have

$$p_i \circ \mu(p, r) > t_i, \forall i = 1, 2, \dots, m \quad (1)$$

In a similar way, let s is the last vertex before q in the path P . Again, we also have

$$p_i \circ \mu(s, q) > t_i, \forall i = 1, 2, \dots, m \quad (2)$$

Since, G be complete $mPFG$, therefore we have $p_i \circ \mu(p, q) = \min\{p_i \circ \sigma(p), p_i \circ \sigma(q)\}$, $\forall i = 1, 2, \dots, m$ as well as $\forall (p, q) \in E$. Therefore, at least one of $p_i \circ \sigma(p)$ or $p_i \circ \sigma(q)$ be t_i , $\forall i = 1, 2, \dots, m$.

Therefore, (1) will contradict if $p_i \circ \sigma(p) = t_i$, $\forall i = 1, 2, \dots, m$ and (2) will contradict if $p_i \circ \sigma(q) = t_i$, for $i = 1, 2, \dots, m$.

Hence, the theorem.

Theorem 4.10. *Suppose $G = (V, \sigma, \mu)$ is an $mPFG$. An arc (a, c) be an mPF bridge iff it is α -strong.*

Proof. Suppose (a, c) be an mPF bridge. Then we have from the definition of mPF bridge, $p_i \circ CONN_{G-(a,c)}(a, c) < p_i \circ CONN_G(a, c)$, for $i = 1, 2, \dots, m$ (1)

Again, from Theorem 3.11 of [13],

$$\text{we have } p_i \circ \mu(a, c) = p_i \circ \text{CONN}_G(a, c), \forall i = 1, 2, \dots, m \quad (2)$$

From (1) and (2), we get $p_i \circ \mu(a, c) > p_i \circ \text{CONN}_{G-(a,c)}(a, c)$, $\forall i = 1, 2, \dots, m$. Hence, (a, c) be α -SE.

Conversely, suppose (a, c) is an α -SE. Then, we have (a, c) is the one and only one strongest path in between a and c and removal of (a, c) will decrease the SC of a and b . Therefore, (a, c) is a bridge.

Theorem 4.11. *A complete mPFG has at most one α -SEs.*

Proof. We know that complete mPFG have at most one mPF bridge. Again, from Theorem 4.10, we have an arc (a, b) be an mPF bridge iff it is α -SE. Hence, a complete mPFG has at most one α -SEs.

Proposition 3. *Every complete mPFG has at most $\binom{n}{2}$ or $\binom{n}{2}-1$ β -SEs.*

Theorem 4.12. *If G is a complete mPFG having n vertices, the following inequalities hold.*

$$(i) \quad 0 \leq p_i \circ \alpha_V(G) \leq \frac{1}{n}.$$

$$(ii) \quad \frac{n^2-n-2}{2n} \leq p_i \circ \beta_V(G) \leq \frac{n-1}{2}.$$

$$\forall i = 1, 2, \dots, m.$$

Proof. With the help of Theorem 4.11, we have G have at most one α -SEs. Hence, we have $p_i \circ \alpha_V(G) \leq \frac{1}{n}$, for each $i = 1, 2, \dots, m$. Again, clearly $p_i \circ \alpha_V(G) \geq 0$, for each $i = 1, 2, \dots, m$. Therefore, $0 \leq p_i \circ \alpha_V(G) \leq \frac{1}{n}$, for each $i = 1, 2, \dots, m$.

Again, from Proposition 3, we know that the minimum number of β -SEs are $\binom{n}{2}-1$. Therefore,

$$\begin{aligned} p_i \circ \beta_V(G) &\geq \frac{n(\frac{n-1}{2}) - 1}{n} \\ &\geq \frac{n^2 - n - 2}{2n} \end{aligned}$$

Thus, $\frac{n^2-n-2}{2n} \leq p_i \circ \beta_V(G) \leq \frac{n-1}{2}$, for each $i = 1, 2, \dots, m$.

Next, we will try to find out the upper limit of α -node count for a block in mPFG.

Theorem 4.13. *If $G = (V, \sigma, \mu)$ is an mPF blocks, we have $p_i \circ \alpha_V(G) \leq 0.5$, $\forall i = 1, 2, \dots, m$.*

Proof. To prove this, we first try to find out the maximum count of α -SEs of G . Let $|V| = n$. We know that if more than one α -SEs are connected with a common node then the node is a mPF cut node. Since, G is an mPF block, therefore it has no mPF cut node. Therefore, the maximum count of α -SEs of G is $\frac{n}{2}$. Thus, $p_i \circ \alpha_V(G) \leq \frac{\frac{n}{2}}{n} = 0.5$, $\forall i = 1, 2, \dots, m$.

Theorem 4.14. *An mPF block G be α -saturated then $p_i \circ \alpha_V(G) = 0.5$, $\forall i = 1, 2, \dots, m$.*

Proof. Let G be α -saturated. Since, G is an mPF block, therefore it has no mPF cut vertex. Hence, every vertex incident with exactly unique α -SE. Therefore, G contains exactly $\frac{n}{2}$ count of α -SEs. Thus, $p_i \circ \alpha_V(G) = \frac{\frac{n}{2}}{n} = 0.5$, for $i = 1, 2, \dots, m$.

5 Algorithms

From Dijkstra's algorithm [5], we first trace $G^* = (V, E)$, where V as well as E indicates the set of all nodes as well edges where $|V| = n$.

Algorithm 1 An algorithm to find α -saturation in m PFG

Input: A m PFG $G = (V, \sigma, \mu)$.

Output: Finding α -saturated m PFG.

Step 1: Put the membership value of vertices $a_j, j = 1, 2, \dots, n$.

Step 2: Put the membership value of edges which satisfied $p_i \circ \mu(a_j, a_k) \leq \inf\{p_i \circ \sigma(a_j), p_i \circ \sigma(a_k)\}, i = 1, 2, \dots, m$.

Step 3: Calculate $p_i \circ CONN_{G - ((a_j, a_k))}(a_j, a_k)$, for $i = 1, 2, \dots, m, \forall (a_j, a_k) \in E$.

Step 4: Verify $p_i \circ \mu(a_j, a_k) > p_i \circ CONN_{G - (a_j, a_k)}(a_j, a_k), \forall (a_j, a_k) \in E$.

Step 5: Select all α -SEs in G .

Step 6: Check whether every node is connected with at least one α -SEs or not.

Algorithm 2 An algorithm to find β -saturation in m PFG

Input: A m PFG $G = (V, \sigma, \mu)$.

Output: Finding β -saturated m PFG.

Step 1: Put the membership value of vertices $a_j, j = 1, 2, \dots, n$.

Step 2: Put the membership value of edges which satisfied $p_i \circ \mu(a_j, a_k) \leq \inf\{p_i \circ \sigma(a_j), p_i \circ \sigma(a_k)\}, i = 1, 2, \dots, m$.

Step 3: Calculate $p_i \circ CONN_{G - ((a_j, a_k))}(a_j, a_k)$, for $i = 1, 2, \dots, m, \forall (a_j, a_k) \in E$.

Step 4: Verify $p_i \circ \mu(a_j, a_k) = p_i \circ CONN_{G - (a_j, a_k)}(a_j, a_k), \forall (a_j, a_k) \in E$.

Step 5: Select all β -SEs in G .

Step 6: Check whether every node is connected with at least one β -SEs or not.

Algorithm 3 An algorithm to find saturation in m PFG

Input: A m PFG $G = (V, \sigma, \mu)$.

Output: Finding saturated m PFG.

Step 1: Put the membership value of vertices $a_j, j = 1, 2, \dots, n$.

Step 2: Put the membership value of edges which satisfied $p_i \circ \mu(a_j, a_k) \leq \inf\{p_i \circ \sigma(a_j), p_i \circ \sigma(a_k)\}, i = 1, 2, \dots, m$.

Step 3: Using Algorithms 1 and 2 identified all α -strong as well as β -SEs in G .

Step 4: Check whether every vertex is connected with at least one α -SEs as well as β -SEs or not.

6 Application

The m PFG is an essential mathematical structure representing the facts in real-life connected through graphical systems, in which nodes and edges lie in an m -polar fuzzy information. In this section, by using saturation in m PFG, we solve one particular allocation problem.

6.1 Model construction

In modern days, education is an essential topic for every person. Under the rule of the Right to Education (RTE) in 2005, everybody has the opportunity to read and write. In the education system, IIT(Indian Institute of Technology) is one of the most important institutions for higher studies in India. Therefore, establishing an IIT in a town among some towns is not an easy task for any Government.

Here, we consider nine towns $a_1, a_2, \dots, a_9$ as nodes. There will be an edge between two nodes if there exists road connection in between two nodes. Here, we use saturation in 3PFG $G = (V, \sigma, \mu)$ to solve the allocation problem. Since, the towns are fixed in nature therefore we can assign the membership value of each node $(1, 1, 1)$, that is $\sigma(a_i) = (1, 1, 1)$, for $i = 1, 2, \dots, 9$. The edge membership value is calculated depending on three criteria. Those criteria are as follows: {Condition of roads, traffic jams on the roads, communication system between two cities}. All the indicators of an edge between two towns are uncertain in nature. We can calculate the edge membership values by remembering the relation $p_i \circ \mu(a, c) \leq \inf\{p_i \circ \sigma(a), p_i \circ \sigma(c)\}$, $\forall i = 1, 2, \dots, m$. The model 3PFG is shown in Fig. 5. Here, the edge membership value is

Figure 5: Model 3PFG G

given in tabular form.

Edge	Membership Value
(a_1, a_2)	$(0.5, 0.4, 0.3)$
(a_2, a_3)	$(0.5, 0.4, 0.3)$
(a_3, a_4)	$(0.3, 0.2, 0.1)$
(a_4, a_5)	$(0.3, 0.2, 0.1)$
(a_5, a_7)	$(0.6, 0.5, 0.4)$
(a_7, a_8)	$(0.5, 0.4, 0.3)$
(a_8, a_9)	$(0.3, 0.2, 0.1)$
(a_1, a_9)	$(0.7, 0.6, 0.5)$
(a_7, a_9)	$(0.6, 0.5, 0.4)$
(a_6, a_9)	$(1, 0.9, 0.8)$
(a_6, a_7)	$(0.6, 0.5, 0.4)$
(a_5, a_6)	$(0.6, 0.5, 0.4)$
(a_2, a_6)	$(0.8, 0.7, 0.6)$
(a_3, a_6)	$(0.5, 0.4, 0.3)$
(a_4, a_7)	$(0.7, 0.6, 0.5)$

6.2 Illustration of membership values

Here, the model network system contains nine nodes and fifteen edges. It can be seen from the given 3PFG that every town is connected to others through some paths. So, first, we want to check whether the connections between towns are α -strong, β -strong or δ -strong. Next, we find out the saturation vertex in Fig. 5. After calculating $CONN_{G-(a,b)}$, for all $(a, b) \in E$, where E is the set of edges of the model 3PFG, we find out which edges are α -strong, β -strong or δ -strong. Then, by the routine computations, we get the classification of edges. Here, the classified edges are given in tabular form.

Edge	Classification
(a_1, a_2)	δ -strong
(a_2, a_3)	δ -strong
(a_3, a_4)	δ -strong
(a_4, a_5)	δ -strong
(a_5, a_7)	δ -strong
(a_7, a_8)	α -strong
(a_8, a_9)	δ -strong
(a_1, a_9)	α -strong
(a_7, a_9)	δ -strong
(a_6, a_9)	α -strong
(a_6, a_7)	δ -strong
(a_5, a_6)	δ -strong
(a_2, a_6)	α -strong
(a_3, a_6)	β -strong
(a_4, a_7)	α -strong

Here, only one β -SEs are present in the model 3PFG G . The node a_6 is the only saturation node in the model 3PFG G as it is an incident with at least one α -SE and one β -SEs.

6.3 Decision making

Since a_6 is the only saturation node in the model 3PFG G , we can say that the town a_6 is the most suitable place to establish the IIT (Indian Institute of Technology) among all other towns considered in this proposed model.

We know that saturation in m PFG plays an essential role in this type of allocation problem through the above discussion. Moreover, we also recognize that saturation in m PFG is more applicable than saturation in FG in allocation problems.

7 Conclusion

In this paper, α -saturation as well as β -saturation in m PFG along with its several properties are initiated. Vertex and edge saturation count in m PFG and a few of its facts on some well known m PFG is also introduced. The upper and lower bound of a vertex as well as edge saturation count in m PFG are also investigated. Saturation in m PFG by using α -saturation as well as β -saturation are also discussed here along with some of its intersecting properties. Using saturation in m PFG, an application is also given in the last part of this paper. Our research work will be extended depending on m PFG to find more characteristics and applications.

Compliance with ethical standards

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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Authors' contributions

All the authors contribute equally in this work.

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