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Nonlinear and Dual-Parameter Chaotic Vibrations of Lumped Parameter Model in Blisk under Combined Aerodynamic Force and Varying Rotating Speed

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Abstract

In this paper, the nonlinear and dual-parameter chaotic vibrations are investigated for the blisk structure with the lumped parameter model under combined the aerodynamic force and varying rotating speed. The varying rotating speed and aerodynamic force are respectively simplified to the parametric and external excitations. The nonlinear governing equations of motion for the rotating blisk are established by using Hamilton's principle. The free vibration and mode localization phenomena are studied for the tuning and mistuning blisks. Due to the mistuning, the periodic characteristics of the blisk structure are destroyed and uniform distribution of the energy is broken. It is found that there is a positive correlation between the mistuning variable and mode localization factor to exhibit the large vibration of the blisk in a certain region. The method of multiple scales is applied to derive four-dimensional averaged equations of the blisk under 1:1 internal and principal parametric resonances. The amplitude-frequency response curves of the blisk are studied, which illustrate the influence of different parameters on the bandwidth and vibration amplitudes of the blisk. Lyapunov exponent, bifurcation diagrams, phase portraits, waveforms and Poincare maps are depicted. The dual-parameter Lyapunov exponents and bifurcation diagrams of the blisk reveal the paths leading to the chaos. The influences of different parameters on the bifurcation and chaotic vibrations are analyzed. The numerical results demonstrate that the parametric and external excitations, rotating speed and damping determine the occurrence of the chaotic vibrations and paths leading to the chaotic vibrations in the blisk.

Keywords: Blisk structure, parametric and external excitations, lumped parameter model, amplitude-frequency response curves, dual parameter chaotic vibrations

1. Introduction

With the development of the aviation industry, the aero-engines are developing towards the light weight, high thrust-weight ratio and high reliability. The blisk is a novel structure of the aero-engine. The blisk structures of the fan and compressors are the effective technologies to improve the performance of the durability and reliability for the aero-engines [1]. Comparing with the traditional blade and disk structures, the advantages of the blisk structures lead to the widely application in the aero-engines. However, due to the light and thin structural characteristics and high rotating speed of the blisk, the vibration amplitudes increase rapidly with the change of the aerodynamic force. The thinner the blisk is, the larger the vibration amplitudes of the blisk become, which may even result in the chaotic vibrations. The chaotic vibrations are harmful to the stability and safety of the aero-engines. Therefore, in order to avoid the chaotic vibrations of the aero-engines, we must investigate the chaotic dynamics of the blisk thoroughly. The novelty of this paper is that the nonlinear and dual-parameter chaotic vibrations are investigated for the blisk with the lumped parameter model under combined the varying rotating speed and aerodynamic force. We focus on the dual-parameter chaotic dynamic responses of the blisk. The influences of several parameters on the nonlinear dynamics are mainly analyzed for the blisk structure

Based on the research results about the free vibration [2,3] and stability analysis [4,5] of the rotary blade, scholars pay more attentions to study the influences of a single parameter on the chaotic vibrations [6,7] of the blade. Especially, when considering the complexity of the operating condition for the blisk structure, the current researches are far from the requirements for analyzing the complex nonlinear dynamics of the blisks. If we want to investigate the chaotic vibrations of the blisks systematically, we must consider the combination of different parameters as much as possible, which may lead to the chaotic vibrations of the blisks. The dual-parameters are referred to any combination of the parametric and external excitation, damping and rotating speed in the blisk structure.

The chaos, a seemingly random motion, highly depends on the initial conditions of the nonlinear dynamical systems [8]. When the systems subject to the complex excitations, the chaotic vibrations more likely happen in the rotary machines [9]. Researchers devoted more attentions to the nonlinear vibrations of the blades [10-15]. However, these studies ignored the influences caused by the interactions among different parameters on the

nonlinear vibrations of the blade structure. In order to accurately predict and investigate the nonlinear dynamic responses of the blisk structures, it is necessary for us to discuss modal interactions of the blisk structure. The free vibration of the blisk structure is more complex than that of a single blade [16]. Thus, as the basis of the nonlinear coupling vibrations for the blisk structures, the modal analysis is of great importance. Generally, three different methods were used to study the natural vibration characteristics of the blisk structures, namely, the reduced-order modeling (ROM) techniques [17-19], experiment methods [20,21] and numerical simulations [22,23]. The harmonic balance method [24], eigenmode method [25,26] and parameter identification method [27,28] were used to establish the reduced-order model of the blisk structures. Tang et al. [29,30] presented an extension of the reduced-order model technique based on the component mode mistuning. Sun et al. [31,32] utilized the energy-based finite element method (FEM) to investigate the free vibration of the hard-coating blisk. Zhang et al. [33] studied the vibrations of the rotating pretwisted composite tapered blade with the graphene coating layers. Zhang et al. [34] investigated the multi-pulse jumping double-parameter chaotic dynamics of the eccentric rotating ring truss antenna under combined the parametric and external excitations. Unfortunately, up till now, almost all ROM methods, including system identification and eigenmode methods, are utilized to analyze the data-driven empirical models. When the parameters change slightly, the accuracy of the ROM will be greatly reduced. Since the dimension of the equations constructed by the reduced-order model is still too high for the analysis of the chaotic vibration for the blisk structures, few researchers carried out the qualitative investigation about the chaotic vibrations of the blisk structures based on the ROM technique.

In fact, the vibrations of the blisks are mostly nonlinear. To understand and solve engineering problems, researchers begin to study the nonlinear vibrations of the blisk structures. Salas et al. [35] used two methods to comparatively study the validity of the noncyclic reduced lumped parameter models and the forced vibrations of the mistuning blisk structures. Considering the effects of the vibration reduction for the damping coating, Yan et al. [36] discussed the forced vibrations of the coated aero-engine blisk. Han and Mignolet [37] predicted the forced dynamic responses of the damped mistuned disk. Willeks et al. [38] studied the response reduction of the bladed disks with the aerodynamic damping by using the harmonic mistuning. Gao et al. [39] gave the application of the hard-coating damper to the mistuned blisk for the passive vibration reduction. Chen [40] investigated the vibration reduction of the blisk by the damping hard coating and gave its

mistuning design. Feng and Jing [41] found that at the low frequencies, the vibration isolation of the structures can be remarkably improved by using the nonlinear properties.

Usually, the contact state among the components of the mechanical structures is not consummate. Therefore, the impact of the contact and friction can not be ignored when studying the nonlinear vibrations of the blisk structures. Laxalde et al. [42] presented the qualitative analysis of the forced dynamic responses for the blisk with the friction ring dampers. Salas et al. [43] gave a mistuned analysis of the forced dynamic responses for an embedded compressor blisk using a reduced-order model. Beirow et al. [44] utilized the equivalent model to explore the influence of the aerodynamic load on the dynamic responses of a blisk. Sarrouy et al. [45] simplified the blades to six identical beams clamped at the disk and explored the complete bifurcation phenomena of the blisk. Hoskoti et al. [46] studied the instability of the vortex induced vibrations for a blade. Gu et al. [47] investigated the free vibration of the rotating cantilever pre-twisted panel with the geometric imperfection of the initial exponential function type. Najafi [48] researched the stability and nonlinear vibrations of a rotating bladed disk at the critical speed.

Although many researches are focused on the nonlinear dynamics of the blades, the literatures related to the chaotic dynamics of the lumped parameter model for the blisk structures are less. Researches seem to be blank on the dual-parameter chaotic vibrations of the blisk structures. Considering the coupling influences of various parameters on the nonlinear vibrations of the blisk, the dual-parameter nonlinear vibrations of the lumped parameter model are investigated for the blisk. The governing equations of motion for the lumped parameter model of the blisk are established by applying Hamilton's principle. The influences of different parameters on the bifurcations are analyzed by using numerical simulations. The free vibration and mode localization phenomenon [49,50] of the mistuning blisk are investigated. When the structure has not the tuning, the energy of the system is no longer uniformly distributed. The larger the mode localization factor is, the larger the vibration amplitude is in a certain region. The amplitude-frequency response curves of the blisk are investigated based on the averaged equations in the polar coordinate under 1:1 internal resonance and principal parametric resonance [51]. The dual-parameter Lyapunov exponents and bifurcation diagrams of the blisk are depicted. The effects of the damping, perturbation rotating speed, steady-state rotating speed and external excitation on the nonlinear dynamics are numerically investigated for the blisk structure. The phase portraits, waveforms and Poincare maps are given to demonstrate the nonlinear dynamic responses of the blisk structure.

2. Free Vibration of Blisk

The lumped parameter model of the blisk structure is shown in Figure 1(a). A single sector of two-degree-of-freedom lumped parameter blisk structure is simplified to a dual-mass model. This model is developed by introducing the nonlinear part of the stiffness for the beam model of the blade and disk [52]. The introduction of the nonlinear terms for the stiffness makes the nonlinear dynamic analysis of the blisk to be possible. The blade and disk sectors are simplified to the lumped masses and massless nonlinear elastic beams. There are the linear springs between two mass blocks of the simplified blisk structure, which represent the coupling effect between two blades. The equivalent masses of the blade and disk are m_b and m_d . The stiffness terms of the blade include the linear part k_b and nonlinear part $\varepsilon_1 k_b$. The stiffness terms of the disk include the linear part k_d and nonlinear part $\varepsilon_2 k_d$. The elastic coefficient of the linear coupling spring is k_l . The displacements of the blade and disk sectors in the circular direction are assumed to be $x_b^i(t)$ and $x_d^i(t)$. The rotating speed for the blisk is considered as $\Omega(t) = \Omega_0 + f \cos \Omega_1 t$, where Ω_0 is the steady-state rotating speed and $f \cos \Omega_1 t$ is the periodic perturbation rotating speed. Cartesian coordinate system of the blisk rotates around the point o. The unit vectors of the coordinate system are $\mathbf{i}$ and $\mathbf{j}$, as shown in Figure 1(b).

It is known that from Figure 1 that the equation of the undamped free vibration for the blisk structure is obtained as follows

$$[\mathbf{M}]\ddot{[\mathbf{X}]} + [\mathbf{K}][\mathbf{X}] = 0, \quad (1)$$

where $[\mathbf{M}]$ and $[\mathbf{K}]$ are the blisk mass and stiffness matrices, and $[\mathbf{X}]$ is the displacement vector.

The matrices $[\mathbf{M}]$, $[\mathbf{K}]$ and $[\mathbf{X}]$ are expressed as follows

$$[\mathbf{M}] = \begin{bmatrix} m_b^1 & & & & \\ & m_d^1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & m_b^n \\ & & & & & m_b^n \end{bmatrix}, \quad (2a)$$

$$[\mathbf{K}] = \begin{bmatrix} k_b^1 & -k_b^1 & \Lambda & \Lambda & \Lambda & 0 \\ -k_b^1 & k_b^1 + k_d^1 + (2 - \cos u)k_l & 0 & -k_l & & k_l \\ M & 0 & k_b^2 & -k_b^2 & 0 & M \\ M & 0 & 0 & 0 & 0 & 0 \\ M & & 0 & 0 & k_b^n & -k_b^n \\ 0 & -k_l & 0 & \Lambda & -k_b^n & k_b^n + k_d^n + (2 - \cos u)k_l \end{bmatrix}, \quad (2b)$$

$$[\mathbf{X}] = [x_b^1 \quad x_d^1 \quad \Lambda \quad x_b^n \quad x_d^n]^T. \quad (2c)$$

where u is the phase angle between two adjacent blades.

The disk vibrates freely with the frequency ω . The following formula is obtained as

$$([\mathbf{K}] - \omega^2 [\mathbf{M}])[\mathbf{X}] = 0. \quad (3)$$

Equation (2) is expressed by the eigenvalues and eigenvectors

$$([\mathbf{A}] - \lambda [\mathbf{I}])[\mathbf{X}] = 0, \quad (4)$$

where $[\mathbf{A}] = [\mathbf{M}]^{-1}[\mathbf{K}]$, and $[\mathbf{I}]$ is the identity matrix.

The aforementioned parameters are respectively selected as $m_b = 0.05\text{kg}$, $m_d = 0.1\text{kg}$, $k_b = 7.5 \times 10^5 \text{ N/m}$, $k_d = 6 \times 10^5 \text{ N/m}$, $k_l = 4 \times 10^5 \text{ N/m}$.

It is assumed that the mistuned mass and mistuned stiffness of the blades respectively follow the random normal distributions with the mean value 0. We consider α as the change of the random mistuned mass and β as the varying of the random mistuned stiffness for the blades. The natural frequencies and mode shapes of the blades are investigated. Table 1 gives the first fifth natural frequencies of the blades without mistuning, which means that α and β are taken as 0 in this case. Figure 2 demonstrates the first fifteen mode shapes of the blades without the mass mistuning or stiffness mistuning. The vibration modes of the blades present the standard sinusoidal manner, which indicates that the distribution of the energy on the blades is completely uniform.

Figure 3 exhibits the first fifteen vibration modes of the blades with the mass mistuning, where the change of the random mistuned mass for the blades is $\alpha = 2\%$. The mass mistuning values Δm_b of the blades are indicated in Table 2. In Figure 3, it is seen that vibration modes of the blades are no longer sinusoidal. Since the mass mistuning destroys the periodic properties of the blisk, which leads to the transfer of the energy between the blades, and finally leads to the mode localization. The aforementioned results illustrate that the stiffness mistuning also leads to the mode localization of the blades.

When the changes of the random mistuned mass are respectively chosen as $\alpha = 1\%$,

$\alpha = 3\%$ and $\alpha = 5\%$, the natural frequencies of the blades without the mistuning $\alpha = 0$ and with mass mistuning in four cases are given, as shown in Figure 4(a). It is found that mass mistuning leads to the reduction of the first eleven natural frequencies for the blade. However, the mass mistuning almost has no the effect on the natural frequencies which are higher than the eleventh-order natural frequencies. Different changes of the mistuned masses for the blades have the same influence on the natural frequencies of the blade. When the changes of the random mistuned stiffness are $\beta = 1\%$, $\beta = 3\%$ and $\beta = 5\%$, the natural frequencies of the tuning $\beta = 0$ and detuning blades in four stiffness forms are depicted, as shown in Figure 4(b), which illustrates that stiffness detuning will increase the eleventh-order to sixteenth-order natural frequencies of the blades. However, the natural frequencies higher than the sixteenth-order keep unchanged when the stiffness mistuning. It is pointed out that the larger the value of the change of the mistuned stiffness is, the more obviously the natural frequencies of blades decrease or increase.

Using Euclidean norm to define the mode localization factors, we have

$$L = \sqrt{\frac{A_{max}^2 - \frac{1}{n-1} \sum_{i=1, i \neq j}^n A_i^2}{\frac{1}{n-1} \sum_{i=1, i \neq j}^n A_i^2}}, \quad (5)$$

where A_{max} is the largest vibration amplitude of the blades, n is the quantity of the blades, j is the number of the blade with the largest amplitude, A_i is the amplitude of the i th blade.

When the changes of the random mistuned mass are $\alpha = 1\%$, $\alpha = 3\%$ and $\alpha = 5\%$, Figure 5(a) demonstrates the mode localization factors of the blades without the mistuning $\alpha = 0$ and with mass mistuning in four cases. Figure 5(b) illustrates the mode localization factors of the blades without the mistuning and with stiffness mistuning in four cases when $\beta = 0, 1\%, 3\%$ and 5% . In Figure 5, the mode localization factors are less than or equal to 1 when the blisk structure vibrates without the mistuning. By contrary, when the blisk structure vibrates with the mistuning, the mode localization factors are greater than or equal to 1. When the mode shape of the blade is sinusoidal and mode localization factor is less than 1, the mode localization phenomenon will not appear. Correspondingly when the mode shapes of the blades do not distribute sinusoidally and mode localization factor is greater than 1, the mode localization phenomenon occurs. The larger α or β are, the

larger the mode localization factors of the blades become. As a combination, the small damping of the blisk structure will leads to the large amplitude vibration. Thus, it is urgent for us to study the nonlinear dynamical behaviors of the blisk.

3. Nonlinear Governing Equations of Motion

Based on the lumped parameter model of the blisk in Figure 1, we establish the dynamic model of the blisk. The displacements of the blade m_b and disk m_d are expressed as

$$\mathbf{R}_{bi} = x_i(t)\mathbf{i} + R\mathbf{j}, \quad \mathbf{R}_{di} = y_i(t)\mathbf{i} + r\mathbf{j}. \quad (6)$$

The velocity and acceleration of the blade and disk are obtained as follows

$$\dot{\mathbf{R}}_{bi} = (\dot{x}_i + \Omega R)\mathbf{i} - x_i\Omega\mathbf{j}, \quad \dot{\mathbf{R}}_{di} = (\dot{y}_i + \Omega r)\mathbf{i} - y_i\Omega\mathbf{j}, \quad (7)$$

and

$$\ddot{\mathbf{R}}_{bi} = (\ddot{x}_i + \Omega\dot{R} - \dot{x}_i\Omega^2)\mathbf{i} - (2\Omega\dot{x}_i + \Omega^2 R + \ddot{x}_i)\mathbf{j}, \quad (8a)$$

$$\ddot{\mathbf{R}}_{di} = (\ddot{y}_i + \Omega\dot{r} - \dot{y}_i\Omega^2)\mathbf{i} - (2\Omega\dot{y}_i + \Omega^2 r + \ddot{y}_i)\mathbf{j}. \quad (8b)$$

The variation of the kinetic energy is given by

$$\begin{aligned} \delta K &= - \left(\sum_{i=1}^n m_b \dot{\mathbf{R}}_{bi} \delta \mathbf{R}_{bi} + m_d \dot{\mathbf{R}}_{di} \delta \mathbf{R}_{di} \right) \\ &= - \sum_{i=1}^n [m_b (\dot{x}_i + \Omega R - x_i \Omega^2) \delta x_i + m_d (\dot{y}_i + \Omega r - y_i \Omega^2) \delta y_i]. \end{aligned} \quad (9)$$

The potential energy U consists of the elastic potential energy U_1 of the blades, elastic potential energy U_2 of the disks and elastic potential energy U_3 of the linear springs. These potential energy expressions are given as follows

$$U = U_1 + U_2 + U_3, \quad (10a)$$

$$U_1 = \sum_{i=1}^n \left[\frac{k_b}{2} (x_i - y_i)^2 + \frac{\varepsilon_1 k_b}{4} (x_i - y_i)^4 \right], \quad (10b)$$

$$U_2 = \sum_{i=1}^n \left[\frac{k_d}{2} (y_i)^2 + \frac{\varepsilon_1 k_d}{4} (y_i)^4 \right]. \quad (10c)$$

$$U_3 = \sum_{i=1}^n \left[\frac{k_c}{2} (y_i - y_{i+1})^2 \right]. \quad (10d)$$

The variation of the potential energy is obtained as

$$\begin{aligned} \delta U = \sum_{i=1}^n [& k_d y_i \delta y_i + \varepsilon_2 k_d y_i^3 \delta y_i + \varepsilon_1 k_b (x_i - y_i)^3 (\delta x_i - \delta y_i) \\ & + k_b (x_i - y_i) (\delta x_i - \delta y_i) + k_c (y_i - y_{i+1})^3 (\delta y_i - \delta y_{i+1})]. \end{aligned} \quad (11)$$

The aerodynamic force subjects to the mass block of the blade. Therefore, we have

$$p_i(t) = p_{0i} + p_{1i} \sin \Omega_2 t + p_{2i} \cos \Omega_2 t. \quad (12)$$

The variation of the virtual work done by the first type of the aerodynamic force is denoted as follows

$$\delta W = \sum_{i=1}^n p_i(t) \delta x_{1i}. \quad (13)$$

The differential governing equation of motion for the rotating blisk is established by adopting Hamilton principle

$$\int_0^t (\delta K - \delta U + \delta W) dt = 0, \quad (14)$$

where t is time and δ is the variation.

Substituting δK , δU and δW into equation (14), the differential governing equations of motion for the rotating blisk are derived in terms of the displacements

δx_i :

$$\sum_{i=1}^n \left(-m_b \left(\ddot{x}_i + \Omega^2 x_i \right) - k_b (x_i - y_i) - \varepsilon_1 k_b (x_i - y_i)^3 + p_i(t) + \mu_1 \dot{x}_i \right) \delta x_i = 0, \quad (15a)$$

δx_{2i} :

$$\begin{aligned} \sum_{i=1}^n \left(-m_d \left(\ddot{y}_i + \Omega^2 y_i \right) - k_d y_i - \varepsilon_2 k_d y_i^3 + k_b (x_i - y_i) + \varepsilon_1 k_b (x_i - y_i)^3 \right) \delta y_i \\ - \sum_{i=1}^n k_c (y_i - y_{i+1}) (\delta y_i - \delta y_{i+1}) + \mu_2 \dot{y}_i \delta y_i = 0, \end{aligned} \quad (15b)$$

where μ_1 and μ_2 are the structural damping terms of the blisk.

The displacement relationships between two adjacent blades and two adjacent disk sectors are described as follows

$$x_{i+1} = e^{iu} x_i, \quad y_{i+1} = e^{iv} y_i, \quad (16)$$

where u is the phase angle between two adjacent blades, and v is the phase angle between two adjacent disk sectors.

We introduce the following dimensionless transformation

$$\begin{aligned}\bar{x}_1 &= \frac{x_1}{R}, \quad \bar{y}_1 = \frac{y_1}{R}, \quad \bar{r} = \frac{r}{R}, \quad \bar{t} = t \sqrt{\frac{RE}{m_d}}, \quad \bar{\Omega} = \Omega \sqrt{\frac{m_d}{RE}}, \\ \bar{m}_d &= \frac{m_d}{m_b}, \quad \bar{k}_d = \frac{k_d}{k_b}, \quad \bar{k}_c = \frac{k_c}{k_b}, \quad \bar{\varepsilon}_1 = \varepsilon_1 R^2, \quad \bar{\varepsilon}_2 = \varepsilon_2 R^2, \quad \bar{P}_i = \frac{P_i}{k_b R}.\end{aligned}\quad (17)$$

Substituting equations (16) and (17) into equation (15), two-degree-of-freedom dimensionless differential governing equations of motion for the blisk are obtained

$\delta x_{11}(t)$:

$$\begin{aligned}& \sum_{n=1}^m e^{i(n-1)u} (p - m_b \Omega^2 R) + \sum_{n=1}^m e^{i2(n-1)u} (-m_b \Omega^2 + m_b \Omega^2 x_1 - x_1 + \mu_1) \\ & + \sum_{n=1}^m k_b e^{i(n-1)(u+v)} y_1 + \sum_{n=1}^m e^{i4(n-1)u} (-\varepsilon_1 k_b x_1^3) + \sum_{n=1}^m e^{i(n-1)(3u+v)} (3\varepsilon_1 k_b x_1^2 y_1) \\ & + \sum_{n=1}^m e^{i(n-1)(2u+2v)} (3\varepsilon_1 k_b x_1 y_1^2) + \sum_{n=1}^m e^{i(n-1)(u+3v)} (\varepsilon_1 k_b y_1^3) = 0,\end{aligned}\quad (18a)$$

$\delta x_{21}(t)$:

$$\begin{aligned}& \sum_{n=1}^m e^{i(n-1)v} (-m_d \Omega^2) + \sum_{n=1}^m e^{i2(n-1)v} (-m_d \Omega^2 y_1 - k_d y_1 - k_c y_1 - k_b y_1 + \mu_2) \\ & + \sum_{n=1}^m e^{i(n-1)(u+v)} k_b x_1 + \sum_{n=1}^m e^{i4(n-1)v} (-\varepsilon_2 k_d y_1^3 - \varepsilon_1 k_b y_1^3) + \sum_{n=1}^m e^{i(n-1)(3u+v)} (\varepsilon_1 k_b x_1^3) \\ & + \sum_{n=1}^m e^{i(n-1)(2u+2v)} (-3\varepsilon_1 k_b x_1^2 y_1) + \sum_{n=1}^m e^{i(n-1)(u+3v)} (-3\varepsilon_1 k_b y_1^2 x_1) \\ & + \sum_{n=1}^m e^{i2(n-1)v} (2k_c y_1) + \sum_{n=1}^m e^{i2nv} (-k_c y_1) = 0,\end{aligned}\quad (18b)$$

where m is the total number of the blades.

Substituting Ω , $\mathfrak{E}$ and $e^{ix} = \cos x + i \sin x$ into equation (18) and separating the real and imaginary parts, the nonlinear governing equations of motion for describing the transverse vibrations of the rotating blisk are obtained as follows

$$\begin{aligned}& \mathfrak{E}_1 + \gamma_1 \mu_1 \mathfrak{E}_1 + (\alpha_{13} - \alpha_{12} \Omega_0^2) x_1 - \alpha_{12} (f^2 \cos^2 \Omega_1 t + 2\Omega_0 f \cos \Omega_1 t) x_1 \\ & + \alpha_{14} y_1 + \alpha_{15} x_1^3 + \alpha_{16} y_1^3 + \alpha_{17} x_1 y_1^2 + \alpha_{18} x_1^2 y_1 \\ & = \alpha_{19} (p_0 + p_1 \sin(\Omega_2 t) + p_2 \cos(\Omega_2 t)) + \alpha_{20} \Omega_1 f \sin \Omega_1 t,\end{aligned}\quad (19a)$$

$$\begin{aligned} & \mathfrak{X}_1 + \gamma_2 \mu_2 \mathfrak{X}_1 + (\alpha_{23} - \alpha_{22} \Omega_0^2) y_1 - \alpha_{22} (f^2 \cos^2 \Omega_1 t + 2\Omega_0 f \cos \Omega_1 t) y_1 + \alpha_{22} x_1 \\ & + \alpha_{25} x_1^3 + \alpha_{26} y_1^3 + \alpha_{27} x_1^2 y_1 + \alpha_{28} x_1 y_1^2 = \alpha_{29} \Omega_1 f \sin \Omega_1 t, \end{aligned} \quad (19b)$$

where the coefficients of equation (19) are calculated based on simplifying equations obtained by separating the real part and imaginary part of equation (18).

We introduce the following transformations

$$x_1 = x_1, \quad y_1 = x_3, \quad \mathfrak{X}_1 = x_2, \quad \mathfrak{Y}_1 = x_4. \quad (20)$$

Equation (19) is transformed into the first-order ordinary differential equations. Therefore, we obtain the dimensionless first-order ordinary differential governing equations of motion for the blisk

$$\mathfrak{X}_1 = x_2. \quad (21a)$$

$$\begin{aligned} \mathfrak{X}_2 = & -\gamma_1 \mu_1 \mathfrak{X}_1 - (\alpha_{13} - \alpha_{12} \Omega_0^2) x_1 + \alpha_{12} (f^2 \cos^2 \Omega_1 t + 2\Omega_0 f \cos \Omega_1 t) x_1 \\ & - \alpha_{14} x_3 - \alpha_{15} x_1^3 - \alpha_{16} x_3^3 - \alpha_{17} x_1 x_3^2 - \alpha_{18} x_1^2 x_3 \\ & + \alpha_{19} (p_0 + p_1 \sin(\Omega_2 t) + p_2 \cos(\Omega_2 t)) + \alpha_{20} \Omega_1 f \sin \Omega_1 t, \end{aligned} \quad (21b)$$

$$\mathfrak{X}_3 = x_4, \quad (21c)$$

$$\begin{aligned} \mathfrak{X}_4 = & -\gamma_2 \mu_2 \mathfrak{X}_2 - \alpha_{22} x_1 - (\alpha_{23} - \alpha_{22} \Omega_0^2) x_2 + \alpha_{22} (f^2 \cos^2 \Omega_1 t + 2\Omega_0 f \cos \Omega_1 t) x_2 \\ & - \alpha_{25} x_1^3 - \alpha_{26} x_2^3 - \alpha_{27} x_1^2 x_2 - \alpha_{28} x_1 x_2^2 + \alpha_{29} \Omega_1 f \sin \Omega_1 t. \end{aligned} \quad (21d)$$

4. Perturbation Analysis and Amplitude-Frequency Curves

The perturbation analysis is performed by using the method of multiple scales [53]. The scale transformations are introduced as follows

$$\begin{aligned} \mu_1 & \rightarrow \varepsilon \mu_1, \quad \alpha_{13} \rightarrow \varepsilon \alpha_{13}, \quad \alpha_{12} \rightarrow \varepsilon \alpha_{12}, \quad \alpha_{14} \rightarrow \varepsilon \alpha_{14}, \quad \alpha_{15} \rightarrow \varepsilon \alpha_{15}, \quad \alpha_{16} \rightarrow \varepsilon \alpha_{16}, \\ \alpha_{17} & \rightarrow \varepsilon \alpha_{17}, \quad \alpha_{18} \rightarrow \varepsilon \alpha_{18}, \quad \alpha_{19} \rightarrow \varepsilon \alpha_{19}, \quad \alpha_{20} \rightarrow \varepsilon \alpha_{20}, \quad \mu_2 \rightarrow \varepsilon \mu_2, \quad \alpha_{23} \rightarrow \varepsilon \alpha_{23}, \\ \alpha_{22} & \rightarrow \varepsilon \alpha_{22}, \quad \alpha_{25} \rightarrow \varepsilon \alpha_{25}, \quad \alpha_{26} \rightarrow \varepsilon \alpha_{26}, \quad \alpha_{27} \rightarrow \varepsilon \alpha_{27}, \quad \alpha_{28} \rightarrow \varepsilon \alpha_{28}, \quad \alpha_{29} \rightarrow \varepsilon \alpha_{29}, \end{aligned} \quad (22)$$

where ε is a small perturbation parameter.

Substituting the scale transformations into equation (19), the following expressions are obtained

$$\begin{aligned} & \mathfrak{X}_1 + \varepsilon \gamma_1 \mu_1 \mathfrak{X}_1 + (\alpha_{13} - \alpha_{12} \Omega_0^2) x_1 - \alpha_{12} \varepsilon (f^2 \cos^2 \Omega_1 t + 2\Omega_0 f \cos \Omega_1 t) x_1 \\ & + \varepsilon \alpha_{14} y_1 + \varepsilon \alpha_{15} x_1^3 + \varepsilon \alpha_{16} y_1^3 + \varepsilon \alpha_{17} x_1 y_1^2 + \varepsilon \alpha_{18} x_1^2 y_1 \\ & = \varepsilon \alpha_{19} (p_0 + p_1 \sin \Omega_2 t + p_2 \cos \Omega_2 t) + \varepsilon \alpha_{20} \Omega_1 f \sin \Omega_1 t, \end{aligned} \quad (23a)$$

$$\begin{aligned}
& \frac{1}{2} \varepsilon \gamma_2 \mu_2 x_1^2 + \varepsilon \alpha_{22} x_1 + (\alpha_{23} - \alpha_{22} \Omega_0^2) y_1 - \alpha_{22} \varepsilon (f^2 \cos^2 \Omega_1 t + 2 \Omega_0 f \cos \Omega_1 t) y_1 \\
& + \varepsilon \alpha_{25} x_1^3 + \varepsilon \alpha_{26} y_1^3 + \varepsilon \alpha_{27} x_1^2 y_1 + \varepsilon \alpha_{28} x_1 y_1^2 \\
& = \varepsilon \alpha_{29} \Omega_1 f \sin \Omega_1 t.
\end{aligned} \tag{23b}$$

The method of multiple scales is adopted to obtain averaged equations. Let the linear solution of equation (23) be in the following form

$$x_{11}(t, \varepsilon) = x_{10}(T_0, T_1) + \varepsilon x_{11}(T_0, T_1), \tag{24a}$$

$$y_1(t, \varepsilon) = x_{20}(T_0, T_1) + \varepsilon x_{21}(T_0, T_1), \tag{24b}$$

where $T_0 = t$ and $T_1 = \varepsilon t$.

The resonant phenomena easily happen for the blade and disk. Due to the energy transfer between different modes, the relationship between two linear natural frequencies leads to the large amplitude nonlinear vibrations. Considering 1:1 internal resonance and principal parametric resonance, the resonant relations are presented as follows

$$\omega_1^2 = \Omega_2^2 - \varepsilon \sigma_1, \quad \omega_2^2 = \Omega_2^2 - \varepsilon \sigma_2, \quad \omega_1 \approx \omega_2, \tag{25}$$

where ω_1 and ω_2 are the first two natural frequencies, σ_1 and σ_2 are two detuning parameters.

Thus, the derivatives with respect to t become

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial}{\partial T_1} \frac{\partial T_1}{\partial t} + \dots = D_0 + \varepsilon D_1 + \dots, \tag{26a}$$

$$\frac{d^2}{dt^2} = (D_0 + \varepsilon D_1 + \dots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \dots, \tag{26b}$$

where $D_n = \frac{\partial}{\partial T_n}$, $n = 0, 1$.

Assuming $\Omega = 1$, substituting equations (24)-(26) into equation (23) and balancing coefficients of the similar power of ε yield

ε^0 order:

$$D_0^2 x_{10} + x_{10} = 0, \tag{27a}$$

$$D_0^2 x_{20} + x_{20} = 0. \tag{27b}$$

ε^1 order:

$$\begin{aligned}
D_0^2 x_{11} + x_{11} = & -2D_0 D_1 x_{10} - \gamma_1 \mu_1 x_{10}^2 + \sigma_1 x_{10} + \alpha_{12} (f^2 \cos^2 \Omega_1 t + 2 \Omega_0 f \cos \Omega_1 t) x_{10} \\
& - k_{12} x_{30} - \alpha_{15} x_{10}^3 - \alpha_{16} x_{30}^3 - \alpha_{17} x_{10} x_{30}^2 - \alpha_{18} x_{10}^2 x_{30} \\
& + \alpha_{19} (p_0 + p_1 \sin(\Omega_2 t) + p_2 \cos(\Omega_2 t)) + \alpha_{20} \Omega_1 f \sin \Omega_1 t,
\end{aligned} \tag{28a}$$

$$\begin{aligned}
D_0^2 x_{21} + x_{21} = & -2D_0 D_1 x_{20} - \gamma_2 \mu_2 x_{20} + \sigma_2 x_{20} + \alpha_{22} (f^2 \cos^2 \Omega_1 t + 2\Omega_0 f \cos \Omega_1 t) x_{20} \\
& - k_{21} x_{10} - \alpha_{25} x_{10}^3 - \alpha_{26} x_{20}^3 - \alpha_{27} x_{10}^2 x_{20} \\
& - \alpha_{28} x_{10} x_{20}^2 + \alpha_{29} \Omega_1 f \sin \Omega_1 t.
\end{aligned} \tag{28b}$$

The general solution of equation (27) is expressed as

$$x_{10} = A_1(T_1) e^{iT_0} + \bar{A}_1(T_1) e^{-iT_0}, \quad x_{20} = A_2(T_1) e^{iT_0} + \bar{A}_2(T_1) e^{-iT_0}. \tag{29}$$

where $\bar{A}_1$ and $\bar{A}_2$ are the complex conjugate terms of A_1 and A_2 .

Substituting equation (29) into equation (28) yields

$$\begin{aligned}
D_0^2 x_{11} + x_{11} = & (-2iD_1 A_1 - \gamma_1 \mu_1 i A_1 + \sigma_1 A_1 + \alpha_{12} \left(\frac{1}{2} f^2 A_1 + \frac{1}{4} f \bar{A}_1 \right) A_1 - k_{12} A_2 \\
& - 3\alpha_{15} A_1^2 \bar{A}_1 - 3\alpha_{16} A_2^2 \bar{A}_2 - 2\alpha_{17} A_1 A_2 \bar{A}_2 - \alpha_{18} \bar{A}_2 A_1^2 \\
& + \alpha_{19} \left(-\frac{i}{2} p_1 + \frac{1}{2} p_2 \right) - \frac{i}{2} \alpha_{20} f) e^{iT_0} + cc + NST
\end{aligned} \tag{30a}$$

$$\begin{aligned}
D_0^2 x_{21} + x_{21} = & (-2iD_1 A_2 - \gamma_2 \mu_2 i A_2 + \sigma_2 A_2 + \alpha_{22} \left(\frac{1}{2} f^2 A_2 + \frac{1}{4} f^2 \bar{A}_2 \right) A_2 \\
& - k_{21} A_1 - 3\alpha_{25} A_1^2 \bar{A}_1 - 3\alpha_{26} A_2^2 \bar{A}_2 - 2\alpha_{27} A_2 A_1 \bar{A}_1 - \alpha_{27} A_1^2 \bar{A}_2 \\
& - 2\alpha_{28} A_1 A_2 \bar{A}_2 - \alpha_{28} A_2^2 \bar{A}_1 - \frac{i}{2} \alpha_{29} f) e^{iT_0} + cc + NST,
\end{aligned} \tag{30b}$$

where cc denotes the complex conjugate of the right terms in equation (30), and NST is the non-secular part.

Letting the secular term in equation (30) to be zero and eliminating the secular term, the following equations are derived as

$$\begin{aligned}
D_1 A_1 = & -\frac{1}{2} \gamma_1 \mu_1 A_1 - \frac{1}{2} i \sigma_1 A_1 - \frac{1}{2} i \alpha_{12} \left(\frac{1}{2} f^2 A_1 + \frac{1}{4} f^2 \bar{A}_1 \right) A_1 + \frac{1}{2} i k_{12} A_2 + \frac{3}{2} i \alpha_{15} A_1^2 \bar{A}_1 \\
& + \frac{3}{2} i \alpha_{16} A_2^2 \bar{A}_2 + i \alpha_{17} A_1 A_2 \bar{A}_2 + \frac{1}{2} i \alpha_{18} \bar{A}_2 A_1^2 - \alpha_{19} \left(\frac{1}{4} p_1 + \frac{i}{4} p_2 \right) - \frac{1}{4} \alpha_{20} f,
\end{aligned} \tag{31a}$$

$$\begin{aligned}
D_1 A_2 = & -\frac{1}{2} \gamma_2 \mu_2 A_2 - \frac{1}{2} i \sigma_2 A_2 - \frac{1}{2} i \alpha_{22} \left(\frac{1}{2} f^2 A_2 + \frac{1}{4} f^2 \bar{A}_2 \right) A_2 + \frac{1}{2} i k_{21} A_1 + \frac{3}{2} i \alpha_{25} A_1^2 \bar{A}_1 \\
& + \frac{3}{2} i \alpha_{26} A_2^2 \bar{A}_2 + i \alpha_{27} A_2 A_1 \bar{A}_1 + \frac{1}{2} i \alpha_{27} A_1^2 \bar{A}_2 + i \alpha_{28} A_1 A_2 \bar{A}_2 + \frac{1}{2} i \alpha_{28} A_2^2 \bar{A}_1 - \frac{1}{4} \alpha_{29} f.
\end{aligned} \tag{31b}$$

To obtain the averaged equations in the polar coordinate, A_1 and A_2 are expressed as follows

$$A_1(T_1) = \frac{1}{2}a_1(T_1)e^{i\phi_1(T_1)}, \quad A_2(T_1) = \frac{1}{2}a_2(T_1)e^{i\phi_2(T_1)}. \quad (32)$$

Substituting equation (32) into equation (31) and separating the real part and imaginary part, the average equations in the polar coordinate system are derived as follows

$$\begin{aligned} \frac{da_1}{dT_1} = & -\frac{1}{2}\gamma_1\mu_1a_1 - \frac{1}{8}\alpha_{12}f^2a_1\sin(2\phi_1) + \left(\frac{1}{2}\alpha_{18}a_2a_1^2 + \frac{1}{2}k_{12}a_2 + \frac{3}{2}\alpha_{16}a_2^3\right)\sin(\phi_1 - \phi_2) \\ & + \frac{1}{2}\alpha_{17}a_1a_2^2\sin(2\phi_1 - 2\phi_2) - \frac{1}{4}\alpha_{19}p_1 - \frac{1}{4}\alpha_{20}\alpha_{12}f, \end{aligned} \quad (33a)$$

$$\begin{aligned} a_1 \frac{d\phi_1}{dT_1} = & -\frac{3}{2}\alpha_{15}a_1^3 - \frac{1}{2}\sigma_1a_1 + \alpha_{17}a_1a_2^2 + \frac{1}{8}a_1\alpha_{12}f^2\cos(2\phi_1) + \frac{1}{2}\alpha_{17}a_1a_2^2\cos(2\phi_1 - 2\phi_2) \\ & + \frac{1}{4}a_1\alpha_{12}f^2 + \left(\frac{3}{2}\alpha_{18}a_2a_1^2 + \frac{1}{2}k_{12}a_2 + \frac{3}{2}\alpha_{16}a_2^3\right)\cos(\phi_1 - \phi_2) - \frac{1}{4}\alpha_{19}p_2, \end{aligned} \quad (33b)$$

$$\begin{aligned} \frac{da_2}{dT_1} = & -\frac{1}{2}\gamma_2\mu_2a_2 + \frac{1}{4}\alpha_{22}\alpha_{24}a_2f^2 - \left(\frac{1}{2}\alpha_{28}a_1a_2^2 + \frac{1}{2}k_{21}a_1 + \frac{3}{2}\alpha_{25}a_1^3\right)\sin(\phi_1 - \phi_2) \\ & - \frac{1}{2}\alpha_{27}a_2a_1^2\sin(2\phi_1 - 2\phi_2) - \frac{1}{8}a_2\alpha_{22}f^2\sin(2\phi_2) - \frac{1}{4}\alpha_{22}\alpha_{29}f, \end{aligned} \quad (33c)$$

$$\begin{aligned} a_2 \frac{d\phi_2}{dT_1} = & \frac{3}{2}\alpha_{26}a_2^3 + \alpha_{27}a_2a_1^2 + \left(\frac{3}{2}\alpha_{28}a_1a_2^2 + \frac{1}{2}k_{21}a_1 + \frac{3}{2}\alpha_{25}a_1^3\right)\cos(\phi_1 - \phi_2) \\ & - \frac{1}{2}\sigma_2a_2 + \frac{1}{2}\alpha_{27}a_2a_1^2\cos(2\phi_1 - 2\phi_2) - \frac{1}{8}a_2\alpha_{22}f^2\cos(2\phi_2). \end{aligned} \quad (33d)$$

Based on equation (33), the amplitude-frequency response curves are investigated for the rotating blisk. Some parameters obviously affect the nonlinear dynamic behaviors of the lumped parameter model for the blisk. We mainly study influences of the rotating speed, aerodynamic force, damping, coupling stiffness and cubic nonlinear terms on the amplitude-frequency response curves of the blisk.

Letting $\varphi = \phi_1 - \phi_2$, when $\alpha_1 = \alpha_2 = \phi_1 = \phi_2 = 0$, the equations of the steady-state vibration for the blisk are obtained as follows

$$\begin{aligned} & \left(\frac{1}{2}\gamma_1\mu_1a_1 + \frac{1}{4}\alpha_{19}p_1 + \frac{1}{4}\alpha_{20}\alpha_{12}f\right)^2 + \left(\frac{3}{2}\alpha_{15}a_1^3 + \frac{1}{2}\sigma_1a_1 - \alpha_{17}a_1a_2^2 + \frac{1}{4}a_1\alpha_{12}f^2 + \frac{1}{4}\alpha_{19}p_2\right)^2 \\ & = \left(-\frac{1}{8}\alpha_{12}f^2a_1\sin(2\phi_1) + \left(\frac{1}{2}\alpha_{18}a_2a_1^2 + \frac{1}{2}k_{12}a_2 + \frac{3}{2}\alpha_{16}a_2^3\right)\sin(\varphi) + \frac{1}{2}\alpha_{17}a_1a_2^2\sin(2\varphi)\right)^2 \\ & + \left[\frac{1}{2}\alpha_{17}a_1a_2^2\cos(2\varphi) - \frac{1}{8}a_1\alpha_{12}f^2\cos(2\phi_1) + \left(\frac{3}{2}\alpha_{18}a_2a_1^2 + \frac{1}{2}k_{12}a_2 + \frac{3}{2}\alpha_{16}a_2^3\right)\cos(\varphi)\right]^2, \end{aligned}$$

(34a)

$$\begin{aligned}
& \left(\frac{1}{2} \gamma_2 \mu_2 a_2 + \frac{1}{4} a_2 \alpha_{22} f^2 + \frac{1}{4} \alpha_{29} \alpha_{22} f \right)^2 + \left(\frac{3}{2} \alpha_{26} a_2^3 - \frac{1}{2} \sigma_2 a_2 + \alpha_{27} a_2 a_1^2 \right)^2 \\
& = \left(- \left(\frac{1}{2} \alpha_{28} a_1 a_2^2 + \frac{1}{2} k_{21} a_1 + \frac{3}{2} \alpha_{25} a_1^3 \right) \sin(\phi) - \frac{1}{2} \alpha_{27} a_2 a_1^2 \sin(2\phi) - \frac{1}{8} a_2 \alpha_{22} f^2 \sin(2\phi_2) \right)^2 \\
& + \left[\left(\frac{3}{2} \alpha_{28} a_1 a_2^2 + \frac{1}{2} k_{21} a_1 + \frac{3}{2} \alpha_{25} a_1^3 \right) \cos(\phi) + \frac{1}{2} \alpha_{27} a_2 a_1^2 \cos(2\phi) - \frac{1}{8} a_2 \alpha_{22} f^2 \cos(2\phi_2) \right]^2.
\end{aligned}
\tag{34b}$$

The amplitude-frequency response curves of the blisk are analyzed when the parameters of equation (34) are respectively selected as

$$\begin{aligned}
& \gamma_1 = 1.0, \mu_1 = 0.2, \alpha_{12} = -1, \alpha_{15} = 0.082, \alpha_{16} = 0.006, \alpha_{17} = -0.02, \alpha_{18} = -0.01, \\
& \alpha_{19} = -1, k_{12} = 1, f = 1, \sigma_2 = 2, p_1 = 2, p_2 = 0.025, \gamma_2 = 1.0, \mu_2 = 0.2, \\
& \alpha_{22} = -1, \alpha_{25} = 0.01, \alpha_{26} = 0.05, \alpha_{27} = -0.01, \alpha_{28} = -0.01, \alpha_{29} = 1.0, k_{21} = 1.
\end{aligned}$$

Letting $a_2 = 1$ in equation (34a) and $a_1 = 1$ in equation (34b), the amplitude frequency response curves of the blades and disk subjected to the aerodynamic force are obtained, as shown in Figures 6 and 7. In Figures 6 and 7, other parameters remain unchanged except for the control parameters. The typical multi-valued phenomena occur, which results in the jumping of the solution in equation (34).

Figure 6(a) demonstrates that the amplitudes and bandwidths of the blades decrease with the increase of the damping μ_1 . The increase of the perturbation rotating speed f and coupling stiffness k_{12} results in the increase of the amplitude and bandwidth for the blades, as shown in Figures 6(b) and 6(c). The right-ward shift of the amplitude-frequency curve occurs for the blades, as shown in Figure 6(b). Figure 6(d) illustrates that as the aerodynamic force p_1 increases, the amplitudes of the amplitude-frequency response curves for the blades increase, but the bandwidth decreases. From Figures 6(a)-6(d), it is shown that the blades have the hard spring properties. As the frequency σ_1 changes, the multi-valued phenomena occur. When the cubic term $|a_{15}|$ increases, Figure 6(e) depicts that the amplitude-frequency response curves bend and are away from zero solution, and the amplitudes of the dynamic responses decreases. When the nonlinear parameter a_{15} is positive, the blades exhibit the hard spring properties. However, when the nonlinear parameter a_{15} is negative, the blades have the soft spring properties. When value of a_{15}

is opposite, the amplitude-frequency response curves are symmetric.

Figure 7 illustrates the influences of several parameters on the amplitude-frequency response curves of the disk, including the damping μ_2 , perturbation rotating speed f and coupling stiffness k_{21} . It is illustrated that the disk has the hard spring properties. The results show that damping μ_2 and coupling stiffness k_{21} have almost the same effect on the amplitude-frequency curves for the disk, as shown in Figures 7(a) and 7(c). In Figure 7(a), the damping μ_2 has no difference to the response bandwidth of the disk. Figure 7(b) demonstrates that the right-ward shift or left-ward shift of the amplitude-frequency response curves for the disk do not occur with the change of the perturbation rotating speed f . It is pointed out that a smaller f results in the remarkable narrowing in the response bandwidth and the enhancement of the amplitude for the disk.

5. Numerical Simulation of Dual-Parameter Chaos

In recent decades, a large variety of numerical methods have been developed to detect the periodic and chaotic vibrations of the nonlinear dynamical systems. The bifurcation diagrams and Lyapunov exponents have been the common choices for investigating the chaotic vibrations of the structures with the geometric nonlinearity and subjected to the various kinds of excitations. The bifurcation diagrams are the effective tools detecting the periodic and chaotic vibrations with a single controlling parameter. However, only depending on two-dimensional bifurcation diagrams is insufficient to illustrate the existence of the chaotic dynamics for the dual-parameter nonlinear system in three-dimensional (3D) space. Lyapunov exponents are used to distinguish the periodic and chaotic vibrations. However, it is difficult to identify which type of the periodic vibration appears in the nonlinear dynamical system according to Lyapunov exponents. Therefore, to exceed the limitations of the traditional bifurcation diagrams and Lyapunov exponent, the dual-parameter bifurcation diagrams are considered to study the periodic and chaotic vibrations of the structure.

Based on equation (21), numerical simulations are performed to study the dynamic behaviors of the lumped parameter model for the rotating blisk subjected to the aerodynamic force. The dual-parameter Lyapunov exponents are numerically obtained by using Benettin-Wolf algorithm. The numerical results of the dual-parameter bifurcation diagrams are constructed based on Runge-Kutta algorithm and Poincare map. Figure 8

shows 3D dual-parameter Lyapunov exponents and bifurcation diagrams of the blisk versus the perturbation rotating speed f and aerodynamic force p_0 . Obviously, there is an excellent correspondence between Lyapunov exponents and bifurcation diagrams. Based on 3D dual-parameter Lyapunov exponents, we give the influences of one parameter on the nonlinear dynamics of the blisk structure when another parameter is regarded as a controlling parameter, as shown in Figures 9. These figures reveal clearly how the system evolves between the periodic and chaotic vibrations.

The initial conditions and parameters of the blisk structure are respectively selected $\gamma_1 = 1.0$, $\alpha_{12} = -0.3$, $\alpha_{13} = 39174.7368$, $\alpha_{14} = 0.08$, $\alpha_{15} = 1.92$, $\alpha_{16} = 1$, $\alpha_{17} = 0.2$, $\alpha_{18} = 0.76$, $\alpha_{19} = 0.5$, $\alpha_{20} = 0.9$, $\gamma_2 = 1.0$, $\alpha_{22} = 0.6$, $\alpha_{23} = 1491.2$, $\alpha_{25} = 1.5$, $\alpha_{26} = 20.09$, $\alpha_{27} = 2.7$, $\alpha_{28} = -1.28$, $\alpha_{29} = 0.1$, $\Omega_1 = 1.0$, $\Omega_2 = 1.0$, $p_1 = 50$, $p_2 = 10$, $x_{10} = 1.0$, $x_{20} = 0.101$, $x_{30} = 1.0$, $x_{40} = 0.1$.

In the following analysis, the values of these parameters remain unchanged. Some important parameters of the blisk, such as the structural damping, aerodynamic force and rotating speed, are considered to analyze dual-parameter chaotic vibrations, respectively.

5.1 Effect of Perturbation Rotating Speed

Considering the effect of the perturbation rotating speed f and aerodynamic force p_0 , the dual-parameter chaotic vibrations are studied for the blisk, and the paths to chaos are explored. Based on 3D dual-parameter bifurcation diagrams and Lyapunov exponents, single parameter bifurcation phenomena of the blisk are obtained by using the variable controlling method. Selecting the perturbation rotating speed f and aerodynamic force p_0 as two controlling parameters, the dual-parameter Lyapunov exponents exhibit the regions where the dual-parameter chaotic vibrations occur, as shown in Figures 8(a) and 8(b), where damping terms μ_1 and μ_2 , and steady-state rotating speed Ω_0 are given as $\mu_1 = 0.46$, $\mu_2 = 0.46$, $\Omega_0 = 200$. In order to show the evolution process of the periodic and chaotic vibrations, 3D dual-parameter bifurcation diagram is given in Figure 8(c).

In Figure 8(a), two horizontal axes respectively are the perturbation rotating speed f and aerodynamic force p_0 , and the vertical axis is the dual-parameter Lyapunov exponents. We calculate four Lyapunov exponents of each point. Therefore, there are four

surfaces along the vertical axis in Figure 8(a). According to the colors on the right side of Figure 8(a), Lyapunov exponents in red are greater than zero, which indicates that the blisk performs the dual-parameter chaotic vibrations under combined the perturbation rotating speed f and aerodynamic force p_0 when the parameters are located at the regions in red. The striking red regions in Figures 8(a) and 8(b) are called as the chaotic regions. To achieve the transformation from one chaotic region to next chaotic region, the periodic vibration between two chaotic regions must happen, where Lyapunov exponents are less than or equal to 0. Meanwhile, 3D dual-parameter bifurcation diagram in Figure 8(c) is given to further verify that this phenomenon occurs in Figures 8(a) and 8(b). The results demonstrate that these figures correspond to each other perfectly.

The evolution paths of the dual-parameter chaotic vibrations are studied in Figure 9 when p_0 is located in the interval $(0,100)$. Choosing the aerodynamic force p_0 as the controlling parameter, the effect of the perturbation rotating speed f on the dynamics of the blisk is analyzed. Figure 9(a) presents the dual parameter Lyapunov exponents. Figures 9(b)-9(d) depict three bifurcation diagrams when the perturbation rotating speeds are $f = 2$, $f = 10$ and $f = 25$. As the perturbation rotating speeds f increase, the periodic vibration of the blisk turns to the chaotic vibrations gradually, as shown in Figures 9(b)-9(d). When $f = 2$, the points of the bifurcation diagram in Figure 9(b) is sparse, which means that the blisk only has the periodic vibration. When $f = 10$, the intensive points appear in Figure 9(c), which implies that the chaotic vibrations arise in the blisk. When f increases to $f = 25$, the intensive points occupy the larger range, which means that the chaotic vibrations occur in a wider interval of the aerodynamic force p_0 , as shown in Figure 9(d). The amplitudes of the chaotic vibrations increase. Moreover, the larger the rotating speed is, the larger the amplitude of the chaotic vibrations is. The results indicate that the perturbation rotating speed f decides whether the chaotic vibrations of the blisk occur under combined the perturbation rotating speed f and aerodynamic force p_0 when p_0 is in the interval $p_0 = 0-100$.

Three-dimensional bifurcation diagrams of the perturbation rotating speed f versus the first-order and second-order modes x_1 and x_3 are presented in the interval $f = 0 \sim 100$, as shown in Figure 10. Where the aerodynamic force p_0 is $p_0 = 50$, and other parameters remain the same as these given in Figure 8. Projecting 3D bifurcation

diagram, namely, the black part in Figure 10(a), to the planes (f, x_1) and (x_1, x_2) , the two-dimensional bifurcation diagram and Poincare map are obtained. Two-dimensional bifurcation diagram and corresponding Poincare map for the second-order mode x_3 are given, as shown in Figure 10(b). Due to figures obtained by numerical simulations, an accurate judgment on the nonlinear vibrations of the blisk is realized.

The enlarged version of the bifurcation diagrams for the perturbation rotating speed f versus the first-order and second-order modes x_1 and x_3 are respectively displayed in Figures 11(a) and 11(b). Figure 11(c) depicts Lyapunov exponents of the rotating blisk when considering the perturbation rotating speed f as a variable. Lyapunov exponents and bifurcation diagrams have high consistency. Lyapunov exponents generated by the system can be positive, negative or zero. The positive Lyapunov exponents imply that the system must be the chaotic or hyperchaotic vibrations. If a chaotic system has the multiple positive exponents in high-dimensional phase space, it is hyperchaotic.

It is judged from Figure 11 that the periodic and chaotic vibrations of the rotating blisk appear repeatedly. The numerical results demonstrate that the perturbation rotating speed f has a major effect on the nonlinear dynamics of the blisk. Both negative Lyapunov exponents in Figure 11(c) and sparse points in Figures 11(a) and 11(b) illustrate the periodic vibrations occur in the blisk. The high density points in the interval $f = 28-65$ show that the chaotic vibrations occur, which is proved by Lyapunov exponents in the corresponding interval in Figure 11(c). When the perturbation rotating speed f is $f = 72$, the max Lyapunov exponent of the blisk is positive, as shown in Figure 11(c). The nonlinear vibrations of the large amplitude appear when $f = 72$, as shown in Figure 12, which illustrates that there exist the chaotic vibrations the blisk. Figures 12(a) and 12(c) are the phase portraits on the planes $(x_1, \dot{x}_1)$ and $(x_3, \dot{x}_3)$. Figures 12(b) and 12(d) give the waveforms on the planes (t, x_1) and (t, x_3) . Figure 12(e) is 3D phase portrait in space $(x_1, \dot{x}_1, x_3)$. The blue and black parts in Figure 12(f) depict Poincare maps on the plane $(x_1, \dot{x}_1)$ and $(x_3, \dot{x}_3)$. In the aforementioned analysis, the phase portraits, waveforms and Poincare map are highly consistent.

5.2 Effects of External Excitation and Steady-State Rotating Speed

The dynamic behaviors of the rotating blisk under the aerodynamic force p_0 are presented in Figures 13 and 14 when the perturbation rotating speed f is $f = 70$. Other parameters keep unchanged. It is seen obviously from Figure 14 that the chaotic vibrations mainly occur in two intervals, in which the plenty of points are presented, and there are positive of Lyapunov exponents. It is hard to decide whether there are the periodic vibration in these two intervals depending on Figures 14(a) and 14(b). However, Lyapunov exponents in Figure 14(c) illustrates that there are exactly the period-1, multi-periodic and chaotic vibrations with the increase of the aerodynamic force p_0 .

In the same way, the dual-parameter chaotic vibrations are investigated for the blisk under combined the steady-state rotating speed Ω_0 and aerodynamic force p_0 when the perturbation rotating speed f is $f = 70$, as shown in Figure 15. The initial condition and other parameters keep the same as Figure 8. Figure 15(a) is 3D dual-parameter Lyapunov exponents of the blisk. Projecting Figure 15(a) on the horizontal plane, we obtain Figure 15(b). Figure 15(c) is the corresponding 3D dual-parameter bifurcation diagram for the blisk structure. These diagrams correspond to each other completely.

Figure 15(b) gives that the left side of the dual-parameter chaotic region extends along 45 degrees and there is an obvious boundary between the chaotic and periodic regions. To study the relation between the boundary and dual-parameter chaotic vibrations of the blisk under combined the steady-state rotating speed Ω_0 and aerodynamic force p_0 , we obtain Figure 16. Figures 16(b) to 16(f) are five bifurcation diagrams about the aerodynamic force when the steady-state rotating speeds respectively are $\Omega_0 = 180, 190, 195, 210$ and 220 . According to the motion of the vertical red line in these diagrams, we see clearly that the chaotic region moves towards right along the horizontal axis p_0 with almost the same maximum vibration amplitude. Single period vibration only appears on the right side in Figures 16(b)-16(d). Figures 16(e) and 16(f) show when Ω_0 increases to a certain value, the vibration of the blisk becomes more complex that the periodic vibrations alternate to appear with the chaotic vibrations, which indicates the aerodynamic load p_0 has an important effect on the dual-parameter chaotic dynamics for the blisk.

The effect of the steady-state rotating speed on the nonlinear dynamics of the blisk is studied when the aerodynamic force is $p_0 = 50$ and other parameters are the same as Figure 15. 3D bifurcation diagrams of the steady-state rotating speed Ω_0 for the blisk in

the interval $\Omega_0 = 160 \sim 260$ are given, as shown in Figure 17. Two-dimensional bifurcation and Poincare maps are obtained by projecting 3D bifurcation diagram to the specific planes. Two-dimensional bifurcation diagrams are shown using the blue and red. Poincare maps are described in the purple. The bifurcation diagrams of Ω_0 versus the first- and second-order modes x_1 and x_3 are represented, as shown in Figures 18(a) and 18(b). Figure 18(c) gives Lyapunov exponents of the blisk. The numerical results illustrate that the periodic and chaotic vibrations appear. When the steady-state rotating speed of the blisk is taken as $\Omega_0 = 225$, the chaotic vibrations are obtained, as shown in Figure 19.

5.3 Effect of Damping on Nonlinear Dynamics

In order to explain the effect of the damping on the dual-parameter vibrations of the blisk, three-dimensional dual-parameter Lyapunov exponents, two-dimensional dual-parameter Lyapunov exponents and three-dimensional dual-parameter bifurcation diagram of the blisk with two damping terms μ_1 and μ_2 are respectively provided, as shown in Figure 20. In this case, the aerodynamic force p_0 , the steady-state rotating speed Ω_0 and the perturbation rotating speed f are respectively selected as $p_0 = 50$, $\Omega_0 = 200$ and $f = 20$. Four layers in Figure 20(a) represent four groups of Lyapunov exponents for the blisk when μ_1 and μ_2 are considered as the controlling parameters. The colors in Figure 20(a) indicate that the blisk presents the complex nonlinear dynamic behaviors. The four quadrants in Figures 20(b) and 20(c) are called 1, 2, 3 and 4, respectively. It is observed that Figure 20(b) has good agreement with Figure 20(c).

When the damping terms μ_1 and μ_2 are limited to the quadrant 1, the blisk only performs the periodic vibration. Otherwise, the chaotic vibrations happen. The blisk is most likely to perform the chaotic vibrations with a small damping. This fully explains why in the quadrant 3 has the largest proportion of the red region among four quadrants. The dense points in Figure 20(c) show that the blisk performs the chaotic vibrations.

In Figure 20, five bifurcation diagrams of the blisk are obtained when the damping terms are $\mu_2 = 0.1, 0.15, 0.2, 0.3$ and 1.0 . To reveal the variation rule between the chaotic phenomena and bifurcation, Figure 21 is given. Four straight lines with the arrows are marked along the boundary of the chaotic regions in Figure 21(a). Figures 21(b)-21(d)

demonstrate that the chaotic regions moves to the left as μ_2 increases in a certain range. The nonlinear vibrations of the blisk give the regular changes: period-1 $\rightarrow$ multi-periodic $\rightarrow$ chaotic vibrations when μ_1 is selected in the interval between the red and black lines. There exist also the period-2, multi-periodic and chaotic vibrations in the region between the green and black lines in Figure 21(d). With μ_2 becomes larger in the quadrants 3 and 4, the nonlinear vibrations of the blisk become more and more diversification. The periodic vibrations occupy a wider interval μ_1 , which means that the blisk is less likely to perform the large amplitude nonlinear vibrations when μ_2 is large enough.

The influences of the damping μ_1 on the nonlinear vibrations of the blisk are presented, as shown in Figure 22. The bifurcation diagrams and Lyapunov exponents of the blisk are depicted. The damping μ_2 is $\mu_2 = 0.46$. Other parameters are as same as Figure 20. It is found that the dynamic responses of the blisk exhibit the following evolution law: chaotic $\rightarrow$ periodic vibrations, with the increase of the damping μ_1 . It is seen that the periodic vibration happens when the damping μ_1 increases to $\mu_1 = 0.44$. The damping μ_1 has significantly effect on the nonlinear dynamics of the blisk.

6. Conclusions

The dual-parameter chaotic vibrations of the lumped parameter model for the rotating blisk are studied under combined the varying rotating speed and aerodynamic force. The free vibration and mode localization phenomenon of the lumped parameter model are investigated for the tuning and detuning blisk structure. The nonlinear dynamic model of the rotating blisk subjected to the aerodynamic force is presented by using Hamilton's principle. The amplitude-frequency response curves are obtained by applying the method of multiple scales under 1:1 internal resonance and principal parametric resonance. The evolution process to the chaotic vibrations for the rotating blisk is described by the dual-parameter Lyapunov exponents and bifurcation diagrams. The influences of different parameters on the nonlinear responses of the blisk are analyzed based on the amplitude frequency response curves of the first two modes. The main conclusions are obtained.

(1) Comparing with the mode shapes of the tuning blades, the mode shapes of the mass detuning or stiffness detuning blades are not sinusoidal. There is a positive correlation between the mistuning variable and mode localization factor. The larger the

mode localization factor is, the more irregular the mode shape becomes. The large amplitude vibrations appear in a certain part and lead to the chaotic vibrations of the blisk.

(2) Some parameters seriously affect the vibration amplitudes, response bandwidths and stiffness properties of the blisk. The spring property of the blisk remains unchanged when the perturbation rotating speed, coupling stiffness and aerodynamic force are changed. The nonlinear terms determine the spring property of the blisk. If it is positive, the blisk shows the hard spring property. When it becomes negative, the blisk presents the soft spring property. The corresponding amplitude-frequency curves are perfectly symmetrical about the longitudinal axis as the values of the nonlinear term are opposite.

(3) The dual-parameter Lyapunov exponents and bifurcation diagrams of the blisk illustrate that the evolution paths leading to the chaotic vibrations and varying pattern of the chaotic regions are determined three different parameters of the blisk, including the rotating speed, aerodynamic force and damping.

(4) The phase portraits, waveforms and Poincare maps are depicted to verify the nonlinear dynamic behaviors of the blisk according to the dual-parameter Lyapunov exponents and bifurcation diagrams when one of the controlling parameters is constant.

The aforementioned conclusions indicate that chaotic vibrations are closely related to the comprehensive effect of the various factors for the blisk. Due to the small damping and approximate stiffness of the disk and blade, the large amplitude coupling vibrations of the blisk are more likely to occur, and even cause a disaster. Thus, our studies have the practical significance for the nonlinear vibration control of the blisk.

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Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Data Availability Statement

All data generated or analysed during this study are included in this published article.

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Figure Captions

- Figure 1 The lumped parameter model of the rotating blisk is given by considering the phase angle, (a) lumped parameter model of blisk, (b) rotating Cartesian coordinate system of blisk.
- Figure 2 The first fifteen modals of the blades are obtained for the tuning blisk.
- Figure 3 The first fifteen vibration modes of the blades are given with the mass mistuning, where the change of the mistuned mass for the blades is $\alpha = 2\%$.
- Figure 4 The natural frequencies of the blisk are obtained without mistuning and with mistuning conditions, (a) influence of masses without mistuning and with mistuning when $\alpha = 0$, $\alpha = 1\%$, $\alpha = 3\%$ and $\alpha = 5\%$, (b) influence of stiffness terms without mistuning and with mistuning when $\beta = 0$, $\beta = 1\%$, $\beta = 3\%$ and $\beta = 5\%$.
- Figure 5 The mode localization factors of the blisk are obtained without mistuning and with mistuning conditions, (a) influence of masses without mistuning and with mistuning when $\alpha = 0$, $\alpha = 1\%$, $\alpha = 3\%$ and $\alpha = 5\%$, (b) influence of stiffness terms without mistuning and with mistuning when $\beta = 0$, $\beta = 1\%$, $\beta = 3\%$ and $\beta = 5\%$.
- Figure 6 The amplitude-frequency response curves are obtained for the blades, (a) influence of damping, (b) influence of amplitude of perturbation rotating speed, (c) influence of coupling stiffness parameter, (d) influence of aerodynamic force, (e) influence of cubic term.
- Figure 7 The amplitude-frequency response curves are obtained for the disk, (a) influence of damping, (b) influence of perturbation rotating speed, (c) influence of coupling stiffness parameter.
- Figure 8 The dual-parameter Lyapunov exponents and bifurcation diagram versus f and p_0 are obtained, (a) 3D dual-parameter Lyapunov exponents, (b) projection of dual-parameter Lyapunov exponents diagram on plane (f, p_0) , (c) dual-parameter bifurcation diagram on space (f, p_0, x_1) .
- Figure 9 The dual-parameter Lyapunov exponents and three bifurcation diagrams with f as control parameter are given, (a) dual-parameter Lyapunov exponents on plane (f, p_0) , (b) bifurcation diagram when $f = 2$, (b) bifurcation diagram

when $f = 10$, (b) bifurcation diagram when $f = 25$.

Figure 10 3D bifurcation diagrams of the perturbation rotating speed for the blisk are given in the interval $f = 0 \sim 100$, (a) 3D bifurcation diagram in space (x_1, x_2, f) , (b) 3D bifurcation diagram in space (x_3, x_4, f) .

Figure 11 The bifurcation diagrams and Lyapunov exponents of the perturbation rotating speed for the blisk are given in interval $f = 0 \sim 100$, (a) bifurcation diagram on plane (x_1, f) ; (b) bifurcation diagram on plane (x_3, f) , (c) Lyapunov exponents.

Figure 12 The chaotic vibrations of the lumped parameter model are studied for the blisk when $f = 72$, (a) phase portrait on plane (x_1, x_2) , (b) waveform on plane (t, x_1) , (c) phase portrait on plane (x_3, x_4) ; (d) waveform on plane (t, x_3) , (e) 3D phase portrait in space (x_1, x_2, x_3) ; (f) Poincare map.

Figure 13 3D bifurcation diagrams of the external excitation for the blisk are given in the interval $p_0 = 0 \sim 100$, (a) 3D bifurcation diagram in space (x_1, x_2, p_0) , (b) 3D bifurcation diagram on space (x_3, x_4, p_0) .

Figure 14 The bifurcation diagrams and Lyapunov exponents of the external excitation are given in the interval $p_0 = 0 \sim 100$, (a) bifurcation diagram on plane (x_1, p_0) , (b) bifurcation diagram on plane (x_3, p_0) , (c) Lyapunov exponents.

Figure 15 The dual-parameter Lyapunov exponents and bifurcation diagrams of Ω_0 and p_0 are obtained, (a) dual-parameter Lyapunov exponents on space $(\Omega_0, p_0, \text{LEs})$, (b) projection of dual-parameter Lyapunov exponents on plane (Ω_0, p_0) , (c) dual-parameter bifurcation diagram on space (Ω_0, p_0, x_1) .

Figure 16 The dual-parameter Lyapunov exponents and five bifurcation diagrams are given with Ω_0 as the control parameter, (a) dual-parameter Lyapunov exponents on plane (Ω_0, p_0) , (b) bifurcation diagram on plane (p_0, x_1) when $\Omega_0 = 180$, (c) bifurcation diagram on plane (p_0, x_1) when $\Omega_0 = 190$, (d) bifurcation diagram on plane (p_0, x_1) when $\Omega_0 = 220$, (e) bifurcation diagram on plane (p_0, x_1) when $\Omega_0 = 195$, (f) bifurcation diagram on plane (p_0, x_1) when $\Omega_0 = 210$.

Figure 17 3D bifurcation diagrams of the steady-state rotating speed for the blisk are given in the interval $\Omega_0 = 160 \sim 260$, (a) 3D bifurcation diagram on space (x_1, x_2, Ω_0) , (b) 3D bifurcation diagram on space (x_3, x_4, Ω_0) .

Figure 18 The bifurcation diagrams and Lyapunov exponents of the steady-state rotating speed for the blisk are given in the interval $\Omega_0 = 160 \sim 260$, (a) bifurcation diagram on plane (x_1, Ω_0) , (b) bifurcation diagram on plane (x_3, Ω_0) , (c) Lyapunov exponents.

Figure 19 The chaotic vibrations of the lumped parameter model for the rotating blisk are studied when $\Omega_0 = 225$, (a) phase portrait on plane $(x_1, \dot{x}_1)$, (b) waveform on plane (t, x_1) , (c) phase portrait on plane $(x_3, \dot{x}_3)$, (d) waveform on plane (t, x_3) , (e) 3D phase portrait in space (x_1, x_2, x_3) , (f) Poincare map.

Figure 20 The dual-parameter Lyapunov exponents and bifurcation diagrams of μ_1 and μ_2 are given for the blisk, (a) dual-parameter Lyapunov exponents on space $(\mu_1, \mu_2, \text{LEs})$, (b) projection of dual-parameter Lyapunov exponents on plane (μ_1, μ_2) , (c) dual-parameter bifurcation diagram on space (μ_1, μ_2, x_1) .

Figure 21 The dual-parameter Lyapunov exponents and five bifurcation diagrams are obtained for the blisk with the damping μ_2 as the control parameter, (a) dual-parameter Lyapunov exponents on plane (μ_1, μ_2) , (b) bifurcation diagram on plane (μ_1, x_1) when $\mu_2 = 0.1$, (c) bifurcation diagram on plane (μ_1, x_1) when $\mu_2 = 0.15$, (d) bifurcation diagram on plane (μ_1, x_1) when $\mu_2 = 0.2$, (e) bifurcation diagram on plane (μ_1, x_1) when $\mu_2 = 0.3$, (f) bifurcation diagram on plane (μ_1, x_1) when $\mu_2 = 1.0$.

Figure 22 3D bifurcation diagrams of the damping are given for the blisk in the interval $\mu_1 = 0 \sim 1$, (a) 3D bifurcation diagram in space (x_1, x_2, μ_1) , (b) 3D bifurcation diagram in space (x_3, x_4, μ_1) .

(a)

(b)

Figure 1

Figure 2

Figure 3

Figure 4

(a)

(b)

Figure 5

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10

Figure 11

Figure 12

Figure 13

Figure 14

Figure 15

Figure 16

Figure 17

Figure 18

Figure 19

Figure 20

Figure 21

Figure 22

Table Caption

Table 1 The first 15 natural frequencies of the blades are obtained.

Table 2 The mass mistuning values Δm_b of the blades are given.

Table 1 The first 15 natural frequencies of the blades are obtained.

Mode	Frequency/Hz	Mode	Frequency/Hz	Mode	Frequency/Hz
1	302.6839	6	381.0069	11	449.3072
2	304.8771	7	396.8619	12	458.8425
3	321.2693	8	411.8959	13	466.9087
4	334.1239	9	425.7699	14	482.9309
5	348.9305	10	438.2748	15	485.7750

Table 2 Mass mistuning values Δm_b of the blades are given.

Number of blades	Δm_b	Number of blades	Δm_b	Number of blades	Δm_b
1	-0.0094	15	0.0067	29	-0.0175
2	-0.0025	16	-0.0181	30	0.0064
3	0.0296	17	-0.0058	31	-0.0112
4	-0.0172	18	0.0070	32	-0.0062
5	0.0157	19	-0.0367	33	-0.0114
6	0.0062	20	0.0207	34	-0.0205
7	-0.0047	21	0.0485	35	-0.0182
8	-0.0211	22	0.0192	36	-0.0042
9	-0.0057	23	-0.0063	37	-0.0340
10	-0.0017	24	0.0086	38	0.0122
11	-0.0294	25	-0.0207	39	-0.0024
12	0.0038	26	0.0376	40	0.0140
13	-0.0164	27	0.0188	—	—
14	-0.0019	28	0.0157	—	—