Prime counting function, first Chebyshev function and the Lambert W function

Abhishek Das (✉ abhik.das31@gmail.com)

B M Birla Science Centre

Research Article

Keywords:

Posted Date: February 10th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-891356/v1>

License: This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Prime counting function, first Chebyshev function and the Lambert W function

Abhishek Das
B. M. Birla Science Centre
Adarsh Nagar, Hyderabad
India, 500063
abhik.das31@gmail.com

Abstract

In this paper, we prove two theorems that relate the prime counting function with the first Chebyshev function and the Lambert W function and thereby a new equivalent expression for the prime counting function is derived.

1 Introduction

The special functions $\pi(x)$ (prime counting function) [1, 2, 3] and $v(x)$ (first Chebyshev function) [4, 5] are functions that are fundamental in number theory, pertinent to primes. On the other hand $W(x)$ (Lambert W function) also has widespread applications in various fields of mathematics and physics [6, 7, 8, 9]. The aim of this paper is to find relations between these functions that could provide useful insights tantamount to current literature in number theory or even broaden the spectrum of existing knowledge.

The second section deals with the proof of a lemma that concerns the upper bound of the product of primes up to the n -th prime. In the third section, **Theorem 1** is proved resorting to the aforementioned lemma and **Theorem 2** gives an alternative expression for the prime counting function with regards to the first Chebyshev function and Lambert W function.

2 An upper bound of the product of primes up to the k -th prime

Lemma 2.1. *The product of primes up to the k -th prime has the following upper bound:*

$$\prod_{n=2}^k p_n < (k!)^2$$

for $k \geq 6$.

Proof. Now, on account of the prime number theorem [10, 11, 12] and Rosser's theorem [13] we know that

$$p_n > n \ln n$$

for $n \geq 1$, where, p_n is the n -th prime number. This implies

$$j = \prod_{n=1}^k p_n > \prod_{n=1}^k n \ln n \tag{1}$$

On the other hand, it is obvious that for $k > 2$ we have the following inequality:

$$\prod_{n=2}^k n \ln n < \prod_{n=2}^k n^2$$

where, $\prod_{n=2}^k n^2 = (k!)^2$. Let us write $r = (k!)^2$ and $q = \frac{j}{2} = \prod_{n=2}^k p_n$. So, for $k > 2$, we conjecture that

$$q < r \tag{2}$$

always (which seems evident computationally, but a proof is necessary). Equation (2) is explicitly written as

$$\prod_{n=2}^k p_n < (k!)^2 \tag{3}$$

for $k > 2$. Again, we know that the n -th prime has the upper bound [14]

$$p_n \leq n \ln(n \ln n)$$

for, $n \geq 6$. Thus, we can write

$$\prod_{n=6}^k p_n \leq \prod_{n=6}^k n[\ln(n \ln n)]$$

Therefore, in order that (3) be satisfied (for $k \geq 6$) we must have

$$\prod_{n=6}^k n[\ln(n \ln n)] < (k!)^2$$

Every product term of the left hand side and the corresponding product term on the right hand side can be expressed through the following inequality:

$$(k-i)[\ln(k-i) + \ln \ln(k-i)] < (k-i) \times (k-i)$$

where, i ranges from 0 to $k-6$. So, another alternative and equivalent expression for equation (3) is given as

$$t > \ln t + \ln \ln t \tag{4}$$

for $t \geq 6$. The inequality (4) essentially implies (3). Incidentally, although there is no direct general proof of the inequality (3) at the moment, the inequality (4) can be easily proved using calculus and the *racetrack principle*. As a consequence, the lemma holds true. ■

Remark. Now, we derive another equivalent form of the inequality (3). By virtue of a lemma due to Erdős' proof of Bertrand's postulate [15], we know the following upper bound for the first Chebyshev function

$$v(k) = \ln \prod_{n=1}^k p_n \leq k \ln 4$$

Also, from (3) we have

$$\ln \prod_{n=1}^k p_n < 2 \ln(\sqrt{2}k!)$$

since, $p_1 = 2$. In order that the inequality (3) be satisfied, one must have from these two equations

$$k \ln 4 < 2 \ln(\sqrt{2}k!)$$

and, from this one can easily obtain the alternative expression for (3)

$$2^{\frac{2t-1}{2}} < t! \quad (5)$$

for $t \geq 6$. So, on account of the lemma and the fact that the inequality (5) is satisfied for $2 < t < 6$, we can conclude that (5) holds true for all $t > 2$.

3 $\pi(x)$, $v(x)$ and $W(x)$

Theorem 1. *If $v(x)$ is the first Chebyshev function, $W(x)$ is the Lambert W function and the prime counting function $\pi(x)$ is large for a sufficiently large real number x , then the following inequality holds:*

$$\pi(x) > \frac{1}{2} \frac{v(x)}{W[v(x)/2]}$$

Proof. Now, considering **Lemma 2.1** and the fact that k is essentially the number of primes up to some real number x , we can write from (3)

$$[\pi(x)!]^2 > \prod_{n=2}^k p_n$$

And, for sufficiently large values of x and that of $\pi(x)$, Stirling's approximation can be used to obtain (with $\ln \pi(x) \gg 1$)

$$\ln \pi(x)! \approx \pi(x) \ln \pi(x)$$

Using this in the previous equation we get

$$\pi(x) \ln \pi(x) > \ln \beta$$

where, $\beta = \sqrt{\prod_{n=1}^{\pi(x)} p_n} = e^{v(x)/2}$ (since, according to the definition of the first Chebyshev function, $v(x) = \ln[\prod_{n=1}^{\pi(x)} p_n]$). Now, suppose $\pi(x) \approx e^{\alpha(x)}$, where $\alpha(x)$ is some function of x . Then

$$\alpha(x) e^{\alpha(x)} > \ln \beta$$

Again, for the function $f(y) = ye^y$, we have

$$f^{-1}(y) = W(y)$$

where, $W(y) = \frac{y}{e^{W(y)}}$ is the Lambert W function. So, putting $y = \alpha(x)$ we have

$$f(y) > \ln \beta$$

$$f^{-1}(\ln \beta) = W(\ln \beta) < y$$

or,

$$\alpha(x) > W(\ln \beta) \quad (6)$$

Using the relation, $W(\ln \beta)e^{W(\ln \beta)} = \ln \beta$ and simplifying, we finally derive

$$\pi(x) > \frac{1}{2} \frac{v(x)}{W[v(x)/2]} \quad (7)$$

■

Theorem 2. *If $y = \frac{v(x)}{2}$, $v(x)$ being the first Chebyshev function, then the prime counting function can be equivalently expressed as:*

$$\pi(y) \sim \frac{W[e^y]}{\ln y} + \ln_y W[e^y]$$

where, $W[x]$ is the Lambert W function. Again, considering the real branch $W_0[x]$ of the Lambert W function for large values of x , the prime counting function can also be expressed as:

$$\pi(y) \sim \frac{y}{\ln y} + \ln_y[y - \ln y + o(1)] + \frac{o(1)}{\ln y} - 1$$

where, 'o' is small o-notation.

Proof. Let us write

$$\alpha = \frac{e^y}{W[e^y]} = e^{W[e^y]}$$

where, $y = \frac{v(x)}{2}$. From this we have

$$\alpha \ln \alpha = e^y$$

or,

$$y = \ln \alpha + \ln \ln \alpha$$

Thus, we derive the following

$$\frac{v(x)}{2} = W[e^{v(x)/2}] + \ln W[e^{v(x)/2}] \quad (8)$$

Now, using the prime number theorem for a real number $y (= \frac{v(x)}{2})$, we have

$$\pi(y) \sim \frac{y}{\ln y} = \frac{W[e^y] + \ln W[e^y]}{\ln y}$$

which finally yields

$$\pi(y) \sim \frac{W[e^y]}{\ln y} + \ln_y W[e^y] \quad (9)$$

where, the base in the second term of the right hand side has been changed from the Euler number to y . Again, if one considers the real branch W_0 of the Lambert W function for large values of x , then one has the known result

$$W_0(x) = \ln x - \ln \ln x + o(1)$$

where, $o(1)$ is the small 'o'-notation. Using this in (9), one finds that

$$\pi(y) \sim \frac{y}{\ln y} + \ln_y[y - \ln y + o(1)] + \frac{o(1)}{\ln y} - 1 \quad (10)$$

Essentially, the real branch $W_0(x)$ allows us to derive another novel expression for the prime counting function, where the term $\frac{y}{\ln y}$ is iterative. ■

4 Discussion

The author would like to mention that in the proof of **Lemma 2.1**, Rosser's theorem has been considered, but the stronger bound discovered by Dusart [16] has not been investigated. That, however, can be the aspect of future research on this particular topic.

On the other hand, there is another paper appertaining to relating $\pi(x)$ and $W(x)$ [17], that comprises results starkly different from the results of the current article.

5 Acknowledgements

The author is expressly indebted to Dr. Ken Roberts (University of Western Ontario, London, Canada) and Prof. Jeffrey O. Shallit (University of Waterloo, Canada) for some pertinent suggestions and comments.

References

- [1] S. Ramanujan, Highly composite numbers, *Proc. London Math. Soc.*, Series 2. 14: 347–409, 1915.
- [2] Bruce C. Berndt, *Ramanujan's Notebooks - Part IV*, Springer Science & Business Media, pp. 112–113, ISBN 9781461269328.
- [3] G. H. Hardy and E. M. Wright, 'The Functions $\theta(x)$ and $\psi(x)$ ' and 'Proof that $\theta(x)$ and $\psi(x)$ are of Order x ', *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, pp. 340–342, 1979.
- [4] J. B. Rosser and L. Schoenfeld, Sharper Bounds for Chebyshev Functions $\theta(x)$ and $\psi(x)$, *Math. Comput.*, 29, 243–269, 1975.
- [5] L. Schoenfeld, Sharper Bounds for Chebyshev Functions $\theta(x)$ and $\psi(x)$, II, *Math. Comput.*, 30, 337–360, 1976.
- [6] R. M. Corless et. al., On the Lambert W function, *Advances in Computational Mathematics*, 5: 329–359, 1996.
- [7] S. Valluri, D.J. Jeffrey & R.M. Corless, Some applications of the Lambert W function to physics, *Can.J.Phys.*, 78, 823, 2000.
- [8] K. Roberts and S. Valluri, Tutorial: The quantum finite square well and the Lambert W function, *Can.J.Phys.*, 95, 2017.
- [9] E. Floratos & G. Linardopoulos, Large-spin and large-winding expansions of giant magnons and single spikes, *Nuclear Physics B*, Volume 897, 2015.
- [10] P. Erdős, On a new method in elementary number theory which leads to an elementary proof of the prime number theorem, *Proc. Nat. Acad. Scis.*, 35, 374–384, 1949.
- [11] A. Selberg, An Elementary Proof of the Prime-Number Theorem, *Annals of Mathematics*, 50 (2): 305–313, 1949.

- [12] D. J. Newman, Simple Analytic Proof of the Prime Number Theorem, *The American Mathematical Monthly*, Vol. 87, No. 9, 1980.
- [13] J. B. Rosser, The n -th Prime is Greater than $n \log n$, *Proceedings of the London Mathematical Society*, 45:21-44, 1939.
- [14] B. Rosser, Explicit bounds for some functions of prime numbers, *American Journal of Mathematics*, 63 (1): 211-232, 1941.
- [15] P. Erdős, Beweis eines Satzes von Tschebyshef, *Acta Sci. Math* (Szeged) 5, 194-198, (1930-1932).
- [16] P. Dusart, The k th prime is greater than $k(\log k + \log \log k - 1)$ for $k \geq 2$, *Mathematics of Computation*, 68 (225): 411-415, 1999.
- [17] Matt Visser, Primes and the Lambert W function, *arXiv:1311.2324v2*, 2017.