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# Set-Invariance-based Interpretations for the $L_1$ Performance of Nonlinear Systems

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**Abstract** This paper is concerned with tackling the $L_1$ performance analysis problem of continuous and piecewise continuous nonlinear systems with non-unique solutions by using the involved arguments of set-invariance principles. More precisely, this paper derives a sufficient condition for the $L_1$ performance of continuous nonlinear systems in terms of the invariant set. However, because this sufficient condition intrinsically involves analytical representations of solutions of the differential equations corresponding to the nonlinear systems, this paper also establishes another sufficient condition for the $L_1$ performance by introducing the so-called extended invariance domain, in which it is not required to directly solving the nonlinear differential equations. These arguments associated with the $L_1$ performance analysis is further extended to the case of piecewise continuous nonlinear systems, and we obtain parallel results based on the set-invariance principles used for the continuous nonlinear systems. Finally, numerical examples are provided to demonstrate the effectiveness as well as the applicability of the overall results derived in this paper.

**Keywords** $L_1$ performance · set invariance · invariant set · invariance domain · external contingent cone · piecewise continuous nonlinear system

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## 1 Introduction

Based on the fact that the $L_\infty$-induced norm of linear time-invariant (LTI) systems for the single-input/ single-output (SISO) case coincides with the $L_1$ norm of the impulse response of the systems, the problem of dealing with this induced norm has been called the $L_1$ problem [1–3]. Here, it is quite difficult to compute the $L_\infty$-induced norm explicitly even for LTI systems, and thus several approximation methods have been developed in [4, 5] and [6, 7] for LTI continuous-time and sampled-data systems, respectively. To put it another way, an upper bound and a lower bound on the $L_\infty$-induced norms can be obtained in those studies within any degree of accuracy.

On the other hand, the $L_1$ analysis problem might be extended for the case of nonlinear systems, but it cannot be formulated in an essentially equivalent fashion to LTI systems. This is because it is intrinsically impossible to define an analytic representation of the $L_\infty$-induced norm for general nonlinear systems. In connection with this, the $L_1$ performance of nonlinear systems is introduced in [8] as an alternative for the $L_\infty$-induced norm. More precisely, a nonlinear system is regarded as satisfying the $L_1$ performance if the $L_\infty$ norm of the output is not larger than 1 for all input bounded by 1 with respect to the $L_\infty$ norm. In connection with this, the idea of set invariance principles used in [9] for discrete-time systems is applied to the $L_1$ analysis problem of continuous-time nonlinear systems [8].

Roughly speaking, the set invariance principles (or equivalently, forward invariance) [10] of dynamical systems imply the property that makes any state variable (or solution) $x(t)$ $(t \geq 0)$ with $x(0) \in K$ always remain in $K$ for all $t \geq 0$. If this property holds for at least one solution $x(t)$ $(t \geq 0)$ with an arbitrary $x(0) \in K$, then we call it a weak invariance or a viability property [11, 12]. Furthermore, it is shown in [12] that the weak invariance of a closed set $K$ is equivalent to the subtangential condition on $K$ [10], by which we mean that all the vector fields on $K$ are contained in the tangent cone of $K$. In contrast, the strong invariance is defined as the property that every solution $x(t)$ $(t \geq 0)$ with an arbitrary $x(0) \in K$ always stays in $K$, and this property is shown in [10] to be naturally established by assuming solution uniqueness together with the subtangential condition. Motivated by this idea to derive the strong invariance, a necessary and sufficient condition for the $L_1$ performance of nonlinear systems has been derived in the aforementioned study [8].

However, if we are in a position to employ the strong invariance in the $L_1$ analysis problem of general nonlinear systems in a similar sense with the previous study [8], the following problems can be occurred. First, the method in [8] cannot be applied to discontinuous dynamical systems [13–15] and differential inclusion systems [16, 17], which could intrinsically have non-unique solutions. Furthermore, this method makes it difficult to widely apply the controller synthesis procedure in [8] for nonlinear systems. To put it another way, the closed-loop system consisting of a nominal plant and a controller obtained through the synthesis procedure in [8] is shown to satisfy the $L_1$ performance if it has unique solutions, but the controllers introduced in that study do not guarantee the solution uniqueness with respect to the closed-loop systems.

To resolve these issues, this paper aims at establishing sophisticated arguments on the $L_1$ performance analysis problem of nonlinear systems, regardless of the solution uniqueness of the systems. More precisely, we tackle the $L_1$ performance analysis problem for continuous nonlinear systems by considering the concept of invariant set [8, 18]. This allows us to arrive at a sufficient condition for the $L_1$ performance of continuous nonlinear systems. We next introduce the involved notion so-called external contingent cone [19] (which could be regarded as an extended version of the conventional contingent cone [11, 12]) for adequately defining an extended version of invariance domain [8, 10], and we call it the extended invariance domain. Another sufficient condition for the $L_1$ performance of continuous nonlinear systems is then established by applying the invariance theorem (i.e., Theorem 5.3.1 in [19]) to the extended invariance domain.

Regarding a wider class of the $L_1$ performance analysis problem, on the other hand, this paper is also concerned with piecewise continuous nonlinear systems [13]. In contrast to the previously mentioned continuous nonlinear systems, the solutions of the systems should be intrinsically described by Filippov's differential equation [20]. This allows us to consider the solutions for those systems, without the solution uniqueness in general. In other words, this paper proposes sufficient conditions for the $L_1$ performance of piecewise continuous nonlinear systems in terms of invariant set and extended invariance domain, which are parallel to the results for continuous nonlinear systems.

The contributions of the results derived in this paper beyond the existing studies associated with the set invariance principles might be briefly summarized as follows. Even though the classical invariance theorem in [12] (i.e., Theorem 2 in pp. 202) enables us to obtain the strong invariance by using only subtangential conditions, it is confined itself to the nonlinear systems such that the vector fields satisfy Lipschitz continuity (see also [21] for details). In contrast to this method, the arguments in this paper are not limited to Lipschitz continuous vector fields and can deal with piecewise continuous nonlinear differential dynamics. Furthermore, the ideas of Lyapunov functions and barrier certificates have been recently developed in [21–24] to establish set invariance results in an equivalent fashion to those in this paper, but they always require an involved task to constructing adequate scalar candidates for the Lyapunov functions and/or barrier certificates. In a comparison with these ideas, this paper can be interpreted as giving a more intuitive and systematic method for solving the $L_1$ analysis problem of nonlinear systems since it is not required in this paper to develop such scalar candidates.

The remaining of this paper is organized as follows. The mathematical preliminaries used in this paper are given in Section 2. The problem definition associated with the $L_1$ performance analysis for continuous nonlinear systems as well as the corresponding results are given in Section 3. The parallel extension of the results for continuous nonlinear systems to piecewise continuous nonlinear systems is discussed in Section 4. Some numerical examples are provided in Section 5 to verify the effectiveness as well as the validity of the results obtained in this paper.

## 2 Mathematical Preliminaries

This section provides the mathematical preliminaries used in this paper.

We use the notations $\mathbb{R}^\nu$ and $\mathbb{N}$ to denote the sets of $\nu$-dimensional real vectors and positive integers, respectively, while the notations $\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{N}_0$ are used to imply the sets of non-negative real numbers, non-positive real numbers and $\mathbb{N} \cup \{0\}$, respectively. The $\infty$-norm of a finite-dimensional vector is denoted by $|\cdot|_\infty$, i.e.,

$$|v|_\infty := \max_i |v_i|$$

Equipped with this notation, we use for $r > 0$ with $x \in \mathbb{R}^n$ the notations $B_r(x)$ and $\overline{B}_r(x)$ to denote the open and closed balls defined respectively as

$$B_r(x) = \{v \in \mathbb{R}^n \mid |v - x|_\infty < r\}$$
$$\overline{B}_r(x) = \{v \in \mathbb{R}^n \mid |v - x|_\infty \leq r\}$$

For the notational simplicity, the notation $\mathbb{BR}^n$ is taken to mean the closed unit ball $\overline{B}_1(0)$ with $0 \in \mathbb{R}^n$. The space of $\nu$-dimensional Lebesgue measurable functions is denoted by $L^\nu_\infty$, such that the corresponding $L_\infty$ norm is well-defined and bounded, i.e.,

$$\|f(\cdot)\|_\infty := \operatorname*{ess\,sup}_{t\geq 0} |f(t)|_\infty < \infty$$

We also denote the space of functions $f \in L^\nu_\infty$ with $\|f\|_\infty \leq 1$ by $W_\nu$, for the notational simplicity, i.e.,

$$W_\nu := \{f \in L^\nu_\infty \mid \|f\|_\infty \leq 1\}$$

For a nonempty subset $A$ of $\mathbb{R}^n$, the notations $\mathrm{Int}(A)$, $\overline{A}$ and $\partial A$ denote the sets of interior points, closure points and boundary points of $A$, respectively. Subsequently, for two nonempty subsets $A, B$ of $\mathbb{R}^n$, the Minkowski sum of $A, B$ and scalar multiplication of $A$ are denoted respectively by $A \oplus B$ and $cA$, i.e.,

$$A \oplus B := \{a + b \in \mathbb{R}^n \mid \forall a \in A,\ \forall b \in B\}$$

$$cA = \{ca \in \mathbb{R}^n \mid \forall a \in A\}$$

If $F : X \rightsquigarrow Y$ is a set-valued function with $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, $F$ is called *upper-semicontinuous* at $x_0 \in X$ [11, 12] if for any open set $O$ in $Y$ containing $F(x_0)$ the following relation holds:

$$\exists V \subseteq X \text{ such that } x_0 \in V \text{ with } F(V) \subseteq O$$

This set-valued function $F$ is also called *Lipschitz continuous* at $x_0 \in X$ [11,12] if there exists a neighborhood $O$ of $x_0$ and a constant $\lambda > 0$ such that

$$F(x_1) \subseteq F(x_2) + \lambda |x_1 - x_2|_\infty \mathbb{BR}^m, \quad \forall x_1, x_2 \in O$$

This definition is applied to a single valued function in an equivalent fashion.

Let $K$ be a nonempty subset of the normed vector space $(\mathbb{R}^n, |\cdot|_\infty)$. Then, the *contingent cone* $T_K(x)$ of $K$ for some $x \in \overline{K}$ is defined as [11, 19]

$$T_K(x) := \{v \in X \mid \liminf_{h\to 0^+} \frac{d_K(x + hv)}{h} = 0\}$$

with the distance function defined as $d_K(x) := \inf_{y\in K} |x - y|_\infty$. It should be remarked that $T_K(x)$ can be defined only on $x \in \overline{K}$. In contrast, the *external contingent cone* $\tilde{T}_K(x)$ [19] is defined for any point of $x \in \mathbb{R}^n$ as

$$\tilde{T}_K(x) := \{v \in X \mid \liminf_{h\to 0^+} \frac{d_K(x + hv) - d_K(x)}{h} \leq 0\}$$

To put it another way, $\tilde{T}_K(x)$ is obviously allows us to take $x$ even in the outside of $\overline{K}$ and is an extended version of $T_K(x)$ in the sense that $\tilde{T}_K(x) = T_K(x)$ for all $x \in \overline{K}$. Therefore, we take the notation $T_K(x)$ to denote the external contingent cone when it is not required to take $x$ in the outside of $\overline{K}$ for the notational simplicity, which will be clear from the context.

## 3 The $L_1$ Performance Analysis for Continuous Nonlinear Systems

Let us consider the following continuous nonlinear system

$$\Sigma_c : \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))w(t) \\ z(t) = h(x(t)) + k(x(t))w(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^p$ is the exogenous input and $z(t) \in \mathbb{R}^m$ is the output. We further assume that $w \in W_p$ and the coefficient functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, $k(\cdot)$ are continuous on $\mathbb{R}^n$.

For a given $w \in W_p$, a solution of the differential equation $\dot{x}(t) = f(x(t)) + g(x(t))w(t)$ in (1) can be described by using Caratheodory's arguments [25]. More precisely, $x$ is called a solution of the differential equation on $[0, \delta)$ with the initial condition $x(0) = x_0$ when $x(t)$ is a absolutely continuous function in $t$ such that the equality

$$x(t) = x_0 + \int_0^t f(x(\tau)) + g(x(\tau))w(\tau)\ d\tau \tag{2}$$

holds for almost everywhere $t \in [0, \delta)$.

Furthermore, such a solution $x(t)$ uniquely exists, if both $f(x)$ and $g(x)$ are assumed to be (locally) Lipschitz continuous in $x$. Otherwise, $x(t)$ might not be uniquely determined. With this in mind, we can define the following $L_1$ performance of $\Sigma_c$ as an *extended* notion of that in [8].

**Definition 1 ($L_1$ performance of $\Sigma_c$)** If *any* solution $x(t)$ of the differential equation in (1) with the initial condition $x(0) = 0$ for an arbitrary $w \in W_p$ can be actually defined on the interval $[0, \infty)$ and every corresponding output $z(t)$ satisfies $\|z\|_\infty \leq 1$, then the continuous nonlinear system $\Sigma_c$ is called to satisfy the $L_1$ performance.

*Remark 1* It should be remarked that the $L_1$ performance is not essentially confined itself to the specific case with respect to the unit magnitude of both $w$ and $z$ (i.e., $\|w\|_\infty \leq 1$ and $\|z\|_\infty \leq 1$), but could be readily extended to the general case such that $\|z\|_\infty \leq \beta$ is considered for all $\|w\|_\infty < \alpha$ for some $\alpha, \beta > 0$ by replacing $g(x)$, $h(x)$ and $k(x)$ with $\alpha \cdot g(x)$, $h(x)/\beta$ and $(\alpha/\beta) \cdot k(x)$, respectively.

*Remark 2* If we are interested in single-input/single-output (SISO) linear time-invariant (LTI) systems as a specific case of nonlinear systems (i.e., $f(x) \equiv A$, $g(x) \equiv B$, $h(x) \equiv C$ and $k(x) \equiv D$ for some constant matrices $A$, $B$, $C$ and $D$), the $L_1$ performance considered in Definition 1 is essentially the same as that the $L_1$ norm of the impulse response is less than or equal to 1.

In a comparison with the previous study [8], the $L_1$ performance in this definition is described without assuming solution uniqueness, and thus it could be interpreted as an extended version of the $L_1$ performance in that study.

On the other hand, one might naturally regard that direct applications of the $L_1$ performance to general nonlinear systems are readily possible, but it is often quite difficult to solve the nonlinear differential equation in (1). In this sense, it could be also quite meaningful to formulate the analysis problem with respect to the $L_1$ performance in Definition 1 independent of the analytic solutions of the differential equations in (1), and we consider the following problem statement.

**Problem 1 ($L_1$ performance analysis for $\Sigma_c$)** Characterize the $L_1$ performance for $\Sigma_c$ without directly solving the differential equation in (1).

In connection with providing a solution procedure to Problem 1 in terms of set invariance principles [11, 19], we first introduce the definition of invariant set, which plays an important role in establishing a sufficient condition for the $L_1$ performance of $\Sigma_c$.

**Definition 2 (Invariant set of $\Sigma_c$)** Suppose that $K$ is a nonempty subset of $\mathbb{R}^n$. Then, $K$ is called an invariant set of $\Sigma_c$ if every solution $x(t)$ of the differential equation in (1) with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$ satisfies $x(t) \in K$ for all $t \in [0, \delta)$ on which the solution can be defined.

With the employment of invariant set for tackling Problem 1 in mind, we are led to the following lemma, which is associated with the solution existence of the differential equation in (1).

**Lemma 1 (Existence of solutions for $\Sigma_c$)** *Let $K$ be a compact subset of $\mathbb{R}^n$. Then, there exists a positive constant $c$ such that the differential equation in (1) has a solution on the interval $[0, c)$ with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$.*

*Proof* For an arbitrary $a > 0$, the compactness assumption on $K$ together with the continuity property of $f$ and $g$ implies that there exist positive constants $M_f$ and $M_g$ such that

$$M_f = \sup_{x \in K \oplus a\mathbb{B}\mathbb{R}^n} |f(x)|_\infty, \quad M_g = \sup_{x \in K \oplus a\mathbb{B}\mathbb{R}^n} |g(x)|_\infty \tag{3}$$

Then, it immediately follows from (3) that

$$\begin{aligned} &\sup_{x \in K \oplus a\mathbb{B}\mathbb{R}^n} |f(x) + g(x)w(t)|_\infty \\ &\leq M_f + M_g \cdot |w(t)|_\infty =: \varphi(t) \end{aligned} \tag{4}$$

Because $\varphi(t)$ is an integrable function on any finite interval, there exists a positive constant $c$ such that

$$\int_0^c \varphi(t) dt \leq a \tag{5}$$

and such a $c$ can be chosen by

$$c = \frac{a}{M_f + M_g} \tag{6}$$

Then, there exists a solution $x(t)$ of the differential equation in (1) on the interval $[0, c)$ by the solution existence theorem in [20]. This together with the fact that $c$ is independent of the choice of the initial condition $x(0) \in K$ completes the proof.

It would be worthwhile to note that the constant $c$ in Lemma 1 is independent of the choice of the initial time $t_0$, and this fact allows us to arrive at the following result, which is relevant to forward completeness of solutions of the differential equation in (1).

**Proposition 1 (Forward completeness of $\Sigma_c$)** *If $K$ is a compact invariant set of $\Sigma_c$, then every solution $x(t)$ of the differential equation in (1) with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$ can be actually defined on the interval $[0, \infty)$ and satisfies $x(t) \in K$ for all $t \geq 0$.*

*Proof* From Lemma 1, there exists a constant $c$ such that solutions of the differential equation in (1) can be defined on $[0, c)$ with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$. Hence, it is obvious from Definition 2 together with the compactness assumption on $K$ that there exists a solution $\tilde{x}_0(t)$ of the differential equation such that $\tilde{x}_0(t) \in K,\ \forall t \in [0, c)$ and the limit $\tilde{x}_0(c) := \lim_{t\to c-} x(t)$ is also contained in $K$.

By essentially the same arguments, there exist solutions $\tilde{x}_1(t)$ of the differential equation such that $\tilde{x}_1(0) = \tilde{x}_0(c)$ and $\tilde{x}_1(t) \in K,\ \forall t \in [0, c)$ as well as $\tilde{x}_1(c) := \lim_{t\to c-} \tilde{x}_1(t) \in K$. Then, repeating this procedure allows us to obtain the function sequences $\tilde{x}_i(t),\ (i \in \mathbb{N}_0)$ such that $\tilde{x}_i(t) \in K,\ \tilde{x}_i(c) = \tilde{x}_{i+1}(0),\ \forall t \in [0, c),\ \forall i \in \mathbb{N}_0$. With these functions, if we define $x(ic + t) := \tilde{x}_i(t),\ (i \in \mathbb{N}_0)$, then $x(t)$ is obviously a solution of the differential equation on $[0, \infty)$ and $x(t) \in K,\ \forall t \geq 0$. This completes the proof.

*Remark 3* If the compactness property is not assumed on $K$ in Proposition 1, there could exist a case such that the assertion in that lemma does not hold. For example, the solution of the differential equation $\frac{dx(t)}{dt} = x^2(t),\ x(0) > 0$ tends to $\infty$ in a finite time.

Based on this proposition, we are led to the following result.

**Theorem 1** *Let us define*

$$\Omega := \{x \in \mathbb{R}^n \mid |h(x) + k(x)w|_\infty \leq 1, \forall w \in \mathbb{BR}^p\} \quad (7)$$

*Then, $\Sigma_c$ satisfies the $L_1$ performance if there exists a compact invariant set $K$ such that*

$$0 \in K \subseteq \Omega \quad (8)$$

*Proof* Since $K(\ni 0)$ is a compact invariant set of $\Sigma_c$, it immediately follows from Proposition 1 that any solution $x(t)$ of the differential equation in (1) with the initial condition $x(0) = 0$ can be actually defined on $[0, \infty)$ and remains in $K$ for all $t \in [0, \infty)$. This together with (8) obviously implies that

$$|z(t)|_\infty = |h(x(t)) + k(x(t))w(t)|_\infty \leq 1,\ \forall t \in [0, \infty) \quad (9)$$

This completes the proof.

*Remark 4* One might argue that the assertions of Theorem 1 are somewhat conservative in a comparison with Proposition 2.6 in [8], in which a necessary and sufficient condition for the $L_1$ performance analysis of $\Sigma_c$ is derived while Theorem 1 only provides the sufficient condition. However, it is intrinsically impossible to establish an exact necessity direction of Theorem 1; the converse of Theorem 1 is not true in general due to the absence of solution uniqueness.

Theorem 1 provides us the sufficient condition for the $L_1$ performance analysis of $\Sigma_c$, but it might be usually difficult to employ this sufficient condition in practical nonlinear systems because an analytical representation of the solutions is required to deal with invariant sets. To resolve this difficulty, we also introduce the following extended notion of invariance domain discussed in [8], by which the $L_1$ performance analysis of $\Sigma_c$ is possible even for the case of the absence of solution uniqueness without directly solving the differential equation.

**Definition 3 (Extended invariance domain of $\Sigma_c$)** Let $K$ be a closed subset of $\mathbb{R}^n$. Then, the set $K$ is called an extended invariance domain of $\Sigma_c$ if there exists an open neighborhood $O$ of $K$ such that

$$f(x) + g(x)w \in \tilde{T}_K(x), \quad \forall w \in \mathbb{BR}^p, \quad \forall x \in O \quad (10)$$

*Remark 5* It should be remarked that $\tilde{T}_K(x)$ used in (10) is an external contingent cone introduced in Section 2, and the employment of this notion allows us to take the dynamic behavior of the differential equation in (1) at the outside of $K$.

In connection with the relation between the invariant set and the extended invariance domain, we next introduce the following lemma by slightly modifying Invariance Theorem in [19] (but without effecting the essentials).

**Lemma 2** *Suppose that $K$ is a nonempty subset of $\mathbb{R}^n$ and $\alpha(t, x)$ is a function defined on $\mathbb{R}_+ \times \mathbb{R}^n$. If $\alpha(t, x)$ is continuous in $x \in \mathbb{R}^n$ and $O$ is an open neighborhood of $K$ such that*

$$\alpha(t, x) \in \tilde{T}_K(x), \quad \forall t \geq 0, \quad \forall x \in O \quad (11)$$

*then any solution $x(t)$ of $\dot{x}(t) = \alpha(t, x(t))$ defined on $[0, \delta)$ with an arbitrary initial condition $x(0) \in K$ satisfies $x(t) \in K,\ \forall t \in [0, \delta)$.*

Based on Lemma 2, we can derive the following result.

**Proposition 2** *If $K$ is an extended invariance domain of $\Sigma_c$, then $K$ is also an invariant set of $\Sigma_c$.*

*Proof* For an arbitrary fixed $w \in W_p$, define $\alpha(t, x)$ as

$$\alpha(t, x) := f(x) + g(x)w(t) \quad (12)$$

Then, it readily follows from (10) that the relation (11) holds since $w(t)$ in (12) can be in fact replaced by $w \in \mathbb{BR}^p$ for each fixed time $t \in \mathbb{R}_+$. By Lemma 2, every solution $x(t)$ of $\dot{x}(t) = \alpha(t, x(t))$ with an arbitrary initial condition $x(0) \in K$ stays in $K$ for all $t$ in $[0, \delta)$ on which the solution can be defined. From this together with Definition 2, $K$ is obviously an invariant set of $\Sigma_c$.

*Remark 6* It should be noted that the converse of Proposition 2 does not hold in general. This is because we are also concerned with the dynamic behavior of $\Sigma_{\rm c}$ at the outside of $O$ (i.e., points in $O - K$). In contrast, the converse is shown in the existing study [8] to be correct under the assumption of solution uniqueness. Thus, it is essential to obtain only sufficient conditions rather than necessary and sufficient conditions for the $L_1$ performance analysis of $\Sigma_{\rm c}$ in this paper for dealing with non-unique solutions.

Finally, we can obtain from Theorem 1 and Proposition 2 the following theorem, which is the main result of this section.

**Theorem 2** *The continuous nonlinear system $\Sigma_{\rm c}$ satisfies the $L_1$ performance if there exists a compact extended invariance domain $K$ such that $0 \in K \subseteq \Omega$, where $\Omega$ is defined as (7).*

This theorem clearly gives us a solution to Problem 1, i.e., a sufficient condition for the $L_1$ performance of $\Sigma_{\rm c}$ without directly solving the differential equation in (1). Furthermore, Theorem 2 can be also interpreted as an extended version of Theorem 2.12 in [8].

On the other hand, it would be worthwhile to remark that Lemma 2 can be extensively used in accordance with the treatment of differential inclusions [19]. With this observation in mind, let us discuss an extension of the results of the present section to piecewise continuous nonlinear systems in the following section.

## 4 Extension to Piecewise Continuous Nonlinear Systems

Let us consider the following piecewise continuous nonlinear system

$$\Sigma_{\rm pc} : \begin{cases} \dot{x}(t) = f_{\rm pc}(x(t)) + g_{\rm pc}(x(t))w(t) \\ z(t) = h_{\rm pc}(x(t)) + k_{\rm pc}(x(t))w(t) \end{cases} \tag{13}$$

where $x(t)$, $w(t)$ and $z(t)$ are the same variables as those in (1) and the coefficient functions $f_{\rm pc}(\cdot)$, $g_{\rm pc}(\cdot)$, $h_{\rm pc}(\cdot)$ and $k_{\rm pc}(\cdot)$ are piecewise continuous and locally bounded functions. It is further assumed that there exist open subsets $D_1, D_2, \ldots, D_N$ of $\mathbb{R}^n$ such that

(a) $\cup_{i=1}^{N} \overline{D_i} = \mathbb{R}^n$.
(b) $D_i \cap D_j = \emptyset$ for all $i \neq j$.
(c) $M := \mathbb{R}^n - \cup_{i=1}^{N} D_i$ has Lebesgue measure zero.
(d) Each $D_i$ becomes domain of continuity for all the coefficient functions $f_{\rm pc}(\cdot)$, $g_{\rm pc}(\cdot)$, $h_{\rm pc}(\cdot)$ and $k_{\rm pc}(\cdot)$.

In contrast to the continuous nonlinear system $\Sigma_{\rm c}$ discussed in the preceding section, the Caratheodory's arguments [25] could not establish the existence of solutions for the differential equation $\dot{x}(t) = f_{\rm pc}(x(t)) + g_{\rm pc}(x(t))w(t)$ in (13) due to the discontinuity of the coefficient functions. To resolve this difficulty, we describe the solutions of the differential equation based on Filippov's arguments [20]. More precisely, a function $x(t)$ is defined as a Filippov's solution of the differential equation of $\Sigma_{\rm pc}$ on $[0, \delta)$ if $x(t)$ is absolutely continuous in $t$ and satisfies the differential inclusion

$$\dot{x}(t) \in F_w(t, x(t)) \tag{14}$$

for almost everywhere $t \in [0, \delta)$, with the set-valued function $F_w(t, x)$ given by

$$\begin{aligned} &F_w(t,x) \\ &= \begin{cases} \{f(x) + g(x)w(t)\}, & \text{if } x \in C \\ \overline{co}\big\{ \lim\limits_{x^* \to x} f(x^*) + g(x^*)w(t) \ \big|\ x^* \in C \big\}, & \text{otherwise} \end{cases} \end{aligned} \tag{15}$$

where $C$ and $\overline{co}(\cdot)$ mean the set of all continuous points of the functions $f(\cdot)$ and $g(\cdot)$ and the closed convex hull of $(\cdot)$, respectively.

On the other hand, the vector field $F_w(t, x)$ is upper-semicontinuous with respect to $x$ but not Lipschitz continuous in general. This fact together with the arguments relevant to solution existence theorem in [20] establishes the local existence of solutions of the differential equation in (13), but the solution uniqueness does not often hold. With this property on the Filippov's arguments-based solution existence in mind, we can naturally extend the $L_1$ performance as well as the corresponding analysis problem defined for the continuous nonlinear system $\Sigma_{\rm c}$ (i.e., Definition 1 and Problem 1) to the piecewise continuous nonlinear system $\Sigma_{\rm pc}$ as follows.

**Definition 4** **($L_1$ performance of $\Sigma_{\rm pc}$)** If any Filippov's solution $x(t)$ of the differential equation in (13) with the initial condition $x(0) = 0$ for an arbitrary $w \in W_p$ can be actually defined on $[0, \infty)$ and every corresponding output $z(t)$ satisfies $\|z\|_\infty \leq 1$, then the piecewise continuous system $\Sigma_{\rm pc}$ is said to satisfy the $L_1$ performance.

**Problem 2** **($L_1$ performance analysis for $\Sigma_{\rm pc}$)** Characterize the $L_1$ performance without directly solving the Filippov's differential equation in (13).

It should be noted that taking Filippov's solution makes the above definition and problem statement intrinsically different to those for the continuous nonlinear system $\Sigma_{\rm c}$, in which the associated solutions are described in terms of Caratheodory's arguments [25]. Thus, it is required to slightly modify the definition of the aforementioned invariant set according to the treatment of Filippov's solution as follows.

**Definition 5 (Invariant set for $\Sigma_{\rm pc}$)** Let $K$ be a nonempty subset of $\mathbb{R}^n$. Then, $K$ is said to be an invariant set of $\Sigma_{\rm pc}$ if every Filippov's solution $x(t)$ of the differential equation in (13) with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$ satisfies $x(t) \in K$ for all $t \in [0, \delta)$ on which the solution $x(t)$ can be defined.

Then, the arguments on existence of solutions and forward completeness of the continuous nonlinear system $\Sigma_{\rm c}$ (i.e., Lemma 1 and Proposition 1) can be naturally established for the piecewise continuous nonlinear system $\Sigma_{\rm pc}$ with the considerations of Filippov's solutions. To put it another way, we could arrive for the piecewise continuous nonlinear system $\Sigma_{\rm pc}$ at the following results.

**Lemma 3 (Existence of solutions for $\Sigma_{\rm pc}$)** *Let $K$ be a compact subset of $\mathbb{R}^n$. Then, there exists a positive constant $c$ such that the differential equation in (13) has a Filippov's solution on the interval $[0, c)$ with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$.*

*Proof* For an arbitrary fixed $a > 0$ and $w \in W_p$, let us take $f(x) + g(x)w(t)$ as a (trivial) single-valued selection of $F_w(t, x)$. Then, it immediately follows from the locally boundedness of $f(\cdot)$ and $g(\cdot)$ together with the compactness of $K$ that this selection is bounded by a function $\varphi(t)$ defined as

$$\varphi(t) := M_f + M_g w(t) \tag{16}$$

where

$$M_f = \sup_{x \in K \oplus a\mathbb{B}\mathbb{R}^n} |f(x)|_\infty, \quad M_g = \sup_{x \in K \oplus a\mathbb{B}\mathbb{R}^n} |g(x)|_\infty \tag{17}$$

If we note the upper-semicontinuity of $F_w(t, x)$, the remaining task is essentially the same as the proof of Lemma 1.

**Proposition 3 (Forward completeness of $\Sigma_{\rm pc}$)** *If $K$ is a compact invariant set of $\Sigma_{\rm pc}$, then every Filippov's solution $x(t)$ of the differential equation in (13) with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$ can be actually defined on the interval $[0, \infty)$ and satisfies $x(t) \in K$ for all $t \geq 0$.*

Even though we provide the details for the proof of Lemma 3 since taking a single-valued selection of $F_w(t, x)$ is intrinsically different to the case of continuous nonlinear systems, we omit the proof of Proposition 3 because it is essentially equivalent to that of Proposition 1. From Proposition 3, we are led to the following theorem, which is parallel to Theorem 1.

**Theorem 3** *The system $\Sigma_{\rm pc}$ satisfies the $L_1$ performance if there exists a compact invariant set $K$ such that $0 \in K \subseteq \Omega$, where $\Omega$ is defined as (7).*

Equivalently to Theorem 1, the converse of Theorem 3 cannot be established. Furthermore, the proof of Theorem 3 is essentially the same as that of Theorem 1, and thus we omit the proof.

With an indirect solution procedure to Problem 2 in mind, we are next concerned with an extended invariance domain of $\Sigma_{\rm pc}$ as a modification of Definition 3. To this end, let us consider the time-independent vector field defined as

$$F_w(x) = \overline{co}\{\lim_{x^* \to x} f(x^*) + g(x^*)w \mid x^* \in C\} \tag{18}$$

for any fixed $w \in \mathbb{B}\mathbb{R}^p$. Then, we can obtain the following involved definition.

**Definition 6 (Extended invariance domain of $\Sigma_{\rm pc}$)** Let $K$ be a closed subset of $\mathbb{R}^n$. Then, the set $K$ is defined as an extended invariance domain of $\Sigma_{\rm pc}$ if there exists a neighborhood $O$ of $K$ such that

$$F_w(x) \subseteq \tilde{T}_K(x), \quad \forall w \in \mathbb{B}\mathbb{R}^p, \quad \forall x \in O \tag{19}$$

Note that the set-valued vector field $F_w(x)$ is employed for describing the extended invariance domain of the piecewise continuous nonlinear system $\Sigma_{\rm pc}$, in contrast to Definition 3. Regarding a relation between invariant set and extended invariance domain with respect to $\Sigma_{\rm pc}$, we also provide the following lemma by modifying Invariance Theorem for differential inclusions [19] without affecting the essentials.

**Lemma 4** *Suppose that $K$ is a nonempty subset of $\mathbb{R}^n$ and $F(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a locally bounded set-valued function with $K \subseteq dom(F(t, \cdot))$. If $O$ is an open (relative to $dom(F(t, \cdot))$) neighborhood of $K$ such that*

$$F(t, x) \subseteq \tilde{T}_K(x), \quad \forall t \geq 0, \quad \forall x \in O \tag{20}$$

*then any solution $x(t)$ of $\dot{x}(t) \in F(t, x(t))$ defined on $[0, \delta)$ with arbitrary initial condition $x(0) \in K$ satisfies $x(t) \in K$ for all $t \in [0, \delta)$.*

By Lemma 4, we can derive the following result.

**Proposition 4** *If $K$ is an extended invariance domain of $\Sigma_{\rm pc}$, then the set $K$ is an invariant set of $\Sigma_{\rm pc}$.*

*Proof* For an arbitrary fixed $w \in W_p$, we define the set-valued function $F(t,x)$ as (15). Then, it immediately follows from Definition 6 and the essentially equivalent procedure for obtaining Proposition 2 that $K$ is an invariant set of $\Sigma_{\rm pc}$.

Similar to the case of continuous nonlinear systems as discussed in Remark 6, the converse of Proposition 4 does not hold in general due to the consideration of dynamic behavior at outside points $O - K$. Finally, we could obtain the following theorem from Theorem 3 and Proposition 4.

**Theorem 4** *The piecewise continuous nonlinear system $\Sigma_{\rm pc}$ satisfies the $L_1$ performance if there exists a compact extended invariance domain $K$ such that $0 \in K \subseteq \Omega$, where $\Omega$ is defined as (7).*

This theorem corresponds to a solution of Problem 2, i.e., a sufficient condition for the $L_1$ performance of $\Sigma_{\rm pc}$ without directly dealing with the differential inclusion relevant to (13). Theorem 4 could be also interpreted as providing an extension of Theorem 2.12 in [8] to the piecewise continuous system $\Sigma_{\rm pc}$.

## 5 Numerical Examples

This section aims at examining the effectiveness of Theorems 2 and 4 through some numerical examples associated with continuous nonlinear and piecewise continuous nonlinear systems, respectively.

### 5.1 Case of Continuous Nonlinear Systems with Non-unique Solutions

Let us consider the continuous nonlinear system

$$\Sigma_c : \begin{cases} \dot{x}(t) = f(x) + w(t) \\ z(t) = \sqrt[3]{x(t)} \end{cases} \tag{21}$$

where the coefficient function $f(x)$ is defined as

$$f(x) = \begin{cases} 1 & \text{if } \ x < -1 \\ 1 + \sqrt{x+1} & \text{if } \ -1 \le x < 0 \\ 2 - 3x^2 & \text{if } \ 0 \le x \end{cases} \tag{22}$$

We first obtain $\Omega$ (defined as (7)) by

$$\Omega = \{x \in \mathbb{R} \mid z = |\sqrt[3]{x}| \le 1\} = [-1, 1] \tag{23}$$

In connection with the application of Theorem 2 to this system, it is required to establish that there exists an corresponding extended invariance domain $K$ such that $0 \in K \subseteq \Omega$. In this regard, the remaining part of this subsection is devoted to showing that such a set $K$ can be taken by $K = [-1, 1]$.

Let us first note that $K$ is an extended invariance domain, if there exists $\epsilon > 0$ such that

$$f(x) + w \in \tilde{T}_K(x), \quad \forall w \in \mathbb{BR}^1 \tag{24}$$

is satisfied for arbitrary $x \in (-1-\epsilon, 1+\epsilon)$. Here, because the external contingent cone $\tilde{T}_K(x)$ is given by

$$\tilde{T}_K(x) = \begin{cases} \mathbb{R}_+ & \text{if } \ -1-\epsilon \le x \le -1 \\ \mathbb{R} & \text{if } \ -1 < x < 1 \\ \mathbb{R}_- & \text{if } \ 1 \le x \le 1+\epsilon \end{cases} \tag{25}$$

it immediately follows from (22) together with (25) that the inclusion of (24) is satisfied for all $x \in (-1-\epsilon, 1+\epsilon)$. Thus, it is obvious from Theorem 2 that the continuous nonlinear system given by (21) and (22) satisfies the $L_1$ performance.

On the other hand, it should be noted that this continuous nonlinear system is one of the examples to which the existing results of the $L_1$ performance analysis in [8] cannot be applied. This is because the differential equation of (21) could have non-unique solutions in accordance with the choice of $w(t)$. For example, if we take $w(t) \equiv -1$, then the differential equation of (21) has non-unique solutions described by

$$x(t; t_0) = \begin{cases} -1 & \text{if } \ 0 \le t \le t_0 \\ -1 + \frac{(t-t_0)^2}{4} & \text{if } \ t > t_0 \end{cases} \tag{26}$$

for arbitrary $t_0 > 0$ with the initial condition $x(0) = -1$. To put it another way, $x(t; t_0)$ given by (26) becomes solutions of (21) for any $t_0 > 0$.

Such a non-uniqueness of solutions as in (26) could be interpreted as arising from the fact that the vector field $f(x)$ of (22) does not satisfy Lipschitz continuity at $x = -1$. This situation is quite crucial for a controller synthesis because the closed-loop system obtained through a feedback connection between a nominal system and a controller cannot satisfy Lipschitz property even if the vector field of the nominal system is Lipschitz continuous as discussed in Theorem 3.9 in [8] and Definition 2.3 in [10].

In connection with this, if the continuous nonlinear system described by (21) is interpreted as obtained by connecting the $L_1$ controller $u(x)$ (which can be computed by using the arguments in [8] and is given by)

$$u(x) = \begin{cases} 2 & \text{if } \ x < -1 \\ 2 + \sqrt{x+1} & \text{if } \ -1 \le x < 0 \\ 3 - 3x^2 & \text{if } \ 0 \le x \end{cases} \tag{27}$$

to the nominal plant with the following Lipschitz continuous vector field

$$\begin{cases} \dot{x}(t) = -1 + w(t) + u(t) \\ z(t) = \sqrt[3]{x(t)} \end{cases} \tag{28}$$

then it is still unclear whether or not the closed-loop system satisfies the $L_1$ performance when we are confined ourselves to the existing arguments in [8]. Hence, this observation undoubtedly implies that the employment of extended invariance domain in this paper allows us to significantly improve the existing results in [8] in the sense that the acceptable class of the $L_1$ admissible controller would be widened.

### 5.2 Case of Piecewise Continuous Nonlinear Systems

Let us consider the helicopter system with a frictional force as shown in Fig. 1. Assume that the dynamics of this system is given by [26]

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= \frac{1}{I_{yy}}\big( - mL_x g\cos(\theta) - mL_z g\sin(\theta) \\ &\qquad - F_{km} sgn(\frac{d\theta}{dt}) - F_{vm}\frac{d\theta}{dt} + u(t)\big) + w(t) \\ z &= \frac{1}{4}\max\{|\theta|, |\frac{d\theta}{dt}|\} \end{aligned} \tag{29}$$

where, $I_{yy}$ is the moment of the inertia, $-mL_x g\cos(\theta)$ and $-mL_z g\sin(\theta)$ are the torques generated from the gravitational force with respect to **x**-axis and **z**-axis, respectively, $-F_{km}sgn(\frac{d\theta}{dt})$ is the frictional force, $-F_{vm}\frac{d\theta}{dt}$ is the damping force and $sgn(\cdot)$ denotes the signum function defined as

$$sgn(x) := \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \tag{30}$$

Furthermore, $w \in W_1$ is the external disturbance and $u(t)$ is the control torque input.

In the following, we take $I_{yy} = 1$, $F_{km} = 1$, $F_{vm} = 5$, $g = 10$, $m = 0.1$ and $L_x = L_z = 10$ for the simplicity of the arguments. If we consider the control torque input $u(t)$ given by

$$u(t) = 10\cos(\theta) + 3sgn(\frac{d\theta}{dt}) \tag{31}$$

then the closed-loop system obtained by connecting (31) with (29) can be described by

$$\Sigma_{\mathrm{pc}} : \begin{cases} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = f(x(t), y(t)) + g(x(t), y(t))w(t) \\ z(t) = \frac{1}{4}\max\{|x(t)|, |y(t)|\} \end{cases} \tag{32}$$

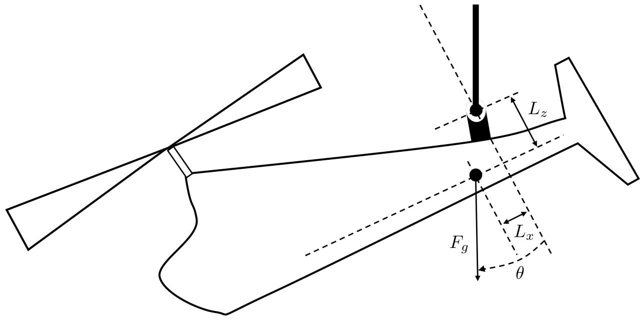


Fig. 1: Helicopter system with a frictional force.

where

$$f(x, y) = \begin{pmatrix} y \\ -10\sin x - 5y + 2sgn(y) \end{pmatrix}, \; g(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{33}$$

For this piecewise continuous nonlinear system $\Sigma_{\mathrm{pc}}$, we first note that $\Omega$ (defined as (7)) is given by

$$\Omega = [-4, 4] \times [-4, 4] \subseteq \mathbb{R}^2 \tag{34}$$

We next show that the set

$$\begin{aligned} K = &\{(x, y) \in \mathbb{R}^2 \mid x \in [-2, -1], y \in [-2x - 2, 2x + 6]\} \\ &\cup \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y \in [-2x - 2, -2x + 2]\} \\ &\cup \{(x, y) \in \mathbb{R}^2 \mid x \in [1, 2], y \in [2x - 6, -2x + 2]\} \end{aligned} \tag{35}$$

becomes an extended invariance domain of the piecewise continuous nonlinear system $\Sigma_{\mathrm{pc}}$ described by (32) such that $K \subseteq \Omega$.

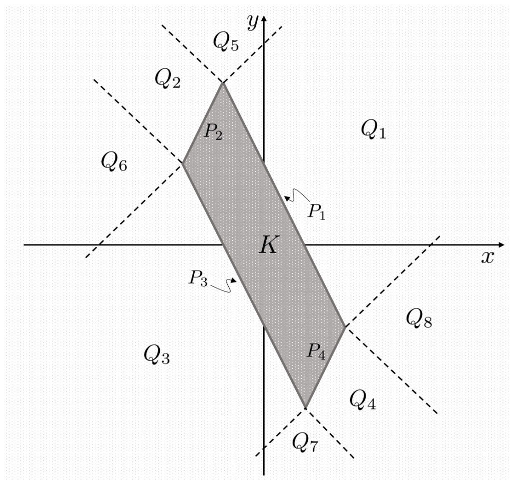


Fig. 2: The extended invariance domain $K$ for $\Sigma_{\mathrm{pc}}$ together with the line segments $P_i$ ($i = 1, 2, 3, 4$) of the boundary $\partial K$ and the open subsets $Q_i$ ($i = 1, 2, \cdots, 8$) of $\mathbb{R}^2 - K$ partitioned by the lines with slopes 1 and $-1$.

To this end, it suffices from Definition 6 to show that there exists a small positive number $\epsilon > 0$ such that the inclusion

$$F_w(x,y) \subseteq \tilde{T}_K(x,y), \quad \forall (x,y) \in K_\epsilon \tag{36}$$

is satisfied, where $F_w(x,y)$ is described by

$$F_w(x,y) = \begin{cases} \begin{pmatrix} y \\ -10\sin x - 5y + 2sgn(y) + w \end{pmatrix} & \text{if } y \neq 0 \\ \left\{ \begin{pmatrix} y \\ -10\sin x - 5y + \alpha + w \end{pmatrix} \middle| \, |\alpha| \leq 2 \right\} & \text{if } y = 0 \end{cases} \tag{37}$$

and $K_\epsilon$ is defined as $K_\epsilon = K \oplus \epsilon\mathbb{B}\mathbb{R}^2$. In other words, for an arbitrary $0 < \epsilon < 1/5$, we divide the cases of $(x,y) \in K_\epsilon$ into $(x,y) \in \mathrm{Int}(K)$, $(x,y) \in \partial K$, $(x,y) \in (\mathbb{R}^2 - K) \cap K_\epsilon$, and establish the relation (36) for each case.

Here, we confine ourselves to the cases of $(x,y) \in \partial K$ and $(x,y) \in (\mathbb{R}^2 - K) \cap K_\epsilon$, since that of $(x,y) \in$ Int(K) can be easily established. Let us divide $\partial K$ and $\mathbb{R}^2 - K$ respectively into

$$\partial K = \cup_{i=1}^4 P_i, \quad \mathbb{R}^2 - K = \cup_{i=1}^8 Q_i \tag{38}$$

where each $P_i (i = 1, \ldots, 4)$ denotes each line segment of the boundary $\partial K$ and each $Q_i (i = 1 \ldots, 8)$ means each open subset of $\mathbb{R}^2 - K$ partitioned by the lines with slopes of 1 and $-1$, and they are depicted in Fig. 2. Because both the vector field $f(x,y)$ in (33) and the extended invariance domain shown in Fig. 2 are symmetric around the origin in $\mathbf{x}-\mathbf{y}$ plane, it is not required to take into account of every region of $\partial K$ and $\mathbb{R}^2 - K$ with respect to dealing with (36), but it suffices to consider $(x,y) \in P_i$ and $(x,y) \in Q_j \cap K_\epsilon$ for $i = 1, 2$ and $j = 1, 2, 5, 8$. For such reduced cases, we can compute the external contingent cone $\tilde{T}_K(x,y)$ consisting of 2-dimensional vector $[u\ v]^T \in \mathbb{R}^2$ as follows.

$$\tilde{T}_K(x,y) = \begin{cases} \{2u + v \leq 0, 2u - v \leq 0\} & \text{if } (x,y) \in \partial P_1 \cap \partial P_4 \\ \{2u + v \leq 0\} & \text{if } (x,y) \in P_1 \cup Q_1 \\ \{2u + v \leq 0, -2u + v \leq 0\} & \text{if } (x,y) \in \partial P_1 \cap \partial P_2 \\ \{-2u + v \leq 0\} & \text{if } (x,y) \in P_2 \cup Q_2 \\ \{v \leq 0\} & \text{if } (x,y) \in Q_5 \\ \{u \leq 0\} & \text{if } (x,y) \in Q_8 \\ \{2u - v \leq 0, u \leq 0\} & \text{if } (x,y) \in \partial Q_4 \cap \partial Q_8 \\ \{2u + v \leq 0, u \leq 0\} & \text{if } (x,y) \in \partial Q_1 \cap \partial Q_8 \\ \{2u + v \leq 0, v \leq 0\} & \text{if } (x,y) \in \partial Q_1 \cap \partial Q_5 \\ \{-2u + v \leq 0, v \leq 0\} & \text{if } (x,y) \in \partial Q_2 \cap \partial Q_5 \end{cases} \tag{39}$$

Then, it readily follows from (37) and (39) that the inclusion (36) holds and thus $K$ defined as (35) is obviously an extended invariance domain of $\Sigma_{\mathrm{pc}}$ given by (32). Hence, by Theorem 4, this piecewise continuous nonlinear system $\Sigma_{\mathrm{pc}}$ satisfies the $L_1$ performance.

On the other hand, it would be also worthwhile to note that piecewise continuous nonlinear systems often have non-unique solutions, as mentioned in Sections 1 and 4. This is also true even for the piecewise continuous nonlinear system $\Sigma_{\mathrm{pc}}$ dealt with in this paper. For example, if we let $(x,y) = (0,0)$ be an initial condition of the differential equation in (32) with $w = 0$, then the solution $(x(t), y(t))$ cannot be uniquely determined since

$$\lim_{y \to 0^+} f(0,y) = 2, \quad \lim_{y \to 0^-} f(0,y) = -2 \tag{40}$$

and this non-uniqueness could be regarded as occurred from the discontinuity of $f(x,y)$ along $y = 0$. Thus, the effectiveness as well as applicability of Theorem 4 to piecewise continuous nonlinear systems with non-unique solutions is verified again through this numerical example.

## 6 Conclusion

Beyond the $L_1$ performance analysis problem discussed in [8], this paper tackled an advanced issue on the $L_1$ performance analysis problem of continuous and piecewise continuous nonlinear systems without assuming solution uniqueness. More precisely, we first provided a sufficient condition for the $L_1$ performance in terms of the invariant set. To resolve the difficulty which arise from the requirement of analytical representations of solutions with respect to the corresponding differential equations in the employment of the sufficient condition, we then introduced the notion of external contingent cone to arrive at the wider class of the existing invariance domain, by which another sufficient condition for the $L_1$ performance analysis of continuous nonlinear systems can be characterized by the so-called extended invariance domain. Here, the extended invariance domain allows us to tackle the $L_1$ performance analysis problem of continuous nonlinear systems with non-unique solutions. Furthermore, we established parallel results with respect to the $L_1$ performance analysis problem of piecewise continuous nonlinear systems by extending the set-invariance-based arguments used for continuous nonlinear systems. Numerical examples were provided to verify the effectiveness as well as the applicability of the results derived in this paper, especially for the possibility of wider applications of the

$L_1$ admissible controller synthesis discussed in [8]. Finally, we believe that the results in this paper could contribute to construct rigorous arguments on the $L_1$ controller synthesis for wider class of nonlinear systems, but this seems a non-trivial task and is left for an interesting future work.

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## Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## Declarations

Not applicable.

## Conflict of interest

The authors declare that they have no conflict of interst.

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