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Finite-time dissipative synchronization of discrete-time semi-Markovian jump complex dynamical networks with actuator faults

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Abstract

This paper is concerned with the problem of finite-time synchronization for a class of discrete-time semi-Markovian jumping complex dynamical networks(CDNs) with actuator faults based on reliable control. The main aim of this paper is to design a state feedback controller such that the resulting closed-loop system is finite-time synchronized under a prescribed dissipativity performance level even in the presence of actuator failures. Moreover, a stochastic nature followed by Bernoulli distribution is described in the considered networks due to the occurrence of probabilistic nature in time-varying delays. By composing a suitable Lyapunov-Krasovskii functional (LKF) containing triple summation terms with the aid of Kronecker product properties and utilizing free weighting matrix approach, sufficient criteria are established in terms of linear matrix inequalities (LMIs) that assures finite-time synchronization and meets the dissipativity performance to the addressed CDN. The usefulness of the presented design scheme is finally verified by numerical examples.

Keywords: Finite-time synchronization, Complex dynamical networks, Dissipativity, Semi-Markovian jumping, Reliable control.

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1 Introduction

Complex network is a large-scale network composed of a large number of nodes joined by edges, and it can mirror several important structure characteristics and dynamic performances of real networks [1,2]. In recent years, the study on CDNs serve as one of the most active research field in control theory that have found successful applications in real world disciplines such as supply chain and manufacturing networks, electricity power grids, transportation systems, water distribution systems, gas transmission, world wide web, social interaction, biological networks, co-authorship and citation of networks of scientists. Due to these utilizations, the researchers have been devoting their interest enormously to investigate the control problems and dynamical behaviors of CDNs [1]- [4].

On the other hand, in practice, synchronization is interesting and significant dynamic phenomena of CDNs compared to other dynamical performance such as auto-waves, spatio-temporal chaos, self-organization and spiral waves etc. Inspired by these advantages, the research communities have investigated synchronization of a CDNs presented in [5]- [7]. Li et al. [6] studied the synchronization problem for a class of discrete-time CDNs with partial mixed impulsive effects. But in practice, the networks might be expected to achieve synchronization in finite-time interval rather than asymptotically. To achieve faster convergence rate in time delay complex networks, an effective method called finite-time synchronization or finite-time boundedness can be used. In addition, finite-time control design have been proposed for the synchronization of different CDNs as authorized by various literatures [8]- [10]. The finite-horizon bounded H_∞ synchronization and state estimation problem for a class of discrete-time complex networks with missing measurements is reported in [8].

Moreover, few actuators may be faulty and uniform of inferior quality, the CDNs structure may be changed as well as the closed loop system becomes unstable. To improve system reliability and security, it is most important to design a reliable controller so that the stability and the performance of the Takagi-Sugeno fuzzy closed-loop system can operate fine, even in the existence of some actuator faults in [11]. Consequently, the study on fault tolerant control or reliable control for dynamical systems has received many results, see for example [12]- [14]. Naturally, the inclusion of time delays are inevitable phenomenon in all kinds of dynamical systems and it arises due to the finite propagation speed between nodes, traffic congestions and memory effects, it has the capability to alter notoriously the dynamic behavior of the system such as stability. Therefore, much work has been done for synchronization of discrete-time networks with time-delay [15]- [17]. The existence of time delay in a stochastic fashion in some practical systems, it consequences the researchers have focused the synchronization analysis of CDNs with probabilistic time-varying delay [18]- [20]. The problem of cluster synchronization in finite time complex networks with probabilistic coupling delays has been addressed in [18].

In most of the practical networks, the abrupt variations often expose in system dynamics due to random failures, changes in the subsystem interconnections and information latching. Therefore, in order to tackle this issues, it is essential to describe CDNs with Markovian jumping parameters, where each state denotes in a discrete-time system with finite discrete jumping mode governed by the Markov process and few interesting results regarding Markov jump systems have been addressed in [21]- [23]. It is pointed that the above literatures have dealt with transition rates of Markov process are constants due to that the sojourn time between two jumps of the Markov chain is governed by exponential distribution. Also, it should be mentioned that the Markov process might consist of time-varying transition rates when modeling practical systems, such kind of process is known as semi-Markov process. Apparently, semi-Markov jump CDNs are comparatively more general than the Markov jump CDNs. Shen et al. [24] obtained the finite-time H_∞ synchronization criterion for complex networks with time-varying delays and semi-Markov jump topology. The existence of sufficient conditions in [25] and convex optimization technique which ensure the L_2-L_∞ synchronization for singularly perturbed complex networks subject to semi-Markov jump topology.

Furthermore, the dissipativity theory was introduced by Willems [26], it serves as a powerful tool in control applications such as robotics, active vibration damping, electromechanical system and circuit theory. Generally speaking, dissipativity which postulates the quantity of energy supplied from the external source is not less than the energy lost inside the dynamic system [27]. The main idea of dissipative theory is the dissipative quality of a system can keep the system internally stable. Also, the dissipativity system theory is a more general criterion when compared with passivity and H_∞ performance. In [28], a set of sufficient conditions is established by using Lyapunov functional methodology and completing square technique for dissipativity control of discrete nonlinear Markovian jump systems with discrete and distributed time-delays. Wang et al. [29], addresses the problem of generalized dissipativity-based synchronization for complex networks with semi-Markov jump topology. In spite, up to now, the finite-time dissipative synchronization has not yet been reported for a class of discrete-time semi-Markovian jump CDNs.

Motivated by the above discussions, this paper analyzes finite-time synchronization problem for discrete-time semi-Markovian jumping CDNs with probabilistic time-varying delay components based on dissipative performance. The reliable control strategy and probability distribution of the time-varying delays are proposed. By constructing a new LKF and Kronecker product technique, delay dependent sufficient conditions under which the CDNs are synchronized to the given finite-time interval. The derived conditions depend not only on the size of the delay but also on the probability of the delay taking values in some intervals. Based on this derived condition, a design algorithm of the proposed state feedback controller which ensure the finite-time synchronization of the CDNs with actuator faults. Finally numerical examples are provided to illustrate the effectiveness of our

theoretical results.

Notation: Throughout this paper, superscripts " T " and " (-1) " stand for matrix transposition and matrix inverse, respectively. $\mathbb{R}^n$ denotes the n -dimensional Euclidean space and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices. $P > 0$ and $P < 0$ represent positive definite and negative definite, respectively. $L_2[0, \infty)$ stands for the space of n -dimensional square integrable function over $[0, \infty)$. I and 0 represent identity matrix and zero matrix with compatible dimensions. The asterisk " $*$ " represents a term that is induced by symmetry. $\text{diag}\{\dots\}$ stands for a block-diagonal matrix. The Kronecker product of matrices $S \in \mathbb{R}^{m \times n}$ and $T \in \mathbb{R}^{p \times q}$ is a matrix in $\mathbb{R}^{mp \times nq}$ and denoted as $S \otimes T$.

2 Network model and Preliminaries

Consider the following discrete-time semi-Markovian jumping CDNs consisting of N nonlinearly coupled nodes, with each node being of n dimensional and have identical dynamic performance. It can be designated as

$$\begin{aligned} x_l(k+1) &= A(\sigma(k))x_l(k) + B(\sigma(k))f(k, x_l(k)) + C(\sigma(k))f(k, x_l(k-d(k))) + \sum_{m=1}^N d_{lm}(\sigma(k))\Gamma_1 \\ &\quad x_m(k) + \sum_{m=1}^N g_{lm}(\sigma(k))\Gamma_2 x_m(k-\tau(k)) + R(\sigma(k))V_l^f(k) + w(k), \\ z_l(k) &= J(\sigma(k))x_l(k), \quad l = 1, 2, \dots, N, \end{aligned} \quad (1)$$

where $x_l(k) \in \mathbb{R}^n$ denotes the state vector of the l^{th} node; $A(\sigma(k))$, $B(\sigma(k))$, $C(\sigma(k))$, $R(\sigma(k))$ and $J(\sigma(k))$ are constant matrices with suitable dimensions at instant k ; $f(k, x_l(k))$ and $f(k, x_l(k-d(k)))$ represent the nonlinear vector-valued functions without and with time delays respectively. $d_{lm}(\sigma(k))$, $g_{lm}(\sigma(k))$ are the elements of the outer coupling configuration matrices $D(\sigma(k))$ and $G(\sigma(k))$ which describes the coupling structure of the CDNs. The positive diagonal matrices Γ_1 and Γ_2 are denotes inner coupling matrices with suitable dimension; $V_l^f(k)$ is the control input of the l^{th} node. If there is a link from node l to node m , then $d_{lm}(\sigma(k)) > 0$, $g_{lm}(\sigma(k)) > 0$; otherwise $d_{lm}(\sigma(k)) = g_{lm}(\sigma(k)) = 0$.

Further, the diagonal elements of the outer coupling matrices are given as follows: $g_{ll}(\sigma(k)) = -\sum_{m=1; l \neq m}^N g_{lm}(\sigma(k))$; $w(k)$ denotes the disturbance input which satisfies $w^T(k)w(k) \leq \delta$; $z(k)$ is the objective output to be estimated.

Let $\sigma(k)$, $k \leq 0$ is a discrete-time semi-Markov process and take values from a finite set $\psi = \{1, 2, \dots, N\}$ with transition probabilities $P\{\sigma(k+1) = q | \sigma(k) = p\} = \Pi_{pq}$, where $0 \leq \Pi_{pq} \leq 1$ for all $p, q \in \psi$

and $\sum_{q \in \psi} \Pi_{pq} = 1$. For notational simplicity, we take $\sigma(k) = p$.

Assumption 1: Considering the information of probability distribution of the time delays $d(k)$ and $\tau(k)$, we define

$$\begin{aligned} P\{d(k) \in [d_m, d_0]\} &= \beta_0, \quad P\{d(k) \in (d_0, d_M]\} = 1 - \beta_0, \\ P\{\tau(k) \in [\tau_m, \tau_0]\} &= \alpha_0, \quad P\{\tau(k) \in (\tau_0, \tau_M]\} = 1 - \alpha_0, \end{aligned}$$

where $0 \leq \alpha_0, \beta_0 \leq 1$ are constants and d_0, τ_0 are integers satisfying $d_m \leq d_0 < d_M, \tau_m \leq \tau_0 < \tau_M$.

Therefore the stochastic variable $\alpha(k), \beta(k)$ can be defined as

$$\alpha(k) = \begin{cases} 1, & \tau(k) \in [\tau_m, \tau_0] \\ 0, & \tau(k) \in (\tau_0, \tau_M] \end{cases} \quad \beta(k) = \begin{cases} 1, & d(k) \in [d_m, d_0] \\ 0, & d(k) \in (d_0, d_M] \end{cases}$$

Assumption 2: $\beta(k)$ and $\alpha(k)$ are the Bernoulli distributed sequences with

$$\begin{aligned} P\{\beta(k) = 1\} &= P\{d(k) \in [d_m, d_0]\} = E\{\beta(k)\} = \beta_0, \\ P\{\beta(k) = 0\} &= P\{d(k) \in (d_0, d_M]\} = 1 - E\{\beta(k)\} = 1 - \beta_0, \\ P\{\alpha(k) = 1\} &= P\{\tau(k) \in [\tau_m, \tau_0]\} = E\{\alpha(k)\} = \alpha_0, \\ P\{\alpha(k) = 0\} &= P\{\tau(k) \in (\tau_0, \tau_M]\} = E\{\alpha(k)\} = \alpha_0. \end{aligned}$$

where $E\{\beta(k)\}$ and $E\{\alpha(k)\}$ are the expectations of $\beta(k)$ and $\alpha(k)$, respectively.

From the Assumption 1, it is easy to say that

$$\begin{aligned} E\{\beta(k) - \beta_0\} &= 0, \quad E\{(\beta(k) - \beta_0)^2\} = \beta_0(1 - \beta_0), \\ E\{\alpha(k) - \alpha_0\} &= 0, \quad E\{(\alpha(k) - \alpha_0)^2\} = \alpha_0(1 - \alpha_0). \end{aligned}$$

Next, the discrete delays $d_1(k), d_2(k), \tau_1(k)$ and $\tau_2(k)$ are introduced in the following manner:

$$d(k) = \begin{cases} d_1(k), & d(k) \in [d_m, d_0] \\ d_2(k), & d(k) \in (d_0, d_M] \end{cases} \quad \tau(k) = \begin{cases} \tau_1(k), & \tau(k) \in [\tau_m, \tau_0] \\ \tau_2(k), & \tau(k) \in (\tau_0, \tau_M]. \end{cases}$$

By using Assumptions 1 and 2, CDNs (1) can be rewritten as

$$\begin{aligned} x_l(k+1) &= A_p x_l(k) + B_p f(k, x_l(k)) + \beta_0 C_p f(k, x_l(k - d_1(k))) + (1 - \beta_0) C_p f(k, x_l(k - d_2(k))) \\ &\quad + (\beta(k) - \beta_0) [C_p f(k, x_l(k - d_1(k))) - C_p f(k, x_l(k - d_2(k)))] + \sum_{m=1}^N d_{lmp} \Gamma_1 x_m(k) \\ &\quad + \alpha_0 \sum_{m=1}^N g_{lmp} \Gamma_2 x_m(k - \tau_1(k)) + (1 - \alpha_0) \sum_{m=1}^N g_{lmp} \Gamma_2 x_m(k - \tau_2(k)) + (\alpha(k) - \alpha_0) \\ &\quad \times \left[\sum_{m=1}^N g_{lmp} \Gamma_2 x_m(k - \tau_1(k)) - \sum_{m=1}^N g_{lmp} \Gamma_2 x_m(k - \tau_2(k)) \right] + R_p V_l^f(k) + w(k), \end{aligned}$$

$$z_l(k) = J_p x_l(k). \quad (2)$$

Assumption 3: For the nonlinear function $f(k, x(k), x(k - d_1(k)), x(k - d_2(k)))$, there exists the known real constant matrix $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ such that $\|f(k, x(k))\| \leq \mathcal{K}_1 \|x(k)\|$, $\|f(k, x(k - d_1(k)))\| \leq \mathcal{K}_2 \|x(k - d_1(k))\|$ and $\|f(k, x(k - d_2(k)))\| \leq \mathcal{K}_3 \|x(k - d_2(k))\|$ for any $x(k) \in \mathbb{R}^n$.

Let $e_l(k) = x_l(k) - s(k)$ be the synchronization error, where $s(k)$ be a solution of an isolated node which is described as

$$\begin{aligned} s(k+1) = & A_p s(k) + B_p f(k, s(k)) + \beta_0 C_p f(k, s(k - d_1(k))) + (1 - \beta_0) C_p f(k, s(k - d_2(k))) \\ & + (\beta(k) - \beta_0) [C_p f(k, s(k - d_1(k))) - C_p f(k, s(k - d_2(k)))]. \end{aligned} \quad (3)$$

Then the corresponding error dynamics of the CDNs (1) can be obtained as follows:

$$\begin{aligned} e_l(k+1) = & A_p e_l(k) + B_p F(k, e_l(k)) + \beta_0 C_p F(k, e_l(k - d_1(k))) + (1 - \beta_0) C_p F(k, e_l(k - d_2(k))) \\ & + (\beta(k) - \beta_0) [C_p F(k, e_l(k - d_1(k))) - C_p F(k, e_l(k - d_2(k)))] + \sum_{m=1}^N d_{lmp} \Gamma_1 e_m(k) \\ & + \alpha_0 \sum_{m=1}^N g_{lmp} \Gamma_2 e_m(k - \tau_1(k)) + (1 - \alpha_0) \sum_{m=1}^N g_{lmp} \Gamma_2 e_m(k - \tau_2(k)) + (\alpha(k) - \alpha_0) \\ & \times [\sum_{m=1}^N g_{lmp} \Gamma_2 e_m(k - \tau_1(k)) - \sum_{m=1}^N g_{lmp} \Gamma_2 e_m(k - \tau_2(k))] + R_p V_l^f(k) + w(k), \\ z_l(k) = & J_p e_l(k), \end{aligned}$$

where $F(k, e_l(k)) = f(k, x_l(k)) - f(k, s(k))$.

Now consider the actuator fault model, for the control input $V_l^f(k)$. We designate $V_l^f(k)$ to describe the signal sent from the actuator and satisfies

$$V_l^f(k) = \mathbb{G} V_l(k), \quad (4)$$

where $\mathbb{G}$ is the actuator fault matrix defined as $\mathbb{G} = \text{diag}\{g_1, g_2, \dots, g_m\}$, $0 \leq \underline{g}_i \leq g_i \leq \bar{g}_i \leq 1$, where $\underline{g}_i$ and $\bar{g}_i$, $i = 1, 2, \dots, m$ are given constants. $g_i = 0$ means that i th actuator fails, $g_i = 1$ means that i th actuator is normal, when $0 < g_i < 1$ denotes the i th actuator meets partial failures. In this paper, we designing a state feedback controller of the form:

$$V_l(k) = K_p e_l(k), \quad (5)$$

where K_p is the state feedback controller gain matrix to be determined. Then, by substituting (5) in (4) and using Kronecker product properties, the resulting closed loop system can be written as

$$e(k+1) = \bar{A}e(k) + \bar{B}F(k, e(k)) + \beta_0 \bar{C}F(k, e(k - d_1(k))) + (1 - \beta_0) \bar{C}F(k, e(k - d_2(k)))$$

$$\begin{aligned}
& + (\beta(k) - \beta_0)[\bar{C}F(k, e(k - d_1(k))) - \bar{C}F(k, e(k - d_2(k)))] + \bar{D}e(k) + \alpha_0\bar{G}e(k - \tau_1(k)) \\
& + (1 - \alpha_0)\bar{G}(e(k - \tau_2(k))) + (\alpha(k) - \alpha_0)[\bar{G}(e(k - \tau_1(k))) - \bar{G}(e(k - \tau_2(k)))] \\
& + \bar{R}GK_p e(k) + w(k), \\
z(k) &= \bar{J}e(k).
\end{aligned} \tag{6}$$

where $\bar{A} = (I \otimes A_p)$, $\bar{B} = (I \otimes B_p)$, $\bar{C} = (I \otimes C_p)$, $\bar{D} = (D_p \otimes \Gamma_1)$, $\bar{G} = (G_p \otimes \Gamma_2)$, $\bar{R} = (I \otimes R_p)$, $\bar{J} = (I \otimes J_p)$, $e(t) = [e_1^T(t), e_2^T(t), \dots, e_N^T(t)]^T$, $F(k, e(k)) = [F^T(k, e_1(k)), F^T(k, e_2(k)), \dots, F^T(k, e_N(k))]^T$, $F(k, e(k - \tau_1(k))) = [F^T(k, e_1(k - \tau_1(k))), F^T(k, e_2(k - \tau_1(k))), \dots, F^T(k, e_N(k - \tau_1(k)))]^T$, $F(k, e(k - \tau_2(k))) = [F^T(k, e_1(k - \tau_2(k))), F^T(k, e_2(k - \tau_2(k))), \dots, F^T(k, e_N(k - \tau_2(k)))]^T$.

Further, we introduce the following lemmas and definitions, which will be important for the derivations of the main results.

Lemma 1 (Schur complement) [30]. Given constant matrices E_{11}, E_{12} and E_{22} with appropriate dimensions where $E_{11}^T = E_{11}$ and $E_{22}^T = E_{22}$ then $E_{11} + E_{12}^T E_{22}^{-1} E_{12} < 0$ iff $\begin{bmatrix} E_{11} & E_{12}^T \\ * & -E_{22} \end{bmatrix} < 0$.

Definition 1 [30]. The error dynamics (6), subject to an exogenous disturbance $w(k)$ satisfy $\sum_{k=0}^N w^T(k) w(k) < \delta$ is said to be stochastically finite-time bounded (SFTB) with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$, where $S_p (p \in \psi)$ is a positive definite matrix, $0 < c_1 < c_2$, $\delta > 0$ if $E\{e^T(k_1)(I \otimes S_p)e(k_1)\} \leq c_1 \implies E\{e^T(k_2)(I \otimes S_p)e(k_2)\} \leq c_2, \forall k_1 \in \{-d, -d+1, \dots, 0\}, \forall k_2 \in \{1, 2, \dots, \mathbb{N}\}$, where $d = \max\{d_M, \tau_M\}$.

Definition 2 [31]. The error dynamics (6) is said to be stochastically finite-time stable (SFTS) with respect to $(c_1, c_2, S_p, \mathbb{N})$,

where S_p is a positive definite matrix, $0 < c_1 < c_2$ and $w(k) = 0$ if $E\{e^T(k_1)(I \otimes S_p)e(k_1)\} \leq c_1 \implies E\{e^T(k_2)(I \otimes S_p)e(k_2)\} \leq c_2, \forall k_1 \in \{-d_M, -d_M+1, \dots, 0\}, \forall k_2 \in \{1, 2, \dots, \mathbb{N}\}$.

Definition 3 [32]. The error dynamics (6) is $(\mathbb{L}, \mathbb{M}, \mathbb{R})$ - γ dissipative with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$, where $0 < c_1 < c_2$, $\gamma > 0$ and S_p is a positive definite matrix, if the system is stochastically finite-time bounded (SFTB) with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$ and under the zero initial condition, the output $z(k)$ satisfies

$$\sum_{k=0}^{\mathbb{N}} [e^T(k) \mathbb{L} e(k) + 2e^T(k) \mathbb{M} w(k) + w^T(k) \mathbb{R} w(k)] \geq \gamma \sum_{k=0}^{\mathbb{N}} w^T(k) w(k),$$

for any non-zero satisfies $\sum_{k=0}^N w^T(k) w(k) < \delta$, here $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{R}$ are real-valued matrices of appropriate dimensions with $\mathbb{L}$ and $\mathbb{Q}$ are symmetric. Without loss of generality, it is assume that $\mathbb{L} \leq 0$, then we have $-\mathbb{L} = (\bar{\mathbb{L}}^{1/2})^2$.

Remark 1 In the view points of the above definitions, it should be noted that the SFTS of the error

system (6) can guarantee the SFTB. That is, in the absence of exogenous disturbance inputs $w(k) = 0$, the concept of SFTB is reduced to SFTS. Thus, SFTB implies SFTS but the converse is not true. Consequently, the system (1) is finite time synchronized under the feedback control law (5).

3 MAIN RESULTS

In this section, to design a state feedback controller which ensures the stochastic synchronization in a finite-time interval of the CDNs (1). First we find the stochastic finite-time boundedness for the closed loop system (6). Next, we find the stochastic finite-time boundedness with dissipativity performance index.

A. Stochastic finite-time boundedness

Theorem 1. Under Assumptions 1 and 2, given scalars $\mu \geq 1$, $\kappa_i > 0 (i = 1, 2)$ and known actuator fault matrix $\mathbb{G}$, the system (6) is stochastic finite-time bounded with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$, if there exists symmetric matrices $P_p > 0 (p \in \psi)$, $Q_b > 0 (b = 1, 2, \dots, 6)$, $T_c > 0$ and $U_c > 0 (c = 1, 2)$, $V_l > 0 (l = 1, 2, 3, 4)$ such that the following inequalities holds:

$$\Psi = \begin{bmatrix} \Gamma_{20 \times 20} & \Xi_1 & \Xi_2 \\ * & -I & 0 \\ * & * & \Xi_3 \end{bmatrix} < 0, \quad (7)$$

$$\zeta_1 c_1 + \lambda_w \delta < \lambda_1 c_2 \mu^{-k},$$

where $\Gamma_{1,1} = -\mu(I \otimes P_p) + (I \otimes Q_1) + (I \otimes Q_2) + (I \otimes Q_3) + (I \otimes Q_4) + (I \otimes Q_5) + (I \otimes Q_6) + (d_0 - d_m + 1)(I \otimes T_1) + (\tau_0 - \tau_m + 1)(I \otimes T_2) + (d_M - d_0 + 1)(I \otimes U_1) + (\tau_M - \tau_0 + 1)(I \otimes U_2) + d_m(I \otimes V_1) + d_M(I \otimes V_3) + \tau_m(I \otimes V_3) + \tau_M(I \otimes V_4) + \kappa_1(I \otimes P_p)(\bar{A} + \bar{R}\mathbb{G}K_p + \bar{D} - 1) + (\bar{A} + \bar{R}\mathbb{G}K_p + \bar{D} - 1)^T(I \otimes P_p)^T \kappa_1$, $\Gamma_{1,10} = \alpha_0 \kappa_1(I \otimes P_p)\bar{G}$, $\Gamma_{1,11} = \kappa_1(1 - \alpha_0)(I \otimes P_p)\bar{G}$, $\Gamma_{1,12} = \kappa_1(I \otimes P_p)\bar{B}$, $\Gamma_{1,13} = \kappa_1 \beta_0(I \otimes P_p)\bar{C}$, $\Gamma_{1,14} = \kappa_1(1 - \beta_0)(I \otimes P_p)\bar{C}$, $\Gamma_{1,19} = -\kappa_1(I \otimes P_p) + (\bar{A} + \bar{R}\mathbb{G}K_p + \bar{D} - 1)^T(I \otimes P_p)^T \kappa_2$, $\Gamma_{1,20} = \kappa_1(I \otimes P_p)$, $\Gamma_{2,2} = -(I \otimes Q_1)$, $\Gamma_{3,3} = -(I \otimes Q_2)$, $\Gamma_{4,4} = -(I \otimes Q_3)$, $\Gamma_{5,5} = -(I \otimes T_1)$, $\Gamma_{6,6} = -(I \otimes U_1)$, $\Gamma_{7,7} = -(I \otimes Q_4)$, $\Gamma_{8,8} = -(I \otimes Q_5)$, $\Gamma_{9,9} = -(I \otimes Q_6)$, $\Gamma_{10,10} = -(I \otimes T_2)$, $\Gamma_{10,19} = \alpha_0 \kappa_2 \bar{G}^T(I \otimes P_p)$, $\Gamma_{11,11} = -(I \otimes U_2)$, $\Gamma_{11,19} = (1 - \alpha_0) \kappa_2 \bar{G}^T(I \otimes P_p)$, $\Gamma_{12,12} = -I$, $\Gamma_{12,19} = \kappa_2 \bar{B}^T(I \otimes P_p)^T$, $\Gamma_{13,13} = -I$, $\Gamma_{13,19} = \kappa_2 \beta_0 \bar{C}^T(I \otimes P_p)$, $\Gamma_{14,14} = -I$, $\Gamma_{14,19} = \kappa_2(1 - \beta_0) \bar{C}^T(I \otimes P_p)$, $\Gamma_{15,15} = -1/d_m(I \otimes V_1)$, $\Gamma_{16,16} = -1/d_M(I \otimes V_2)$, $\Gamma_{17,17} = -1/\tau_m(I \otimes V_3)$, $\Gamma_{18,18} = -1/\tau_M(I \otimes V_4)$, $\Gamma_{19,19} = -\kappa_2(I \otimes P_p) - \kappa_2(I \otimes P_p)^T$, $\Gamma_{20,20} = -\bar{W}$, $\Xi_1 = [(I \otimes W), 0, 0, 0, (I \otimes E), (I \otimes H), \underbrace{0 \cdots 0}_{14}]$, $\Xi_2 = [\Xi_{21} \ \Xi_{22} \cdots \Xi_{2N}]$, $\Xi_{21} = [\sqrt{\Pi_{p1}}(I \otimes$

$$\begin{aligned}
& P_1) \underbrace{0 \cdots 0}_{17}, \sqrt{\Pi_{p1}}(I \otimes P_1), 0]^T, \quad \Xi_{22} = [\sqrt{\Pi_{p2}}(I \otimes P_2) \underbrace{0 \cdots 0}_{17}, \sqrt{\Pi_{p2}}(I \otimes P_2), 0]^T, \quad \cdots, \Xi_{2N} = \\
& [\sqrt{\Pi_{pN}}(I \otimes P_N) \underbrace{0 \cdots 0}_{17}, \sqrt{\Pi_{pN}}(I \otimes P_N), 0]^T, \quad \Xi_3 = -\text{diag}\{(I \otimes P_1), (I \otimes P_2), \cdots, (I \otimes P_N)\}, \\
& \lambda_1 = \min\{\lambda_{\min}(I \otimes \mathcal{P}_p), p \in \psi\}, \quad \lambda_2 = \max\{\lambda_{\max}(I \otimes \mathcal{P}_p), p \in \psi\}, \quad \lambda_3 = \max\{\lambda_{\max}(I \otimes \\
& \mathcal{Q}_{1p}), p \in \psi\}, \quad \lambda_4 = \max\{\lambda_{\max}(I \otimes \mathcal{Q}_{2p}), p \in \psi\}, \quad \lambda_5 = \max\{\lambda_{\max}(I \otimes \mathcal{Q}_{3p}), p \in \psi\}, \\
& \lambda_6 = \max\{\lambda_{\max}(I \otimes \mathcal{Q}_{4p}), p \in \psi\}, \quad \lambda_7 = \max\{\lambda_{\max}(I \otimes \mathcal{Q}_{5p}), p \in \psi\}, \quad \lambda_8 = \max\{\lambda_{\max}(I \otimes \\
& \mathcal{Q}_{6p}), p \in \psi\}, \quad \lambda_9 = \max\{\lambda_{\max}(I \otimes \mathcal{T}_{1p}), p \in \psi\}, \quad \lambda_{10} = \max\{\lambda_{\max}(I \otimes \mathcal{T}_{2p}), p \in \psi\}, \\
& \lambda_{11} = \max\{\lambda_{\max}(I \otimes \mathcal{U}_{1p}), p \in \psi\}, \quad \lambda_{12} = \max\{\lambda_{\max}(I \otimes \mathcal{U}_{2p}), p \in \psi\}, \quad \lambda_{13} = \max\{\lambda_{\max}(I \otimes \\
& \mathcal{V}_{1p}), p \in \psi\}, \quad \lambda_{14} = \max\{\lambda_{\max}(I \otimes \mathcal{V}_{2p}), p \in \psi\}, \quad \lambda_{15} = \max\{\lambda_{\max}(I \otimes \mathcal{V}_{3p}), p \in \psi\}, \quad \lambda_{16} = \\
& \max\{\lambda_{\max}(I \otimes \mathcal{V}_{4p}), p \in \psi\}, \quad \zeta_1 = \lambda_2 + \lambda_3 d_m + \lambda_4 d_0 + \lambda_5 d_M + \lambda_6 \tau_m + \lambda_7 \tau_0 + \tau_8 \tau_M + \lambda_9 d_0 + \\
& \lambda_9 \frac{(d_0 - d_m)(d_m + d_0 - 1)}{2} + \lambda_{10} \tau_0 + \lambda_{10} \frac{(\tau_0 - \tau_m)(\tau_m + \tau_0 - 1)}{2} + \lambda_{11} d_M + \lambda_{11} \frac{(d_0 + d_m)(d_m - d_0 + 1)}{2} + \lambda_{12} \tau_M + \\
& \lambda_{12} \frac{(\tau_0 + \tau_m)(\tau_m - \tau_0 + 1)}{2} + \lambda_{13} \frac{d_m(d_m - 1)}{2} + \lambda_{14} \frac{d_M(d_M - 1)}{2} + \lambda_{15} \frac{\tau_m(\tau_m - 1)}{2} + \lambda_{16} \frac{\tau_M(\tau_M - 1)}{2}.
\end{aligned}$$

Proof: Construct the following LKF for the error system (6) as

$$\mathcal{V}(k) = \sum_{b=1}^5 \mathcal{V}_b(k), \quad (8)$$

where

$$\begin{aligned}
\mathcal{V}_1(k) &= e^T(k)(I \otimes P_p)e(k), \\
\mathcal{V}_2(k) &= \sum_{j=k-d_m}^{k-1} e^T(j)(I \otimes Q_1)e(j) + \sum_{j=k-d_0}^{k-1} e^T(j)(I \otimes Q_2)e(j) + \sum_{j=k-d_M}^{k-1} e^T(j)(I \otimes Q_3)(k)e(j) \\
&+ \sum_{j=k-\tau_m}^{k-1} e^T(j)(I \otimes Q_4)e(j) + \sum_{j=k-\tau_0}^{k-1} e^T(j)(I \otimes Q_5)e(s) + \sum_{j=k-\tau_M}^{k-1} e^T(j)(I \otimes Q_6)e(j), \\
\mathcal{V}_3(k) &= \sum_{j=k-d_1(k)}^{k-1} e^T(j)(I \otimes T_1)e(j) + \sum_{j=d_m}^{d_0-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes T_1)e(l) + \sum_{j=k-\tau_1(k)}^{k-1} e^T(j)(I \otimes T_2)e(j) \\
&+ \sum_{j=\tau_m}^{\tau_0-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes T_2)e(l), \\
\mathcal{V}_4(k) &= \sum_{j=k-d_2(k)}^{k-1} e^T(j)(I \otimes U_1)e(j) + \sum_{j=d_0+1}^{d_m-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes U_1)e(l) + \sum_{s=k-\tau_2(k)}^{k-1} e^T(j)(I \otimes U_2)e(j) \\
&+ \sum_{j=\tau_0+1}^{\tau_m-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes U_2)e(l), \\
\mathcal{V}_5(k) &= \sum_{j=k-d_m}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_1)e(l) + \sum_{j=k-d_M}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_2)e(l) + \sum_{j=k-\tau_m}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_3)e(l)
\end{aligned}$$

$$+ \sum_{j=k-\tau_M}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_4)e(l).$$

Then, taking mathematical expectation of the forward difference formula $\Delta \mathcal{V}(k) = \mathcal{V}(k+1) - \mathcal{V}(k)$ along the trajectories of the system (6), we have

$$\begin{aligned} \mathbb{E}[\Delta \mathcal{V}_1(k)] &= \mathbb{E}[\mathcal{V}_1(k+1) - \mathcal{V}_1(k)] \\ &= \mathbb{E}[e^T(k+1) \sum_{q=1}^N \Pi_{pq} P_q e(k+1) - e^T(k) P_p e(k)] \\ &= \mathbb{E}[\eta(k) + e(k)]^T \sum_{q=1}^N \Pi_{pq} P_q [\eta(k) + e(k)] - e^T(k) P_p e(k), \end{aligned} \quad (9)$$

where $\eta(k) = e(k+1) - e(k)$

$$\begin{aligned} \mathbb{E}[\Delta \mathcal{V}_2(k)] &= \mathbb{E}[\mathcal{V}_2(k+1) - \mathcal{V}_2(k)] \\ &= \mathbb{E}[e^T(k) \sum_{j=1}^6 (I \otimes Q_j) e(k) - e^T(k-d_m)(I \otimes Q_1) e(k-d_m) - e^T(k-d_0)(I \otimes Q_2) \\ &\quad e(k-d_0) - e^T(k-d_M)(I \otimes Q_3) e(k-d_M) - e^T(k-\tau_m)(I \otimes Q_4) e(k-\tau_m) \\ &\quad - e^T(k-\tau_0)(I \otimes Q_5) e(k-\tau_0) - e^T(k-\tau_M)(I \otimes Q_6) e(k-\tau_M)], \end{aligned} \quad (10)$$

$$\begin{aligned} \mathbb{E}[\Delta \mathcal{V}_3(k)] &= \mathbb{E}[\mathcal{V}_3(k+1) - \mathcal{V}_3(k)] \\ &\leq \mathbb{E}[(d_0 - d_m + 1) e^T(k)(I \otimes T_1) e(k) - e^T(k-d_1(k))(I \otimes T_1) e(k-d_1(k)) \\ &\quad + (\tau_0 - \tau_m + 1) e^T(k)(I \otimes T_2) e(k) - e^T(k-\tau_1(k))(I \otimes T_2) e(k-\tau_1(k))], \end{aligned} \quad (11)$$

$$\begin{aligned} \mathbb{E}[\Delta \mathcal{V}_4(k)] &= \mathbb{E}[\mathcal{V}_4(k+1) - \mathcal{V}_4(k)] \\ &\leq \mathbb{E}[(d_M - d_0 + 1) e^T(k)(I \otimes U_1) e(k) - e^T(k-d_2(k))(I \otimes U_1) e(k-d_2(k)) \\ &\quad + (\tau_M - \tau_0 + 1) e^T(k)(I \otimes U_2) e(k) - e^T(k-\tau_2(k))(I \otimes U_2) e(k-\tau_2(k))], \end{aligned} \quad (12)$$

$$\begin{aligned} \mathbb{E}[\Delta \mathcal{V}_5(k)] &= \mathbb{E}[\mathcal{V}_5(k+1) - \mathcal{V}_5(k)] \\ &\leq d_m e^T(k)(I \otimes V_1) e(k) - \frac{1}{d_m} \sum_{j=k-d_m}^{k-1} e^T(k)(I \otimes V_1) \sum_{k-d_m}^{k-1} e(k) - \frac{1}{d_M} \sum_{j=k-d_M}^{k-1} e^T(k) \\ &\quad \times (I \otimes V_2) \sum_{j=k-d_M}^{k-1} e(k) + d_M e^T(k)(I \otimes V_3) e(k) + \tau_m e^T(k)(I \otimes V_3) e(k) - \frac{1}{\tau_m} \\ &\quad \times \sum_{k-\tau_m}^{k-1} e^T(k)(I \otimes V_3) \sum_{k-\tau_m}^{k-1} e(k) + \tau_m e^T(k)(I \otimes V_4) e(k) - \frac{1}{\tau_M} \sum_{j=k-\tau_M}^{k-1} e^T(k)(I \otimes V_4) \\ &\quad \times \sum_{j=k-\tau_M}^{k-1} e(k). \end{aligned} \quad (13)$$

Furthermore, we have

$$\begin{aligned}
\eta(k) &= e(k+1) - e(k) \\
&= (\bar{A} + \bar{R}\mathbb{G}K_p + \bar{D} - 1)e(k) + \bar{B}F(k, e(k)) + \beta_0\bar{C}F(k, e(k - d_1(k))) + (1 - \beta_0)\bar{C} \\
&\quad \times F(k, e(k - d_2(k))) + (\beta(k) - \beta_0)[\bar{C}F(k, e(k - d_1(k))) - \bar{C}F(k, e(k - d_2(k)))] \\
&\quad + \alpha_0\bar{G}e(k - \tau_1(k)) + (1 - \alpha_0)\bar{G}(e(k - \tau_2(k))) + (\alpha(k) - \alpha_0)[\bar{G}(e(k - \tau_1(k))) \\
&\quad - \bar{G}(e(k - \tau_2(k)))] + w(k).
\end{aligned} \tag{14}$$

If we take $\mathcal{Z}_p = [\alpha_1(I \otimes P_p)^T \quad \alpha_2(I \otimes P_p)^T]^T$ with $\alpha_i > 0 (i = 1, 2)$, the following equation holds:

$$\begin{aligned}
2E\{\chi^T(k)\mathcal{Z}_p[(\bar{A} + \bar{R}\mathbb{G}K_p + \bar{D} - 1)e(k) + \bar{B}F(k, e(k)) + \beta_0\bar{C}F(k, e(k - d_1(k))) + (1 - \beta_0)\bar{C} \\
F(k, e(k - d_2(k))) + (\beta(k) - \beta_0)[\bar{C}F(k, e(k - d_1(k))) - \bar{C}F(k, e(k - d_2(k)))] + \alpha_0\bar{G}e(k - \tau_1(k)) \\
+ (1 - \alpha_0)\bar{G}(e(k - \tau_2(k))) + (\alpha(k) - \alpha_0)[\bar{G}(e(k - \tau_1(k))) - \bar{G}(e(k - \tau_2(k)))] + w(k) - \eta(k)]\} = 0,
\end{aligned} \tag{15}$$

where $\chi^T(k) = [e^T(k) \quad \eta^T(k)]$.

From Assumption 3, we can obtain the following inequalities

$$e^T(k)(I \otimes W)(I \otimes W)^T e(k) - F^T(k, x(k))F(k, x(k)) \geq 0, \tag{16}$$

$$e^T(k - d_1(k))(I \otimes E)(I \otimes E)^T e(k - d_1(k)) - F^T(k, x(k - d_1(k)))F(k, x(k - d_1(k))) \geq 0, \tag{17}$$

$$e^T(k - d_2(k))(I \otimes H)(I \otimes H)^T e(k - d_2(k)) - F^T(k, x(k - d_2(k)))F(k, x(k - d_2(k))) \geq 0. \tag{18}$$

Combining (9) - (18), we have

$$\mathbb{E}[\Delta V(k) - (\mu - 1)V(k) - w^T(k)\bar{W}w(k)] \leq \zeta^T(k)\{\Omega_{[20 \times 20]} + \mathbb{E}_1^T \mathbb{E}_1 + \mathbb{E}_2^T \mathbb{E}_2\}\zeta(k), \tag{19}$$

where

$$\begin{aligned}
\zeta(k) &= [e^T(k) \quad e^T(k - d_m) \quad e^T(k - d_0) \quad e^T(k - d_M) \quad e^T(k - d_1(k)) \quad e^T(k - d_2(k)) \quad e^T(k - \tau_m) \\
&\quad e^T(k - \tau_0) \quad e^T(k - \tau_M) \quad e^T(k - \tau_1(k)) \quad e^T(k - \tau_2(k)) \quad F^T(k, e(k)) \quad F^T(k, e(k - d_1(k))) \\
&\quad F^T(k, e(k - d_2(k))) \quad \sum_{k=d_m}^{k-1} e^T(k) \quad \sum_{k=d_M}^{k-1} e^T(k) \quad \sum_{k=\tau_m}^{k-1} e^T(k) \quad \sum_{k=\tau_M}^{k-1} e^T(k) \quad \eta^T(k) \quad w^T(k)],
\end{aligned}$$

$$\mathbb{E}_1^T = [(I \otimes W)^T, 0, 0, 0, (I \otimes E)^T, (I \otimes H)^T, \underbrace{0, \dots, 0}_{14}]$$

$$\mathbb{E}_2^T = [\sqrt{\sum_{q=1}^N \Pi_{pq}(I \otimes P_q)}, \underbrace{0, \dots, 0}_{17}, \sqrt{\sum_{q=1}^N \Pi_{pq}(I \otimes P_q)}, 0]^T.$$

Further, by using schur complement to the right hand side of (19), we get the required LMI in (7). If the matrix inequality in (7) holds, it is obvious that

$$\mathbb{E}[\Delta V(k) - (\mu - 1)V(k) - w^T(k)\bar{W}w(k)] \leq 0.$$

$$\begin{aligned} \mathbb{E}\{\mathcal{V}(k+1) - \mathcal{V}(k)\} &\leq (\mu - 1)\mathbb{E}\mathcal{V}(k) + E\{w^T(k)\bar{W}w(k)\} \\ &< (\mu - 1)\mathbb{E}\mathcal{V}(k) + \lambda_W E\{w^T(k)w(k)\}, \end{aligned}$$

where $\lambda_W = \lambda_{\max}(\bar{W})$. Thus, we can get $\mathbb{E}\{\mathcal{V}(k+1)\} < \mu\mathbb{E}\{\mathcal{V}(k)\} + \lambda_W E\{w^T(k)w(k)\}$.

Further, if $\mu \geq 1$, it follows that

$$\begin{aligned} \mathbb{E}\{\mathcal{V}(k)\} &\leq \mu^k \mathbb{E}\{\mathcal{V}(0)\} + \lambda_W \mathbb{E}\left\{\sum_{n=0}^{k-1} \mu^{k-n-1} w^T(n)w(n)\right\} \\ &\leq \mu^k \mathbb{E}\mathcal{V}(0) + \mu^k \lambda_W \delta. \end{aligned} \quad (20)$$

Next, we define the following parameters:

$$\begin{aligned} \mathcal{P}_p &= S_p^{-\frac{1}{2}} P_p S_p^{-\frac{1}{2}}, \quad \mathcal{Q}_{bp} = S_p^{-\frac{1}{2}} Q_b S_p^{-\frac{1}{2}}, \quad b = 1, \dots, 6, \quad \mathcal{T}_{cp} = S_p^{-\frac{1}{2}} T_c S_p^{-\frac{1}{2}}, \quad c = 1, 2, \\ \mathcal{U}_{jp} &= S_p^{-\frac{1}{2}} U_j S_p^{-\frac{1}{2}}, \quad j = 1, 2, \quad \mathcal{V}_{lp} = S_p^{-\frac{1}{2}} V_l S_p^{-\frac{1}{2}}, \quad l = 1, 2, 3, 4. \end{aligned}$$

Then, from (8), we can have

$$\begin{aligned} \mathbb{E}\{\mathcal{V}(0)\} &= \mathbb{E}\{e^T(0)(I \otimes S_p^{\frac{1}{2}} \mathcal{P}_p S_p^{\frac{1}{2}})e(0)\} + \sum_{j=-d_m}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{Q}_{1p} S_p^{\frac{1}{2}})e(j) + \sum_{j=-d_0}^{-1} e^T(j) \\ &\quad \times (I \otimes S_p^{\frac{1}{2}} \mathcal{Q}_{2p} S_p^{\frac{1}{2}})e(j) + \sum_{j=-d_M}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{Q}_{3p} S_p^{\frac{1}{2}})e(j) + \sum_{j=-\tau_m}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{Q}_{4p} S_p^{\frac{1}{2}}) \\ &\quad \times e(j) + \sum_{j=-\tau_0}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{Q}_{5p} S_p^{\frac{1}{2}})e(j) + \sum_{j=-\tau_M}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{Q}_{6p} S_p^{\frac{1}{2}})e(j) + \sum_{j=k-d_1(0)}^{-1} e^T(j) \\ &\quad \times (I \otimes S_p^{\frac{1}{2}} \mathcal{T}_{1p} S_p^{\frac{1}{2}})e(j) + \sum_{j=d_m}^{d_0-1} \sum_{l=-j}^{-1} e^T(l)(I \otimes S_p^{\frac{1}{2}} \mathcal{T}_{1p} S_p^{\frac{1}{2}})e(l) + \sum_{j=-\tau_1(0)}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{T}_{2p} S_p^{\frac{1}{2}}) \\ &\quad \times e(j) + \sum_{j=\tau_m}^{\tau_0-1} \sum_{l=-j}^{-1} e^T(l)(I \otimes S_p^{\frac{1}{2}} \mathcal{T}_{2p} S_p^{\frac{1}{2}})e(l) + \sum_{j=-d_2(0)}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{U}_{1p} S_p^{\frac{1}{2}})e(j) + \sum_{j=d_0+1}^{d_m-1} \sum_{l=-j}^{-1} \\ &\quad \times e^T(l)(I \otimes S_p^{\frac{1}{2}} \mathcal{U}_{1p} S_p^{\frac{1}{2}})e(l) + \sum_{s=k-\tau_2(0)}^{-1} e^T(j)(I \otimes S_p^{\frac{1}{2}} \mathcal{U}_{2p} S_p^{\frac{1}{2}})e(j) + \sum_{j=\tau_0+1}^{\tau_m-1} \sum_{l=-j}^{-1} e^T(l) \\ &\quad \times (I \otimes S_p^{\frac{1}{2}} \mathcal{U}_{2p} S_p^{\frac{1}{2}})e(l) + \sum_{j=-d_m}^{-1} \sum_{l=j}^{-1} e^T(l)(I \otimes S_p^{\frac{1}{2}} \mathcal{V}_{1p} S_p^{\frac{1}{2}})e(l) + \sum_{j=-d_M}^{-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes S_p^{\frac{1}{2}} \mathcal{V}_{2p} S_p^{\frac{1}{2}})e(l) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=-\tau_m}^{-1} \sum_{l=j}^{-1} e^T(l) (I \otimes S_p^{\frac{1}{2}} \mathcal{V}_{3p} S_p^{\frac{1}{2}}) e(l) + \sum_{j=-\tau_M}^{-1} \sum_{l=j}^{k-1} e^T(l) (I \otimes S_p^{\frac{1}{2}} \mathcal{V}_{4p} S_p^{\frac{1}{2}}) e(l), \\
& \leq c_1 \left[\lambda_2 + \lambda_3 d_m + \lambda_4 d_0 + \lambda_5 d_M + \lambda_6 \tau_m + \lambda_7 \tau_0 + \lambda_8 \tau_M + \lambda_9 d_0 + \lambda_9 \frac{(d_0 - d_m)(d_m + d_0 - 1)}{2} \right. \\
& + \lambda_{10} \tau_0 + \lambda_{10} \frac{(\tau_0 - \tau_m)(\tau_m + \tau_0 - 1)}{2} + \lambda_{11} d_M + \lambda_{11} \frac{(d_0 + d_m)(d_m - d_0 + 1)}{2} + \lambda_{12} \tau_M \\
& + \lambda_{12} \frac{(\tau_0 + \tau_m)(\tau_m - \tau_0 + 1)}{2} + \lambda_{13} \frac{d_m(d_m - 1)}{2} + \lambda_{14} \frac{d_M(d_M - 1)}{2} + \lambda_{15} \frac{\tau_m(\tau_m - 1)}{2} \\
& \left. + \lambda_{16} \frac{\tau_M(\tau_M - 1)}{2} \right] \\
& \leq \zeta_1 c_1.
\end{aligned} \tag{21}$$

Hence $\mathbb{E}\{\mathcal{V}(k)\} \geq \lambda_1 \mathbb{E}\{e^T(k)(I \otimes S_p)e(k)\}$.

By using the inequalities (20) and (21), we can obtain

$$\begin{aligned}
\mathbb{E}\{e^T(k)(I \otimes S_p)e(k)\} & \leq \frac{1}{\lambda_1} (\zeta_1 c_1 + \lambda_W \delta) \mu^k \\
& < c_2, \quad \forall k = 1, 2, \dots, N
\end{aligned}$$

From the above computations, we can conclude that the closed-loop error system (6) is stochastically finite-time bounded with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$. $\square$

B. Stochastic finite-time dissipative

A sufficient condition is given to ensure the finite-time boundedness with dissipative performance of CDNs have analyzed in the following theorem

Theorem 2. Under assumptions 1 and 2, for given scalar $\mu \geq 1$, the error system (6) is stochastic finite-time dissipative with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta, \mathbb{L}, \mathbb{M}, \mathbb{R})$, if there exist symmetric matrices $X_p > 0$ ($p \in \psi$), $\hat{Q}_{bp} > 0$ ($b = 1, 2, \dots, 6$), $\hat{T}_{cp} > 0$, $\hat{U}_{cp} > 0$ ($c = 1, 2$), $\hat{V}_{lp} > 0$, ($l = 1, 2, 3, 4$) such that the following conditions hold:

$$\hat{\Psi} = \begin{bmatrix} \hat{\Gamma}_{20 \times 20} & \hat{\Xi}_1 & \hat{\Xi}_2 & \Upsilon \\ * & -I & 0 & 0 \\ * & * & \hat{\Xi}_3 & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{22}$$

$$\hat{\zeta}_1 c_1 + \lambda_{\mathbb{R}} \delta < \lambda_1^{-1} c_2 \mu^{-k}, \tag{23}$$

where $\hat{\Gamma}_{1,1} = -\mu(I \otimes X_p) + (I \otimes \hat{Q}_{1p}) + (I \otimes \hat{Q}_{2p}) + (I \otimes \hat{Q}_{3p}) + (I \otimes \hat{Q}_{4p}) + (I \otimes \hat{Q}_{5p}) + (I \otimes \hat{Q}_{6p}) + (d_0 - d_m + 1)(I \otimes \hat{T}_{1p}) + (\tau_0 - \tau_m + 1)(I \otimes \hat{T}_{2p}) + (d_M - d_0 + 1)(I \otimes \hat{U}_{1p}) + (\tau_M - \tau_0 + 1)(I \otimes \hat{U}_{2p}) + d_m(I \otimes \hat{V}_{1p}) + d_M(I \otimes \hat{V}_{3p}) + \tau_m(I \otimes \hat{V}_{3p}) + \tau_M(I \otimes \hat{V}_{4p}) + \kappa_1 \bar{A}(I \otimes X_p) + \kappa_1(I \otimes X_p)^T \bar{A}^T + \kappa_1 \bar{R} \bar{G} Y_p + \kappa_1 Y_p^T \bar{G}^T \bar{R}^T + \kappa_1 \bar{D}(I \otimes X_p) + \kappa_1(I \otimes X_p)^T \bar{D}^T - \kappa_1(I \otimes X_p) - \kappa_1(I \otimes X_p)^T$, $\hat{\Gamma}_{1,10} = \alpha_0 \kappa_1 \bar{G}$, $\hat{\Gamma}_{1,11} = \kappa_1(1 - \alpha_0) \bar{G}$, $\hat{\Gamma}_{1,12} = \kappa_1 \bar{B}$, $\hat{\Gamma}_{1,13} = \kappa_1 \beta_0 \bar{C}$, $\hat{\Gamma}_{1,14} = \kappa_1(1 - \beta_0) \bar{C}$, $\hat{\Gamma}_{1,19} = -\kappa_1(I \otimes X_p) + \kappa_2(I \otimes X_p) \bar{A} + \kappa_2 \bar{R} \bar{G} Y_p + \kappa_2(I \otimes X_p) \bar{D} - \kappa_2(I \otimes X_p)$, $\hat{\Gamma}_{1,20} = \kappa_1 - (I \otimes X_p) \bar{J}^T \mathbb{M}$, $\hat{\Gamma}_{2,2} = -(I \otimes \hat{Q}_{1p})$, $\hat{\Gamma}_{3,3} = -(I \otimes \hat{Q}_{2p})$, $\hat{\Gamma}_{4,4} = -(I \otimes \hat{Q}_{3p})$, $\hat{\Gamma}_{5,5} = -(I \otimes \hat{T}_{1p})$, $\hat{\Gamma}_{6,6} = -(I \otimes \hat{U}_{1p})$, $\hat{\Gamma}_{7,7} = -(I \otimes \hat{Q}_{4p})$, $\hat{\Gamma}_{8,8} = -(I \otimes \hat{Q}_{5p})$, $\hat{\Gamma}_{9,9} = -(I \otimes \hat{Q}_{6p})$, $\hat{\Gamma}_{10,10} = -(I \otimes \hat{T}_{2p})$, $\Gamma_{10,19} = \alpha_0 \kappa_2 \bar{G}^T(I \otimes X_p)$, $\hat{\Gamma}_{11,11} = -(I \otimes \hat{U}_{2p})$, $\hat{\Gamma}_{11,19} = (1 - \alpha_0) \kappa_2 \bar{G}^T(I \otimes X_p)$, $\hat{\Gamma}_{12,12} = -I$, $\hat{\Gamma}_{12,19} = \kappa_2 \bar{B}^T$, $\hat{\Gamma}_{13,13} = -I$, $\hat{\Gamma}_{13,19} = \kappa_2 \beta_0 \bar{C}^T$, $\hat{\Gamma}_{14,14} = -I$, $\hat{\Gamma}_{14,19} = \kappa_2(1 - \beta_0) \bar{C}^T$, $\hat{\Gamma}_{15,15} = -1/d_m(I \otimes \hat{V}_{1p})$, $\hat{\Gamma}_{16,16} = -1/d_M(I \otimes \hat{V}_{2p})$, $\hat{\Gamma}_{17,17} = -1/\tau_m(I \otimes \hat{V}_{3p})$, $\hat{\Gamma}_{18,18} = -1/\tau_M(I \otimes \hat{V}_{4p})$, $\hat{\Gamma}_{19,19} = -\kappa_2(I \otimes X_p) - \kappa_2(I \otimes X_p)^T$, $\hat{\Gamma}_{19,20} = \kappa_2$, $\hat{\Gamma}_{20,20} = -\mathbb{R} + \gamma I$, $\hat{\Xi}_1 = [(I \otimes X_p)(I \otimes W), 0, 0, 0, 0, (I \otimes X_p)(I \otimes E), (I \otimes X_p)(I \otimes H), \underbrace{0 \cdots 0}_{13}]$, $\hat{\Xi}_2 = [\hat{\Xi}_{21} \hat{\Xi}_{22} \cdots \hat{\Xi}_{2N}]$, $\hat{\Xi}_{21} = [\sqrt{\Pi_{p1}}(I \otimes X_p) \underbrace{0 \cdots 0}_{17}, \sqrt{\Pi_{p1}}(I \otimes X_p), 0]^T$, $\hat{\Xi}_{22} = [\sqrt{\Pi_{p2}}(I \otimes X_p) \underbrace{0 \cdots 0}_{17}, \sqrt{\Pi_{p2}}(I \otimes X_p), 0]^T$, $\cdots$, $\hat{\Xi}_{2N} = [\sqrt{\Pi_{pN}}(I \otimes X_p) \underbrace{0 \cdots 0}_{17}, \sqrt{\Pi_{pN}}(I \otimes X_p), 0]^T$, $\hat{\Xi}_3 = -diag\{(I \otimes X_1), (I \otimes X_2), \cdots, (I \otimes X_N)\}$, $\Upsilon = [\bar{J}^T \sqrt{-L}, \underbrace{0 \cdots 0}_{19}]$, $\hat{\zeta}_1 = \lambda_2^{-1} + \lambda_1^{-2} \lambda_3 d_m + \lambda_1^{-2} \lambda_4 d_0 + \lambda_1^{-2} \lambda_5 d_M + \lambda_1^{-2} \lambda_6 \tau_m + \lambda_1^{-2} \lambda_7 \tau_0 + \lambda_1^{-2} \lambda_8 \tau_M + \lambda_1^{-2} \lambda_9 d_0 + \lambda_1^{-2} \lambda_9 \frac{(d_0 - d_m)(d_m + d_0 - 1)}{2} + \lambda_1^{-2} \lambda_{10} \tau_0 + \lambda_1^{-2} \lambda_{10} \frac{(\tau_0 - \tau_m)(\tau_m + \tau_0 - 1)}{2} + \lambda_1^{-2} \lambda_{11} d_M + \lambda_1^{-2} \lambda_{11} \frac{(d_0 + d_m)(d_m - d_0 + 1)}{2} + \lambda_1^{-2} \lambda_{12} \tau_M + \lambda_1^{-2} \lambda_{12} \times \frac{(\tau_0 + \tau_m)(\tau_m - \tau_0 + 1)}{2} + \lambda_1^{-2} \lambda_{13} \frac{d_m(d_m - 1)}{2} + \lambda_1^{-2} \lambda_{14} \frac{d_M(d_M - 1)}{2} + \lambda_1^{-2} \lambda_{15} \frac{\tau_m(\tau_m - 1)}{2} + \lambda_1^{-2} \lambda_{16} \frac{\tau_M(\tau_M - 1)}{2}$.
 Moreover, the gain matrices are given by $K_p = Y_p X_p^{-1}$.

Proof: In order to describe the dissipative performance of the closed-loop system (6), we consider the energy function $\mathcal{J}$ as

$$\mathcal{J} = \mathbb{E}\left\{\sum_{k=0}^{\mathcal{N}} [e^T(k) \mathcal{L}e(k) + 2e^T(k) \mathcal{M}w(k) + w^T(k) [\mathcal{R} - \gamma I] w(k)]\right\}. \quad (24)$$

The proof follows from Theorem 1, it follows that $\mathbb{E}\{\Delta \mathcal{V}(k) - (\mu - 1)V(k) - \mathcal{J}\} \leq 0$.

Thus, $\mathbb{E}\{\Delta \mathcal{V}(k)\} < \mathbb{E}\{\mu \mathcal{V}(k) + e^T(k) \mathcal{L}e(k) + 2e^T(k) \mathcal{M}w(k) + w^T(k) [\mathcal{R} - \gamma I] w(k)\}$.

By simple modifications, it is quite easy to get that

$$\begin{aligned} \mathbb{E}\{\Delta V(k)\} &\leq \mathbb{E}\{\mu^k V(0) + \sum_{t=0}^{k-1} \mu^{k-t-1} e^T(k) \mathcal{L}e(k) + 2 \sum_{t=0}^{k-1} \mu^{k-t-1} e^T(k) \mathcal{M}w(k) \\ &\quad + \sum_{t=0}^{k-1} \mu^{k-t-1} w^T(k) [\mathcal{R} - \gamma I] w(k)\}. \end{aligned}$$

Under zero initial condition and the fact $V(k) \geq 0, \forall k = 1, 2, \dots, \mathcal{N}$, we have

$$\mathbb{E}\left\{\sum_{t=0}^{k-1} \mu^{k-t-1} e^T(k) \mathcal{L}e(k) + 2 \sum_{t=0}^{k-1} \mu^{k-t-1} e^T(k) \mathcal{M}w(k) + \sum_{t=0}^{k-1} \mu^{k-t-1} w^T(k) [\mathcal{R} - \gamma I] w(k)\right\} \geq 0.$$

Further, if $\mu \geq 1$, the above inequality implies that

$$\mathbb{E}\left\{\sum_{k=0}^{\mathcal{N}} e^T(k) \mathcal{L}e(k) + 2 \sum_{k=0}^{\mathcal{N}} e^T(k) \mathcal{M}w(k) + \sum_{k=0}^{\mathcal{N}} w^T(k) [\mathcal{R} - \gamma I] w(k)\right\} \geq 0. \quad (25)$$

Then, from (25), it is easily to get the inequality in the Definition 1. To complete the proof of this theorem, let $X_p = P_p^{-1}$, $\hat{Q}_{bp} = X_p Q_p X_p (b = 1, 2, \dots, 6)$, $\hat{T}_{cp} = X_p T_c X_p$, $\hat{U}_{cp} = X_p U_c X_p (c = 1, 2)$, $\hat{V}_{lp} = X_p V_l X_p (l = 1, 2, 3, 4)$, then performing the congruence transformations to (7) by $\text{diag}\{\underbrace{(I \otimes X_p), \dots, (I \otimes X_p)}_{11}, (I \otimes I), (I \otimes I), (I \otimes I), \underbrace{(I \otimes X_p), \dots, (I \otimes X_p)}_5, (I \otimes I), (I \otimes I), \mathbb{U}\}$, where $\mathbb{U} = -\text{diag}\{(I \otimes X_1), (I \otimes X_2), \dots, (I \otimes X_N)\}$ and Letting $K_p X_p = Y_p$, we can easily to get the required LMI (22).

Further, $\lambda_1 \leq (I \otimes S_p^{-\frac{1}{2}} P_p S_p^{-\frac{1}{2}}) \leq \lambda_2$, $0 < (I \otimes S_p^{-\frac{1}{2}} Q_1 S_p^{-\frac{1}{2}}) < \lambda_3$, $0 < (I \otimes S_p^{-\frac{1}{2}} Q_2 S_p^{-\frac{1}{2}}) < \lambda_4$, $0 < (I \otimes S_p^{-\frac{1}{2}} Q_3 S_p^{-\frac{1}{2}}) < \lambda_5$, $0 < (I \otimes S_p^{-\frac{1}{2}} Q_4 S_p^{-\frac{1}{2}}) < \lambda_6$, $0 < (I \otimes S_p^{-\frac{1}{2}} Q_5 S_p^{-\frac{1}{2}}) < \lambda_7$, $0 < (I \otimes S_p^{-\frac{1}{2}} Q_6 S_p^{-\frac{1}{2}}) < \lambda_8$, $0 < (I \otimes S_p^{-\frac{1}{2}} T_1 S_p^{-\frac{1}{2}}) < \lambda_9$, $0 < (I \otimes S_p^{-\frac{1}{2}} T_2 S_p^{-\frac{1}{2}}) < \lambda_{10}$, $0 < (I \otimes S_p^{-\frac{1}{2}} U_1 S_p^{-\frac{1}{2}}) < \lambda_{11}$, $0 < (I \otimes S_p^{-\frac{1}{2}} U_2 S_p^{-\frac{1}{2}}) < \lambda_{12}$, $0 < (I \otimes S_p^{-\frac{1}{2}} V_1 S_p^{-\frac{1}{2}}) < \lambda_{13}$, $0 < (I \otimes S_p^{-\frac{1}{2}} V_2 S_p^{-\frac{1}{2}}) < \lambda_{14}$, $0 < (I \otimes S_p^{-\frac{1}{2}} V_3 S_p^{-\frac{1}{2}}) < \lambda_{15}$, and $0 < (I \otimes S_p^{-\frac{1}{2}} V_4 S_p^{-\frac{1}{2}}) < \lambda_{16}$.

Then, according to the congruence transformations, these above relations can be altered into $\lambda_2^{-1} (I \otimes S_p^{-1}) < (I \otimes X_p) < \lambda_1^{-1} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{Q}_{1p}) < \lambda_1^{-2} \lambda_3 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{Q}_{2p}) < \lambda_1^{-2} \lambda_4 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{Q}_{3p}) < \lambda_1^{-2} \lambda_5 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{Q}_{4p}) < \lambda_1^{-2} \lambda_6 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{Q}_{5p}) < \lambda_1^{-2} \lambda_7 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{Q}_{6p}) < \lambda_1^{-2} \lambda_8 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{T}_{1p}) < \lambda_1^{-2} \lambda_9 (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{T}_{2p}) < \lambda_1^{-2} \lambda_{10} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{U}_{1p}) < \lambda_1^{-2} \lambda_{11} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{U}_{2p}) < \lambda_1^{-2} \lambda_{12} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{V}_{1p}) < \lambda_1^{-2} \lambda_{13} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{V}_{2p}) < \lambda_1^{-2} \lambda_{14} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{V}_{3p}) < \lambda_1^{-2} \lambda_{15} (I \otimes S_p^{-1})$, $0 < (I \otimes \hat{V}_{4p}) < \lambda_1^{-2} \lambda_{16} (I \otimes S_p^{-1})$, it can easily received the inequality (23). Hence, completes the proof. $\square$

Remark 2. By using Definitions 2 and 3, it should be noted that if $w(k) = 0$, the concept of SFTB is reduced to SFTS. Also we can observe that if the error system (6) is stochastically finite time stable then the proposed CDNs (1) is finite time synchronized under the control law (5). In CDNs (1), removing the nonlinear function and external disturbance and considering only one mode in operation, then it can be rewritten as follows:

$$x_l(k+1) = A x_l(k) + B f(k, x_l(k)) + C f(k, x_l(k)) + \sum_{m=1}^N d_{lm} \Gamma_1 x_m(k) + \sum_{m=1}^N g_{lm} \Gamma_2 x_m(k - \tau(k))$$

$$\begin{aligned}
& + RV_l^f(k), \\
z_l(k) &= Jx_l(k), \quad l = 1, 2, \dots, N.
\end{aligned} \tag{26}$$

Then the closed-loop error system (6) becomes

$$\begin{aligned}
e(k+1) &= \bar{A}e(k) + \bar{B}F(k, e(k)) + \beta_0 \bar{C}F(k, e(k)) + \bar{D}e(k) + \alpha_0 \bar{G}e(k - \tau_1(k)) + (1 - \alpha_0) \\
& \quad \times \bar{G}(e(k - \tau_2(k))) + (\alpha(k) - \alpha_0)[\bar{G}(e(k - \tau_1(k))) - \bar{G}(e(k - \tau_2(k)))] + \bar{R}\mathbb{G}Ke(k), \\
z(k) &= \bar{J}e(k).
\end{aligned} \tag{27}$$

Corollary 1. For given matrices $S > 0$, and scalars $\mu \geq 1$, $c_1 > 0$, $c_2 > 0$, $\mathbb{N}$ and $\alpha_i \geq 0$ ($i = 1, 2$) and under the Assumptions 1 and 2, the error system (27) is stochastic finite-time synchronization with respect to $(c_1, c_2, S, \mathbb{N})$, if there exist symmetric matrices $X > 0$, $\hat{Q}_b > 0$ ($b = 1, 2, 3$), $\hat{T}_c > 0$, $\hat{U}_c > 0$ ($c = 1, 2$), $\hat{V}_l > 0$ ($l = 1, 2$) such that the following inequalities hold:

$$\hat{\Psi} = \begin{bmatrix} \hat{\Gamma}_{10 \times 10} & \hat{\Xi}_1 \\ * & -I \end{bmatrix} < 0, \tag{28}$$

$$\hat{\zeta}_1 c_1 < \lambda_1^{-1} c_2 \mu^{-k},$$

where $\hat{\Gamma}_{1,1} = -\mu(I \otimes X) + (I \otimes \hat{Q}_1) + (I \otimes \hat{Q}_2) + (I \otimes \hat{Q}_3) + (\tau_0 - \tau_m + 1)(I \otimes \hat{T}_1) + (\tau_M - \tau_0 + 1)(I \otimes \hat{U}_1) + \tau_m(I \otimes \hat{V}_1) + \tau_M(I \otimes \hat{V}_2) + \kappa_1 \bar{A}(I \otimes X) + \kappa_1(I \otimes X)^T \bar{A}^T + \kappa_1 \bar{R}\mathbb{G}Y + \kappa_1 Y^T \mathbb{G}^T \bar{R}^T + \kappa_1 \bar{D}(I \otimes X) + \kappa_1(I \otimes X)^T \bar{D}^T - \kappa_1(I \otimes X) - \kappa_1(I \otimes X)^T$, $\hat{\Gamma}_{1,5} = \alpha_0 \kappa_1 \bar{G}$, $\hat{\Gamma}_{1,6} = \kappa_1(1 - \alpha_0) \bar{G}$, $\hat{\Gamma}_{1,7} = \kappa_1 \bar{B}$, $\hat{\Gamma}_{1,10} = -\kappa_1(I \otimes X) + \kappa_2(I \otimes X) \bar{A} + \kappa_2 \bar{R}\mathbb{G}Y + \kappa_2(I \otimes X) \bar{D} - \kappa_2(I \otimes X)$, $\hat{\Gamma}_{2,2} = -(I \otimes \hat{Q}_1)$, $\hat{\Gamma}_{3,3} = -(I \otimes \hat{Q}_2)$, $\hat{\Gamma}_{4,4} = -(I \otimes \hat{Q}_3)$, $\hat{\Gamma}_{5,5} = -(I \otimes \hat{T}_1)$, $\Gamma_{5,10} = \alpha_0 \kappa_2 \bar{G}^T(I \otimes X)$, $\hat{\Gamma}_{6,6} = -(I \otimes \hat{U}_1)$, $\hat{\Gamma}_{6,10} = (1 - \alpha_0) \kappa_2 \bar{G}^T(I \otimes X)$, $\hat{\Gamma}_{7,7} = -I$, $\hat{\Gamma}_{7,10} = \kappa_2 \bar{B}^T$, $\hat{\Gamma}_{8,8} = -1/\tau_m(I \otimes \hat{V}_1)$, $\hat{\Gamma}_{9,9} = -1/\tau_M(I \otimes \hat{V}_2)$, $\hat{\Gamma}_{10,10} = -\kappa_2(I \otimes X) - \kappa_2(I \otimes X)^T$, $\hat{\Xi}_1 = [(I \otimes X)(I \otimes W), \underbrace{0 \cdots 0}_9]$ and remaining parameters are zero. Moreover, the gain matrices are estimated by $K = YX^{-1}$.

Proof: Consider the following LKF

$$\begin{aligned}
\mathcal{V}_1(k) &= e^T(k)(I \otimes P)e(k), \\
\mathcal{V}_2(k) &= \sum_{j=k-\tau_m}^{k-1} e^T(j)(I \otimes Q_1)e(j) + \sum_{j=k-\tau_0}^{k-1} e^T(j)(I \otimes Q_2)e(j) + \sum_{j=k-\tau_M}^{k-1} e^T(j)(I \otimes Q_3)e(j), \\
\mathcal{V}_3(k) &= \sum_{j=k-\tau_1(k)}^{k-1} e^T(j)(I \otimes T_1)e(j) + \sum_{j=\tau_m}^{\tau_0-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes T_1)e(l), \\
\mathcal{V}_4(k) &= \sum_{s=k-\tau_2(k)}^{k-1} e^T(j)(I \otimes U_1)e(j) + \sum_{j=\tau_0+1}^{\tau_m-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes U_1)e(l),
\end{aligned}$$

$$\mathcal{V}_5(k) = \sum_{j=k-\tau_m}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_1)e(l) + \sum_{j=k-\tau_M}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_2)e(l),$$

$$\zeta(k) = [e^T(k) \quad e^T(k-\tau_m) \quad e^T(k-\tau_0) \quad e^T(k-\tau_M) \quad e^T(k-\tau_1(k)) \quad e^T(k-\tau_2(k)) \quad F^T(k, e(k)) \\ \sum_{k-\tau_m}^{k-1} e^T(k) \quad \sum_{k-\tau_M}^{k-1} e^T(k) \quad \eta^T(k)].$$

Remaining proof follows from Theorem 2, we can obtain the required result. Thus, the proof is completed. $\square$

4 Numerical Simulations

This section provides a simulation examples to show the effectiveness and superiority of the established criteria for finite-time synchronization of the proposed CDNs.

Example 1. Consider a class of CDNs in the form of (2) with three nodes and dimension of the state vector of each node is two.

Mode 1:

$$A_1 = \begin{bmatrix} -1.43 & 1.14 \\ 1.23 & -1.26 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.25 & 2.0095 \\ 0.15 & -1.45 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.54 & 0.05 \\ 0.42 & 0.23 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 1.01 & 0.02 \\ 0 & 0.01 \end{bmatrix}, \\ R_1 = \begin{bmatrix} 1.021 & 0 \\ 3.86 & 0.011 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} -0.79 & 0.26 \\ 0.31 & 0.13 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.05 & 1.95 \\ -0.05 & 2.05 \end{bmatrix}, \quad C_2 = 0.1 * \begin{bmatrix} -2.14 & 0.05 \\ 0.042 & -1.23 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1.1 & 0.01 \\ 0 & 1.2 \end{bmatrix}, \\ R_2 = \begin{bmatrix} 0.052 & 1.96 \\ 0.01 & -0.1 \end{bmatrix}$$

$$\alpha_0 = 0.6, \beta_0 = 0.4.$$

The inner and outer coupling matrices of non-delayed and delayed terms are taken as

$$\Gamma_1 = \text{diag}\{0.09, 0.09\}, \Gamma_2 = \text{diag}\{0.25, 0.25\} \text{ and } D_1 = D_2 = G_1 = G_2 = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.2 & -0.6 & 0.4 \\ 0 & 0.2 & -0.2 \end{bmatrix}.$$

The non-linear function $f(x_i(k)) = \begin{bmatrix} \tanh(0.2x_{i1}(k)) \\ \tanh(0.2x_{i2}(k)) \end{bmatrix}$, it can easily satisfies Assumption 3 with the

matrices $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 = \text{diag}\{0.4, 0.4\}$ and also the noise signal is chosen as

$$w(k) = 0.05 \exp(-0.1k) \sin k.$$

Further, the time-delays are taken as $d_1(k) = 3.01 + 0.25 \sin(0.1k)$, $d_2(k) = 3.5 + 0.25 \sin(0.1k)$,

$\tau_1(k) = 1.05 + 0.25 \sin(0.5k)$, $\tau_2(k) = 2.01 + 0.25 \sin(0.02k)$ and the actuator failure matrix

is $\mathbb{G} = \text{diag}\{0.8, 0.8\}$. The transition probability matrices are $\begin{bmatrix} 0.2 & 0.8 \\ 0.35 & 0.65 \end{bmatrix}$ and $\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$. Moreover, the other parameters are considered as $c_1 = 0.01$, $c_2 = 13.8344$, $S_1 = S_2 = 0.2$, $\mathbb{N} = 20$, $\delta = 0.1$, $\gamma = 0.4729$, $\mu = 1.1$. Also we choose $\mathbb{L} = -1.15$, $\mathbb{M} = \text{diag}\{1, 1\}$ and $\mathbb{R} = \text{diag}\{1.2, 1.2\}$.

Now by using the aforementioned parameters and applying MATLAB toolbox to solve the LMIs formulated in Theorem 2, the following feedback control gain matrices are estimated

$$K_{11} = \begin{bmatrix} 0.5860 & 3.3082 \\ -32.0133 & 16.1103 \end{bmatrix}, \quad K_{12} = \begin{bmatrix} 0.5860 & 3.3082 \\ -32.0133 & 16.1103 \end{bmatrix}, \quad K_{13} = \begin{bmatrix} 0.5860 & 3.3082 \\ -32.0133 & 16.1103 \end{bmatrix},$$

$$K_{21} = \begin{bmatrix} 174.2709 & -106.9181 \\ -3.3945 & 2.1245 \end{bmatrix}, \quad K_{22} = \begin{bmatrix} 174.2709 & -106.9181 \\ -3.3945 & 2.1245 \end{bmatrix}, \quad K_{23} = \begin{bmatrix} 174.2709 & -106.9181 \\ -3.3945 & 2.1245 \end{bmatrix},$$

For the simulation purposes, the initial conditions for the states of the nodes and the isolated node are taken as $x_1(0) = [4 \ 2]^T$, $x_2(0) = [-1 \ 1]^T$, $x_3(0) = [2 \ -4]^T$ and $s_1(0) = [1 \ 2]^T$.

Based on the above values, simulation results are presented in Figs. 1-7. Specifically, the state responses of the first, second and third nodes together with the isolated node are plotted in Fig.1. It can be seen from this figure that the states of the nodes are exactly synchronized with the isolated node. The error responses with and without control are depicted in Fig.2. It can be observed from Fig.2(b) that the error response without control cannot reach zero within a finite period. Figs.3 and 4 represent simulation of semi-Markov jumping process and disturbance signal, respectively. In addition, the Bernoulli random variable $\alpha(k)$ with $\alpha_0 = 0.6$ and $\beta(k)$ with $\beta_0 = 0.4$ are plotted in Figs.5 and 6. The time history of $e_i^T S_p e_i(k)$ ($i = 1, \dots, 6$) is depicted in Fig.7. From these figures, it can be realized that the CDNs(2) is finite-time bounded with respect to $(0.01, 13.8344, 0.2, 20, 0.1)$ even in the existence of the network induced imperfections such as actuator fault and time-varying delays.

Example 2. Consider the discrete-time system (27) with parameters as follows:

$$A = 2 * \begin{bmatrix} 0.041 & -0.004 \\ -1.001 & 0.001 \end{bmatrix}, \quad B = 1.5 * \begin{bmatrix} 0.0084 & 0 \\ 0 & 0.0001 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0.4106 & 0.984 \\ 0.9456 & 0.6766 \end{bmatrix}.$$

The inner and outer coupling matrices of non-delayed and delayed terms are taken as $\Gamma_1 = \text{diag}\{0.09, 0.09\}$,

$$\Gamma_2 = \text{diag}\{0.25, 0.25\} \text{ and } D_1 = G_1 = \begin{bmatrix} -0.4 & 0.4 & 0 \\ 0.1 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The non-linear function $f(x_i(k)) = \begin{bmatrix} \tanh(0.2x_{i1}(k)) \\ \tanh(0.2x_{i2}(k)) \end{bmatrix}$, it can easily satisfies Assumption 3 with the matrices $\mathcal{K}_1 = \text{diag}\{0.1, 0.1\}$.

Further, the time-delays are taken as $\tau_1(k) = 1.25 + 0.25 \sin(0.5k)$, $\tau_2(k) = 2.5 + 0.25 \sin(0.02k)$ and the actuator failure matrix is $\mathbb{G} = \text{diag}\{0.8, 0.8\}$. Moreover, the other parameters are considered as $c_1 = 1.5$, $c_2 = 9.5379$, $S = 0.1$, $\mathbb{N} = 20$, $\gamma = 0.4729$, $\mu = 1.1$.

Now, by using the aforementioned parameters and applying MATLAB toolbox to solve the LMIs formulated in Corollary 1, the following feedback control gain matrices are estimated

$$K_{11} = K_{12} = K_{13} = \begin{bmatrix} 3.5056 & -0.6347 \\ -1.3886 & 0.4554 \end{bmatrix}.$$

For the simulation purposes, the initial conditions for the states of the nodes and the isolated node are taken as $x_1(0) = [3 \ 2]^T$, $x_2(0) = [-1 \ 1]^T$, $x_3(0) = [1 \ 2]^T$ and $s_1(0) = [1 \ -4]^T$.

Based on the initial condition and system parameter values, simulation results are presented in Figs. 8-10. Specifically, the state responses of the first, second and third nodes together with the isolated node are plotted in Fig.8. It can be seen from this figure that the states of the nodes are exactly synchronized with the isolated node. The error responses with control are depicted in Fig.9 and the time history of $e_i^T S e_i(k)$ ($i = 1, \dots, 6$) is depicted in Fig.10. From these figures, it can be realized that the CDNs (27) is finite-time synchronization with respect to $(1.5, 9.5379, 0.1, 20)$ even in the existence of the network induced imperfections such as actuator fault.

Figure 1: State responses of the system (1)

5 Conclusion

In this paper, the dissipative based finite-time synchronization problem has been investigated for a discrete-time CDNs subject to semi-Markovian jumping parameters, probabilistic time-varying delays and actuator faults through the reliable control. By constructing suitable Lyapunov functional method and Kronecker product properties, the required criteria for ensuring the finite-time synchronization with dissipativity performance index of the CDNs have been obtained in the form of LMIs.

Figure 2: State responses of the error system (6) with and without control

Figure 3: Semi-Markovian switching signal $\sigma(k)$

Figure 4: Disturbance signal

Figure 5: Stochastic parameter $\alpha(k)$ with $\alpha_0 = 0.6$ Figure 6: Stochastic parameter $\beta(k)$ with $\beta_0 = 0.4$

Figure 7: Evolution of $e_i^T(k)S_p e_i(k)$

Figure 8: Response of system states

Figure 9: Error response of CDNs (25)

Figure 10: Evolution of $e_i^T(k)S_e e_i(k)$

Finally, two numerical examples have been exploited to show the effectiveness of the established proposed results.

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants performed by any of the authors.

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