Domination for subordinated positive semigroups

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DOMINATION FOR SUBORDINATED POSITIVE SEMIGROUPS

CHEN-YU LI

ABSTRACT. We first give a necessary and sufficient condition for $-f(B)$ to be dominated by $-f(A)$, where f is a completely Bernstein function, B and A are C_0 -semigroup generators. Then we prove that there is no domination relationship between a semigroup and the subordinated semigroup if additional conditions are satisfied.

1. INTRODUCTION

For a Bernstein function f and a bounded C_0 -semigroup generator $-A$, operator $f(A)$ is defined as follows: suppose $-A$ generates a bounded C_0 -semigroup $\{T^{-A}(t)\}_{t \geq 0}$, (a, b, σ) be the Levy triple of f , then

$$f(A)x = ax + bAx + \int_{0+}^{\infty} (I - T^{-A}(t))x\sigma(dt). \quad x \in D(A),$$

and $-f(A)$ generates a bounded semigroup $\{T^{-f(A)}(t)\}$. If f is a completely Bernstein function then there is a different but equivalent definition of $-f(A)$: suppose (a, b, μ) is the Stieltjes representation of f and A is a sectorial operator, then

$$f_0(A)x = ax + bAx + \int_{0+}^{\infty} A(\lambda + A)^{-1}x\mu(d\lambda). \quad x \in D(A),$$

and

$$f(A) = \overline{f_0(A)}.$$

It was proved by [15, Theorem 4.1] that $f(A)$ is a sectorial operator with spectral angle $\omega(f(A)) = \omega(A)$. This kind of function calculus is sometimes called Hirsch function calculus which has many applications and generalizations, for example, A. Gumilko et al([13], [14] and [16]) use this approach to investigate the approximation theory of semigroups and the ergodicity properties of subordinated semigroups; C. Batty et al([9]) extend the domain of this functional calculus to function space consisting of $\mathcal{NP}_+$ functions and several resolvent representations are given here; the generalization of function calculus in Besov space and some related topics are discussed by C. Batty et al([10] and [11]).

In this paper, we first investigate the domination relationship between $\{T^{-f(B)}(t)\}$ and $\{T^{-f(A)}(t)\}$ with condition $\{T^{-B}(t)\}$ is dominated by $\{T^{-A}(t)\}$. The positivity of subordinated semigroups can be directly deduced from the representation([23, Propositiono

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13.1])

$$T^{-f(A)}(t) = \int_{0+}^{\infty} T^{-A}(s) \mu_t(ds).$$

And a necessary and sufficient condition for positive C_0 -semigroup $\{T^{-A}(t)\}$ dominates C_0 -semigroup $\{T^{-B}(t)\}$ was given by W. Arendt([2, Theorem 4.3]) via Kato' inequality. By using these results we can prove that the positive subordinated C_0 -semigroup $\{T^{-f(A)}(t)\}$ dominates the subordinated C_0 -semigroup $\{T^{-f(B)}(t)\}$ if and only if positive C_0 -semigroup $\{T^{-A}(t)\}$ dominates C_0 -semigroup $\{T^{-B}(t)\}$.

Then we prove that there is no domination relationship between positive C_0 -semigroup $\{T^{-A}(t)\}$ and subordinated positive C_0 -semigroup $\{T^{-f(A)}(t)\}$ if additional conditions are satisfied. To be more precise, we first prove that the irreducibility and mean-ergodicity are preserved by Hirsch functional calculus, then use the result proved by W. Arendt([6, Theorem 1.3]), if $\{T^{-A}(t)\}$ is dominated by $\{T^{-f(A)}(t)\}$ or converse, then $A = f(A)$ and thus $f(x) = x$ by uniqueness of functional calculus and semigroup generator.

2. PRELIMINARIES

2.1. Basic Notations. Throughout this paper, X is a Banach space, E is a Banach lattice, $L(X)$, $L(E)$ are the Banach algebras of all bounded linear operators on X , E . We always assume that A is a closed unbounded operator and densely defined on X (or E). We will denote by $D(A)$, $N(A)$, $R(A)$ the domain, kernel and range of A respectively. And by $\rho(A)$, $\sigma(A)$ we denote the resolvent set and spectrum of A . $R(\lambda, A) := (\lambda - A)^{-1}$ means the resolvent of A at λ if $\lambda \in \rho(A)$. A sector Σ_θ is defined as

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \mid \lambda \neq 0, \quad |\arg \lambda| < \theta\}$$

for $\theta \in (0, \pi)$ and $\Sigma_0 = (0, \infty)$. We write $H^\infty(\Sigma_\theta)$ for the space of all holomorphic function on the open set Σ_θ with super norm $\|f\|_{H^\infty(\Sigma_\theta)} := \sup_{x \in \Sigma_\theta} \|f(x)\|$, and $H_0^\infty(\Sigma_\theta)$ for all holomorphic function on the open set Σ_θ with polynomial limit,

$$H_0^\infty(\Sigma_\theta) := \{f \in H^\infty(\Sigma_\theta) \mid |f(z)| \leq M \min\{|z|^s, |z|^{-s}\}, s > 0, z \in \Sigma_\theta\}$$

and

$$\xi(\Sigma_\theta) := H_0^\infty(\Sigma_\theta) \oplus \langle (1+z)^{-1} \rangle \oplus \langle 1 \rangle.$$

2.2. Sectorial Operator and Functional Calculus. Sectorial operator is widely used in operator theory, it has many equivalent definitions, in this paper we use the following one.

Definition 2.1. Let $\omega \in [0, \pi)$, an operator A on X is called sectorial of angle ω , which denote by $A \in \text{sect}(\omega)$, if $\sigma(A) \subseteq \overline{\Sigma_\omega}$ and

$$M(A, \omega_1) := \sup \{ \|\lambda R(\lambda, A)\| \mid \lambda \in \mathbb{C} \setminus \Sigma_{\omega_1} \} < \infty$$

for all $\omega_1 \in (\omega, \pi)$.

Now we list the definition of functional calculus, the elaborate details, properties and proofs can be found in [15], [17] and [24].

Definition 2.2. [17, Lemma 2.3.1] Let A be a sectorial operator of angel ω , let $\phi \in (\omega, \pi)$ and $f \in H_0^\infty(\Sigma_\phi)$, we define

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma_{\omega_1}} f(z)R(z, A)dz$$

where Γ_{ω_1} is the boundary of sector Σ_{ω_1} oriented in the positive sense and $\omega_1 \in (\omega, \phi)$.

Definition 2.3. Let $g(x) \in H_0^\infty(\Sigma_\phi)$, $f \in \xi(\Sigma_\phi)$ with representation

$$f(x) = a + \frac{b}{1+x} + g(x),$$

and A is a sectorial operator of angle $\omega < \phi$, then

$$f(A) := a + b(1+A)^{-1} + g(A).$$

Definition 2.4. Let A be a sectorial operator of angel ω , f be a holomorphic function on Σ_ϕ , $\omega < \phi$. If there exist a function $e \in \xi(\Sigma_\phi)$ such that $ef \in \xi(\Sigma_\phi)$ and $0 \in \rho(e(A))$, then

$$f(A) = e(A)^{-1}(ef)(A).$$

Since Laplace transform is used in resolvent theory frequently, the following special Laplace inversion formula is quiet useful.

Lemma 2.5. [17, Lemma 3.3.1] *Let $\phi \in (\frac{\pi}{2}, \pi)$ and $\Phi \in H_0^\infty(\Sigma_\phi)$, Then there is a unique $g \in L^1(0, \infty)$ such that*

$$\Phi(t) = \int_0^\infty e^{-ts}g(s)ds.$$

More precisely, g is given by

$$g(t) := \frac{-1}{2\pi i} \int_\Gamma \Phi(z)e^{zt}dz$$

where Γ is the boundary of sector Σ_{ω_1} oriented in the positive sense and $\omega_1 \in (\frac{\pi}{2}, \phi)$ is arbitrary.

2.3. Bernstein Function and Hirsch fcuntional calculus.

Definition 2.6. A function $f \in C^\infty((0, +\infty), \mathbb{R})$ is called completely monotonic if

$$(-1)^n f^{(n)}(\lambda) \geq 0, \quad \forall \lambda > 0.$$

Definition 2.7. A function $\varphi \in C^\infty((0, +\infty), \mathbb{R})$ is a Bernstein function if $\varphi(\lambda) \geq 0$ for any $\lambda > 0$ and φ' is completely monotonic.

We use $\mathcal{BF}$ to denote the space of Bernstein function.

Proposition 2.8. *A function f is a Bernstein function if and only if it admits the representation*

$$f(\lambda) = a + b\lambda + \int_{0+}^\infty (1 - e^{-\lambda t})\mu(dt).$$

where $a, b > 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$, (a, b, μ) is called the *Levy triple* of f .

Definition 2.9. A function f is a completely Bernstein function if and only if it admits the representation

$$f(\lambda) = a + b\lambda + \int_{0+}^\infty (1 - e^{-\lambda t})m(t)dt.$$

where $a, b > 0$ and $m(t)$ is the complete monotone density of μ .

We use $\mathcal{CBF}$ to denote the space of completely Bernstein function.

Proposition 2.10. A completely Bernstein function f has an analytic representation in upper plane which is given by

$$f(\lambda) = a + b\lambda + \int_{0+}^\infty \frac{\lambda}{\lambda + t}\sigma(dt).$$

where $a, b > 0$ and σ is a measure on $(0, \infty)$ satisfying $\int_0^\infty \frac{1}{1+t}\sigma(dt) < \infty$. (a, b, σ) may called the *Stieljes representation* of f .

The following functional calculus definition for Bernstein function can be found in many references such as [9], [15] and [23].

Theorem 2.11. [23, Theorem 13.6] Let $f \in \mathcal{BF}$, (a, b, μ) be the Levy triple of f , $-A$ generates a bounded C_0 -semigroup $\{T^{-A}(t)\}$, then $-f(A)$ is a bounded C_0 -semigroup generator, and for $x \in D(A)$,

$$f(A)x = ax + bAx + \int_{0+}^\infty (I - T^{-A}(t))xd\mu(t).$$

The operator $f(A)$ is called the *Bernstein function* of A . If $\{\mu_t\}$ is the vaguely continuous convolution semigroup corresponding to f ,

$$e^{-tf(\lambda)} = \int_{0+}^\infty e^{-s\lambda}\mu_t(ds).$$

Then the bounded C_0 -semigroup $\{T^{-f(A)}\}$ generated by $-f(A)$ can be written as

$$T^{-f(A)}(t) = \int_{0+}^\infty T^{-A}(s)\mu_t(ds).$$

Theorem 2.12. [15, Section 3.2] Let $f \in \mathcal{CBF}$, (a, b, σ) be the Stieljes representation of f and A is a sectorial operator with spectral angle $\omega(A)$. Then $f(A)$ is a sectorial operator with spectral angle $\omega(f(A)) = \omega(A)$. Moreover, define $f_0(A)$ as follows,

$$f_0(A)x = ax + bAx + \int_{0+}^\infty A(\lambda + A)^{-1}x\sigma(dt), \quad x \in D(A).$$

then $f(A) = \overline{f_0(A)}$. The operator $f(A)$ is called the *complete Bernstein function* of A . Note that by this definition, $D(A)$ is core for $f(A)$

The following result shows that the extended holomorphic functional calculus and the Hirsch functional calculus are compatible.

Theorem 2.13. [8, Theorem 4.12] *Let $f \in \mathcal{CBF}$, A be an injective sectorial operator on X . Then the operator $f(A)$ defined by the Hirsch functional calculus (Theorem 2.12) coincides with $f(A)$ defined via the extended holomorphic functional calculus (Definition 2.4).*

3. DOMINATION FOR SUBORDINATED SEMIGROUPS

Here we give a brief introduction on ordered Banach spaces, positive operators and resolvent positive operators. More details about these topics can be found in [4], [5] and [18].

Definition 3.1. Let E be a Banach space. By a positive cone in E we understand a closed subset E_+ of E satisfying $E_+ + E_+ \subset E_+$ and $\lambda E_+ \subset E_+$ for all $\lambda > 0$. The order relations on E are defined by $x \geq y$ iff $x - y \in E_+$. Moreover, the cone E_+ is said to be generating if $E = E_+ - E_+$; the cone E_+ is said to be normal if there is some constant c such that $z \leq y \leq x$ always implies $\|y\| \leq c \max\{\|z\|, \|x\|\}$.

We say that E has order continuous norm if each decreasing positive sequence in E converges.

Through out this paper, we always assume a Banach lattice E is generating with normal cone.

Definition 3.2. A vector-valued function $f : (0, \infty) \rightarrow E$ is said to be positive if $f(\lambda) \in E_+$ and will be denote by $f \geq 0$.

A function $f \in C^\infty((0, \infty), E)$ is said to be completely monotonic if $(-1)^n f^{(n)}(\lambda) \geq 0$ for every $n \in \mathbb{N}_0$ and $\lambda \in (0, \infty)$.

The following vector-valued Bernstein theorem gives a necessary and sufficient condition for completely monotonic function on ordered Banach space with order continuous norm.

Lemma 3.3. [5, Theorem 2.7.18] *Let E be an ordered Banach lattice with order continuous norm. A function $f \in C^\infty((0, \infty), E)$ is completely monotonic if and only if there is an unique increasing left-continuous function $a : \mathbb{R}_+ \rightarrow E$ satisfying that $a(0) = 0$ and $\sup_{t \geq 0} \{\|e^{\lambda t} a(t)\|\} < \infty$ for every $\lambda > 0$ such that*

$$f(\lambda) = \int_0^\infty e^{-\lambda t} da(t) = \int_0^\infty \lambda e^{-\lambda t} a(t) dt.$$

Definition 3.4. Let E be an ordered Banach lattice. An operator $T \in L(E)$ is said to be positive if $T(E_+) \subset E_+$ and will be denoted by $T \geq 0$.

A closed operator A on E is called resolvent positive if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

A C_0 -semigroup $\{T(t)\}$ is said to be positive if $T(t)$ is positive for every $t > 0$.

We need the following lemma which involves the conception of integrated semigroups, we refer to [5] for more definitions and properties.

Lemma 3.5. [4, Theorem 4.1] *Let A be a densely defined, resolvent positive operator on ordered Banach lattice E satisfying $(\omega, \infty) \subseteq \rho(A)$ and $R(\lambda, A) \geq 0$ for every $\lambda \in (\omega, \infty)$. Then A generates an once-integrated semigroup $\{S(t)\}$ satisfying $S(0) = 0$, $0 \leq S(s) \leq S(t)$ when $s \leq t$. And there exist constants $M, \omega_1 > \omega$ such that $\|S(t)\| \leq Me^{\omega_1 t}$ for every $t > 0$ and*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dS(t), \quad \operatorname{Re} \lambda > \omega_1.$$

Since we know that if $-A$ generates a bounded positive C_0 -semigroup, then $-A$ is resolvent positive, and thus $-f(A)$ is resolvent positive for every $f \in \mathcal{BF}$. Next lemma shows that if $f \in \mathcal{CBF}$, then this conclusion is also hold without the generation of semigroup.

Lemma 3.6. *Let A be an densely defined, injective sectorial operator on ordered Banach lattice E , $\omega(A) = \omega \leq \frac{\pi}{2}$, and $-A$ is resolvent positive. Then for every $f \in \mathcal{CBF}$, $-f(A)$ is resolvent positive.*

Proof. First, by lemma 2.12, $f(A)$ is a sectorial operator. define function $g_\lambda(z)$:

$$g_\lambda(z) = \frac{1}{\lambda + f(z)}, \quad \lambda \in (0, \infty).$$

obviously, $g_\lambda(z)$ is completely monotonic and $g_\lambda(z) \in \xi(\Sigma_\omega)$, $\omega \in (\max\{\frac{\pi}{2}, \omega(A)\}, \pi)$. By composition theorem [13, Theorem 2.4.2] we know that,

$$g_\lambda(A) = \frac{1}{\lambda + f(A)}(A) = \frac{1}{\lambda + f(A)},$$

and for every $\lambda \in (0, \infty)$, $g_\lambda(A)$ is a bounded operator. In order to prove that $-f(A)$ is resolvent operator, we only need to prove that for every $\lambda \in (0, \infty)$, $g_\lambda(A)$ is positive. Since we can choose $\lambda > 0$ be a certain number, so we write $g_\lambda(z)$ as $g(z)$ for convenience.

If $g(z) = 1$ or $g(z) = \frac{1}{1+z}$, the conclusion is obviously true. If $g \in H_0^\infty(\Sigma_\omega)$, let $\frac{\pi}{2} < \phi < \omega$, by [17, Lemma 2.3.1] we know that,

$$g(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_\phi} g(\lambda) R(\lambda, A) d\lambda.$$

Since $-A$ is a densely defined, resolvent positive operator, by lemma 3.5 we know that $-A$ generates a once-integrated semigroup $\{T_1(t)\}$ such that,

$$R(\lambda, -A) = \int_0^\infty e^{-\lambda t} dT_1(t), \quad \operatorname{Re}(\lambda) > 0.$$

Since $g(z)$ is completely monotonic, by using lemma 3.3 we know there exists an increasing function $\Phi(s)$ such that,

$$g(z) = \int_0^\infty e^{-zs} d\Phi(s), \quad s > 0.$$

And since $g \in H_0^\infty(\Sigma_\omega)$, so by using lemma 2.5 we know there exists a function $\phi \in L^1(0, \infty)$ such that,

$$g(z) = \int_0^\infty e^{-zs} \phi(s) ds$$

Combine these results together we have

$$d\Phi(s) = \phi(s) ds,$$

so $\phi(s)$ is non-negative, and

$$\begin{aligned} g(A) &= \frac{1}{2\pi i} \int_\Gamma g(\lambda) R(\lambda, A) d\lambda \\ &= \frac{-1}{2\pi i} \int_\Gamma g(\lambda) \int_0^\infty e^{\lambda t} dT_1(t) d\lambda \\ &= \int_0^\infty \phi(t) dT_1(t). \end{aligned}$$

Since once-integrated semigroup $T_1(t)$ is non-decreasing, this end the proof. $\square$

Example 3.7. If we choose $f(z) = z^\beta, \beta < 1$, then $f(z)$ is a completely Bernstein function. Then if $-A$ is resolvent positive, so does $-f(A) = -A^\beta$.

A problem related to positive semigroup is: How to determin the order raletion of two semigroups on σ -order complete Banach lattice? For such a problem, W.Arendt gives the following result.

Lemma 3.8. [2, Theorem 4.3] *Let E be a σ -order complete Banach lattice, $\{T(t)\}$ is a C_0 -semigroup generated by A , and $\{S(t)\}$ is a C_0 -semigroup generated by B . Then the following statement are equivalent*

(1) $\{S(t)\}$ is dominated by $\{T(t)\}$. That is,

$$|S(t)f| \leq T(t)|f|, \quad t > 0, f \in E,$$

(2) The following inequality holds,

$$(3.1) \quad \operatorname{Re} \langle \operatorname{sign}(f) B f, \phi \rangle \leq \langle |f|, A^* \phi \rangle, f \in D(B), 0 \leq \phi \in D(A^*),$$

where $\operatorname{sign}(f)$ is an unique bounded operator on E satisfies:

- (1) $|\operatorname{sign}(f)g| < |g|, g \in E$.
- (2) $\operatorname{sign}(f)g = 0$ if $\inf\{|f|, |g|\} = 0$.
- (3) $\operatorname{sign}(f)f = |f|$.

The inequality (3.1) also be called the abstract version of Kato's inequality. Therefore, a nature question is: if C_0 -semigroup $\{T^{-B}(t)\}$ is dominated by C_0 -semigroup $\{T^{-A}(t)\}$, under which condition can $\{T^{-f(B)}(t)\}$ dominated by $\{T^{-f(A)}(t)\}$? The following theorem answers this question.

Theorem 3.9. *Let E be a σ -order complete Banach lattice, operator $-A$ generates a bounded positive C_0 -semigroup $\{T^{-A}(t)\}$ and $-B$ igenerates a bounded C_0 -semigroup*

$\{T^{-B}(t)\}$, both A and B are injective. Then $\{T^{-B}(t)\}$ is dominated by $\{T^{-A}(t)\}$ if and only if for every $f \in \mathcal{CBF}$, $\{T^{-f(B)}(t)\}$ is dominated by $\{T^{-f(A)}(t)\}$.

Proof. By using lemma 3.8 we only need to prove that: if the following inequality holds,

$$(3.2) \quad \operatorname{Re}\langle \operatorname{sign}(u)(-B)u, \phi \rangle \leq \langle |u|, (-A)^*\phi \rangle, u \in D(B), 0 \leq \phi \in D(A^*).$$

then

$$\operatorname{Re}\langle \operatorname{sign}(u)(-f(B))u, \phi \rangle \leq \langle |u|, (-f(A))^*\phi \rangle, u \in D(f(B)), 0 \leq \phi \in D(f(A)^*).$$

Because E is a σ -complete order Banach lattice, if the preceding inequality holds in an operator core of $f(B)$ and $f(A)^*$, then it is hold for $u \in D(f(-B))$ and $\phi \in D(f(-A)^*)$.

Now assume inequality (3.2) holds. Then we know that

$$|T^{-B}(t)u| \leq T^{-A}(t)|u|, t > 0, u \in E.$$

Let (a, b, σ) be the Stieljes representation of f , $u \in D(B)$, $\phi \in D(A^*)$, by theorem 2.12 we know that $D(B)$ is core for $f(B)$, $D(A^*)$ is core for $f(A^*) = f^*(A)$, and

$$-f(A)u = -au + b(-A)u + \int_{0+}^{\infty} (-A)(\lambda + A)^{-1}u d\sigma(\lambda).$$

So we only need to prove that

$$\operatorname{Re}\langle \operatorname{sign}(u) \int_{0+}^{\infty} (-B)(\lambda + B)^{-1}u d\sigma(\lambda), \phi \rangle \leq \langle |u|, (\int_{0+}^{\infty} (-A)(\lambda + A)^{-1}d\sigma(\lambda))^*\phi \rangle,$$

where

$$u \in D(B), \quad 0 \leq \phi \in D(A^*).$$

Since $\phi \in D(A^*)$, by [17, Proposition A 4. 2(j)] we know that

$$(\int_{0+}^{\infty} (-A)(\lambda + A)^{-1}d\sigma(\lambda))^*\phi = \int_{0+}^{\infty} (-A)((\lambda + A)^{-1})^*\phi d\sigma(\lambda).$$

So our goal is inequality

$$(3.3) \quad \operatorname{Re}\langle \operatorname{sign}(u)(-B)(\lambda + B)^{-1}u, \phi \rangle \leq \langle |u|, (-A)((\lambda + A)^{-1})^*\phi \rangle,$$

holds for every $\lambda > 0$. Since A is a sectorial operator, $\omega(A) = \frac{\pi}{2}$, and $-A$ is resolvent positive, define a function $G(x)$:

$$G(x) = \frac{x}{\lambda + x}, \quad \lambda > 0, \quad x \in \mathbb{R}_+.$$

This is a completely Bernstein function, then by using lemma 3.6, $-G(A) = -A(\lambda + A)^{-1}$ is resolvent positive. Then by lemma 3.8, inequality (3.3) is equal to inequality

$$|S(t)u| \leq T(t)|u|,$$

where $\{S(t)\}$ and $\{T(t)\}$ are C_0 -semigroups generated by bounded operator $-B(\lambda + B)^{-1}$ and $-A(\lambda + A)^{-1}$ respectively. Then by [2, Section 4], the preceding inequality is equal to inequality

$$|R(\mu, -B(\lambda + B)^{-1})u| \leq R(\mu, -A(\lambda + A)^{-1})|u|,$$

with μ big enough. To prove this inequality, define function $H(x)$:

$$H(x) = \frac{1}{\mu + \frac{x}{\lambda+x}}, \quad \mu, \lambda \in \mathbb{R}_+.$$

$H(x)$ obviously be a completely monotonic function, and $H(0) = \frac{1}{\mu}$, then by Bernstein theorem there exists a increasing function $\Phi(t)$ such that:

$$H(x) = \int_0^\infty e^{-tx} d\Phi(t), \quad \frac{1}{\mu} = H(0) = \int_0^\infty d\Phi(t).$$

Since $H(0) = \frac{1}{\mu} \neq 0$, then

$$H(A)u = R(\mu, -A(\lambda + A)^{-1})u = \int_0^\infty T^{-A}(t)u d\Phi(t),$$

where $\{T^{-A}(t)\}$ is the positive C_0 -semigroup generated by $-A$, Since $-B$ is dominated by $-A$, then we have

$$\begin{aligned} |R(\mu, -B(\lambda + B)^{-1})u| &= |H(B)u| \\ &= \left| \int_0^\infty T^{-B}(t)u d\Phi(t) \right| \\ &\leq \int_0^\infty |T^{-B}(t)u| d\Phi(t) \\ &\leq \int_0^\infty T^{-A}(t)|u| d\Phi(t) \\ &= H(A)|u| = R(\mu, -A(\lambda + A)^{-1})|u|. \end{aligned}$$

This complete the proof. $\square$

One purpose of Kato' inequality is to describe the positivity of semigroup $\{T^{-A}(t)\}$ by using the inequality that the generator $-A$ satisfies. In [19], R. Nagel conjectured that some abstract version of Kato' inequality is equivalent to positivity of semigroups, one form of this equivalenence was proved in [2, Theorem 1.6].

Theorem 3.10. *Let E be a σ -order complete Banach lattice. Operator A generates a bound positive C_0 -semigroup $\{T(t)\}$ if and only if there exists a strictly positive set M^* of sub-eigenvector of A^* such that*

$$\langle \text{sign}(u)Au, \phi \rangle \leq \langle |u|, A^*\phi \rangle, u \in D(A), \phi \in M'.$$

In this characterization, the author use the conception 'sub-eigenvector set'. The element u is the sub-eigenvector of the operator A if $u > 0$, $u \in D(A)$ and there exists a constant $M \in \mathbb{R}$ such that

$$Au < Mu.$$

So we naturally give the following two propositions.

Proposition 3.11. *Let A be a closed injective sectorial operator and $-A$ is resolvent positive, function $f \in \mathcal{CBF}$. If $u \in E$ is a sub-eigenvector of $-A$, then u is a sub-eigenvector of $-f(A)$.*

Proof. Let (a, b, σ) be the Stieljes representation of f . Suppose $u > 0$, $u \in D(A) \subseteq D(f(A))$ and exists a constant M such that

$$-Au < Mu, \quad M \in \mathbb{R}$$

We need to prove that there exists a constant $K \in \mathbb{R}$ such that

$$-f(A)u = -au + b(-A)u + \int_{0+}^{\infty} (-A)R(\lambda, -A)ud\sigma(\lambda) < Ku.$$

Since $-A$ is resolvent positive, for every $\lambda > M + 1$, we have

$$u > R(\lambda, -A)(\lambda - M)u,$$

and

$$(-A)R(\lambda, -A)u = R(\lambda, -A)(-A)u < R(\lambda, -A)Mu < \frac{M}{\lambda - M}u.$$

Because $\int_{0+}^{\infty} \frac{1}{\lambda+1} d\sigma(\lambda) < \infty$, denote $\int_{0+}^{M+1} \sigma(d\lambda) = N$ we have

$$\begin{aligned} -f(A)u &= -au + b(-A)u + \int_{0+}^{\infty} (-A)R(\lambda, -A)ud\sigma(\lambda) \\ &\leq (bM - a)u + \int_{M+1}^{\infty} (-A)R(\lambda, -A)ud\sigma(\lambda) + \int_{0+}^{M+1} (-A)R(\lambda, -A)ud\sigma(\lambda) \\ &\leq (bM - a)u + \int_{M+1}^{\infty} \frac{M}{\lambda - M}ud\sigma(\lambda) + \int_{0+}^{M+1} (u - \lambda R(\lambda, -A))ud\sigma(\lambda) \\ &\leq (bM - a)u + \int_{M+1}^{\infty} \frac{M}{\lambda - M}ud\sigma(\lambda) + Nu - \int_{0+}^{M+1} \lambda R(\lambda, -A)ud\sigma(\lambda) \\ &\leq [(bM - a) + \int_{M+1}^{\infty} \frac{M}{\lambda - M}d\sigma(\lambda) + N]u \\ &:= Ku. \end{aligned}$$

□

Proposition 3.12. *Let A be a closed sectorial operator and $-A$ is resolvent positive, function $f \in \mathcal{CBF}$. Let $\phi \in X_+$, $u \in D(A)$. If*

$$\langle \text{sign}(u)(-A)u, \phi \rangle \leq \langle |u|, (-A)^*\phi \rangle,$$

then

$$\langle \text{sign}(u)(-f(A))u, \phi \rangle \leq \langle |u|, (-f(A))^*\phi \rangle.$$

Proof. Similar to the proof of theorem 3.9, we only need to prove that for every $\lambda > 0$,

$$\langle \text{sign}(u)(-A)(\lambda + A)^{-1}u, \phi \rangle \leq \langle |u|, (-A)^*(\lambda + A^*)^{-1}\phi \rangle.$$

Since $-A(\lambda + A)^{-1}$ is a bounded resolvent positive operator and generates a positive C_0 -semigroup. Then by weak Kato' inequality ([2, Proposition 1.1]) we can easily prove that the target inequality hold. $\square$

Phillips showed that the Bernstein function of a bounded C_0 -semigroup generator still generates a bounded C_0 -semigroup. Because we do not assume that $-A$ is a semigroup generator in proposition 3.11 and proposition 3.12. Therefore, quoting proposition 3.11, proposition 3.12 and [2, Theorem 1.6], we use a completely different method to prove that the completely Bernstein function of a bounded positive C_0 -semigroup generator still generates a bounded positive C_0 -semigroup.

If C_0 -semigroup $\{T(t)\}$ is positive and bounded, then we introduce the concept of irreducibility of $T(t)$.

Definition 3.13. [3, Definition 1.3] A positive semigroup $T(t)$ on Banach lattice E is called irreducible if, for each $0 < f \in E$ and each $0 < g \in E^*$, there exists a time $t \in (0, \infty)$ such that

$$\langle g, T(t)f \rangle > 0.$$

There are two lemmas about properties of subordinated semigroup $\{T^{-f(A)}(t)\}$ inherited from semigroup $\{T^{-A}(t)\}$.

Lemma 3.14. *Let $\{T^{-A}(t)\}$ be a bounded positive C_0 -semigroup on E generated by operator $-A$, and f is a completely Bernstein function, (a, b, σ) be the Stieljes representation of f and $b > 0$, $T^{-f(A)}(t)$ be the C_0 -semigroup generated by $-f(A)$. If $\{T^{-A}(t)\}$ is irreducible, then $\{T^{-f(A)}(t)\}$ is irreducible.*

Proof. By Definition 3.13, since $\{T^{-A}(t)\}$ is irreducible, assume $0 < x \in E$ and $0 < y \in E^*$, there exists a time $t_0 \in (0, \infty)$ such that

$$\langle y, T^{-A}(t_0)x \rangle > 0.$$

Then we need to prove that there exists a time $t_1 \in (0, \infty)$ such that

$$\langle y, T^{-f(A)}(t_1)x \rangle > 0.$$

Let $\{\mu_t\}$ be the vaguely continuous convolution semigroup corresponding to f , then

$$T^{-f(A)}(t) = \int_{0+}^{\infty} T^{-A}(s)\mu_t(ds)$$

and

$$\begin{aligned} \langle y, T^{-f(A)}(t)x \rangle &= \langle y, \int_{0+}^{\infty} T^{-A}(s)x\mu_t(ds) \rangle \\ &= \int_{0+}^{\infty} \langle y, T^{-A}(s)x \rangle \mu_t(ds). \end{aligned}$$

Since $\langle y, T^{-A}(t_0)x \rangle > 0$ and function $f(\cdot) := \langle y, T^{-A}(\cdot)x \rangle : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, then there exist an interval U such that $t_0 \in U$ and $\langle y, T^{-A}(s)x \rangle > 0$ for every $s \in U$. Then

we only need to prove that there exists a time $t_1 \in (0, \infty)$ such that

$$\begin{aligned} \langle y, T^{-f(A)}(t_1)x \rangle &= \int_{0+}^{\infty} \langle y, T^{-A}(s)x \rangle \mu_{t_1}(ds) \\ &\geq \int_U \langle y, T^{-A}(s)x \rangle \mu_{t_1}(ds) > 0. \end{aligned}$$

This is equal to show that there exists a time $t_1 \in (0, \infty)$ such that

$$\mu_{t_1}(U) > 0.$$

By [23, definition 5.4] we know that there exists a subordinator $\{S_t\}$ with state space $[0, \infty)$ such that

$$\mu_t(U) = P(S_t \in U),$$

then we need to show that exists a time $t_1 \in (0, \infty)$ such that

$$P(S_{t_1} \in U) > 0,$$

and this can be deduced from [12, Proposition 1.9(ii)] and assumption $b > 0$. $\square$

Lemma 3.15. *Let $\{T^{-A}(t)\}$ be a bounded C_0 -semigroup on X generated by $-A$, and f is a completely Bernstein function, and $\{T^{-f(A)}(t)\}$ be the C_0 -semigroup generated by $-f(A)$. If $x \in X$ and*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T^{-A}(s)x ds$$

exists, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T^{-f(A)}(s)x ds$$

exists.

Proof. If $f \not\equiv 0$, then by [23, Proposition 7.1], there is a complete Bernstein function f^* , such that

$$f(\lambda)f^*(\lambda) = \lambda.$$

Since both f, f^* are completely Bernstein function, then by Theorem [23, Theorem 13.23], for every sectorial operator A ,

$$A = (f \cdot f^*)(A) = f(A)f^*(A).$$

So for every $x \in R(A)$, $Ay = x$, $y \in X$, we have

$$f(A)f^*(A)y = Ay = x.$$

So $R(A) \subseteq R(f(A))$.

If $x \in N(A)$, then $f(A)x = f(0)x$ and $N(A) \subseteq N(f(A)) \oplus R(f(A))$. So we have

$$N(A) \oplus \overline{R(A)} \subseteq N(f(A)) \oplus \overline{R(f(A))}.$$

Then our conclusion can be proved directly by [5, Corollary 4.3.2] and [5, Proposition 4.3.4]. $\square$

By using preceding two lemmas, we can prove the following theorem, which assert that there is no ordered relationship between $\{T^{-A}(t)\}$ and $\{T^{-f(A)}(t)\}$ if some conditons are satisfied.

Theorem 3.16. *Let $\{T^{-A}(t)\}$ be a bounded positive C_0 -semigroup on X generated by $-A$, f is a completely Bernstein function, (a, b, σ) be the Stieljes representation of f and $b \neq 0, 1$, $\{T^{-f(A)}(t)\}$ be the bounded positive C_0 -semigroup generated by $-f(A)$. If $N(A^*) \neq \{0\}$, $\{T^{-A}(t)\}$ is irreducible, (a, b, σ) be the Stieljes representation of f , $b > 0$ and*

$$(3.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T^{-A}(s) ds$$

exists in operator topology, Then

(1) *inequality*

$$0 \leq T^{-A}(t) \leq T^{-f(A)}(t)$$

do not hold for all $t > 0$.

(2) *inequality*

$$0 \leq T^{-f(A)}(t) \leq T^{-A}(t)$$

do not hold for all $t > 0$.

Proof. In order to prove conclusion (1), we assume and $0 \leq T^{-A}(t) \leq T^{-f(A)}(t)$ hold for all $t > 0$.

Since $\{T^{-A}(t)\}$ is irreducible and limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T^{-A}(s) ds$$

exist in operator topology, then by lemma 3.14 and lemma 3.15 we know that $\{T^{-f(A)}(t)\}$ is irreducible and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T^{-f(A)}(s) x ds$$

exist in operator topology. Since $N(A^*) \neq 0$, then by [6, Theorem 1.3] we know that $T^{-A}(t) = T^{-f(A)}(t)$, $t \in [0, \infty)$, then by uniqueness of semigroup generator and functional calculus we know that $f(A) = A$ and $f(x) = x$, thus the Stieljes representation of f is $(0, 1, 0)$, but this is impossible since $b \neq 1$.

Now we turn to prove conclusion (2) and we can assume $a = 0$. If $a = 0$ and $N(A^*) \neq \{0\}$, let $0 \neq x \in N(A^*)$, then $f^*(A)x = f(A^*)x = ax = 0$, thus $N(f^*(A)) \neq 0$. Then use [6, Theorem 1.3] again we deduce that $T^{-A}(t) = T^{-f(A)}(t)$. This contradicts the assumption that $b \neq 1$. $\square$

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