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Time series measurement based on multiscale high order entropy and roughness grain exponents

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In multiscale time series analysis, multiscale entropy provides a good framework to quantify the information of time series. Multiscale fractional high-order entropy based on roughness grain exponents (MFHER) is able to identify dynamical, scale dependent and oscillation information. In detail, MFHER can be seen as a powerful tool to assess the complex characteristics of time series. A set of synthetic time series and an application of real world data financial series are researched. The results show that high order entropy performs well in distinguishing different time series. It has also been found fractional high order entropy is highly sensitive to parameter variation and thus provides a broad perspective to research the complexity of dynamic systems. This study gains an insight into the measurement of MFHER to demonstrate the wide applicability of entropy measures.

Aiming at the complexity of network and the uncertainty of internal and external environment, this paper proposes MFHER to quantify the time series information on multiscale time scales. It is of great interests in identifying dynamical properties of financial series. The results show that the volatility of the sequence is gradually stable when the scale is greater than four. High order entropy can identify the difference among the time series.

Keywords: Roughness grain exponents; Fractional high-order entropy; Multiscale; Time series

1 Introduction

Information entropy could be considered as a metric to identify signals in different fields, such as the study of cellular automata of computer science, the neural cortex of the brain, social network systems [1–11]. It has the properties of nonnegativity, extremum and additivity, which can reflect the disorder and uncertainty of the system. It is consistent with the physical meaning of information entropy.

It was extended to measure the uncertainty of information in 1948 by Shannon [11, 12]. Let $p = (p_1, p_2, \dots, p_n)$ be a discrete probability distribution, $0 \leq p_i \leq 1$ and $\sum_{i=1}^n p_i = 1$, Shannon entropy is defined as:

$$I(p) = - \sum_{i=1}^n p_i \log_2 p_i. \quad (1)$$

On a discrete probability distribution, in order to characterize the original probability distribution to describe the characteristics of the dynamic system, Beck and Schlogl proposed escort distribution [13]. The escort distribution associated with it is defined by :

$$P_i = \frac{\phi(p_i)}{\sum_j \phi(p_j)} \quad (2)$$

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Fig. 1: The oscillation of a real value function.

where ϕ is a certain positive function. Of particular importance is the case $\phi(s) = s^t (0 \leq s \leq 1, t > 0)$, and correspondingly

$$P_i^{(t)} = \frac{(p_i)^t}{\sum_j (p_j)^t}. \quad (3)$$

The parameter t is regarded as the order of $P_i^{(t)}$. The statistical and probabilistic attributes of a discrete probability distribution, may be scanned using the associated escort distribution and employed in the formulation of Shannon entropy:

$$I(P_i)^t = - \sum_{i=1}^n P_i \log_2 P_i = - \sum_{i=1}^n \frac{(p_i)^t}{\sum_j (p_j)^t} \log_2 \frac{(p_i)^t}{\sum_j (p_j)^t}. \quad (4)$$

The high order (t -order) Shannon entropy is similar to the form of graph entropy proposed by Rashevsky, Trucco, and Mowshowitz [11, 14, 15]. As a complexity measure related to equivalent relations of finite graphs, graph entropy [16] is used to describe the information structure of graphs. It is a numerical value obtained by defining probability distribution [11, 17] on its vertex set [16] in the form of Shannon Entropy. Many researchers expressed intense interest in studying this topic, since J. Koerner first presented a novel approach to determine the structural information content (i.e. graph entropy) of an undirected and connected graph [18]. Although graph entropy has been widely used to characterize the internal structure information of the system in various fields [19, 20], it has few application on financial markets.

With the reform of stock markets and improvement of stock trading system [21–23], more obvious price fluctuations appear. Studying the characteristics of price fluctuations in the stock market is helpful to analyze and further understand the factors that affect stock markets. For this reason, researchers have carried out a variety of studies on the volatility trend of stock indexes. Traditional researches on the volatility trend mainly focus on the trend decomposition and periodicity analysis of stock market, the existence of volatility trend correlation between adjacent data points of the time series [24–28]. The signal's roughness of interval points may more easily reflect the overall price fluctuation characteristics of the stock market. The advantage of using interval data is that it can observe the economic operation more macroscopically and mine more data information, so it can make more accurate judgment. A roughness exponent can reflect the variation amplitude of signal in a dyadic interval. This paper proposes MFHER to measure the dynamic characteristics of stock market.

The rest of the paper is organized as follows. Section 2 presents the definition of MFHER. In Section 3, MFHER is applied to ARFIMA, Logistic map and financial series to support its effectiveness and practicality. Finally, the Section 4 draws the conclusion.

2 Methodology

2.1 Roughness grain exponents

There is a real value function F over a subinterval I of interval $[0, 1]$, and the oscillation is the most directly related to the roughness of the graph of F . It's corresponding oscillation of the function F is defined as:

$$Osc_F(I) = \sup_{s \in I} F(s) - \inf_{s \in I} F(s). \quad (5)$$

Thus, considering any small interval within the range of $(0, 1)$, we can define roughness grain exponents of a function, a signal or a measurement. Given a scale $r > 0$ and $t \in [0, 1]$, that is, $[t - r, t + r] \in [0, 1]$, the corresponding oscillation grain exponent of F is set by:

$$\alpha(t, r) = \frac{\log Osc_F([t - r, t + r])}{\log r}. \quad (6)$$

We consider dyadic scales: for an integer, $n \geq 1$, the dyadic interval, $I_n(k)$, in a dyadic framework is defined as:

$$I_n(k) = [(k - 1)2^{-n}, k2^{-n}]. \quad (7)$$

Here, $1 \leq k \leq 2^n$ and the length $|I_n(k)| = 2^{-n}$. The set of all dyadic intervals at scale n is denoted by $\mathcal{G}_n$. In short, the sequence is divided into non-overlapping subsequences of same length. Similarly, the oscillation of a signal X in the interval $I_n(k)$ is defined as:

$$Osc_X(I_n(k)) = \sup\{|X(u) - X(v)| : u, v \in I_n(k)\}. \quad (8)$$

Fig.1 illustrates the expression of Eq.(8) within a range of function F . Therefore, the roughness grain exponents can be defined in dyadic scales by:

$$\alpha_{X, Osc}(I_n(k)) = \frac{\log Osc_X(I_n(k))}{\log 2^{-n}}. \quad (9)$$

2.2 Multiscale roughness grain exponents (MRGE)

This paper aims at contributing to numerical estimation in entropy analysis by the estimation of the roughness grain exponents. It is an extensive method to measure the oscillation degree of a signal X in relation to its roughness.

First, the original series is standardized to get a new time series as $\{y_i\}_{i=1}^N$. According to Eq.(7), for $\tau \geq 1$, the interval $I_\tau(k)$, $1 \leq k \leq \tau - 1$ is decided as $I_\tau(k) = [(k - 1)\tau, k\tau]$ and its length is $|I_\tau(k)| = \tau$. A roughness grain exponents reflect the range of signal variation in the region. It is convenient to make use of oscillation based on grain exponent of signals. The oscillation of X in the interval $I_\tau(k)$ is illustrated by:

$$Osc_X(I_\tau(k)) = \sup\{|Y(u) - Y(v)| : u, v \in I_\tau(k)\}. \quad (10)$$

As the same time, the corresponding oscillation grain exponent is further regarded as:

$$\alpha_{X, Osc}(I_\tau(k)) = \frac{\log Osc_X(I_\tau(k))}{\log \tau}. \quad (11)$$

This exponent can measure the signals' roughness at points [29].

2.3 Multiscale fractional high order entropy by roughness grain exponents (MFHER)

As mentioned above, the definition of MFHER is based on the roughness grain exponents and complex behavior of information derived from the structure of an event. In this case, taking into account the escort distribution to measure the t -order entropy of an event, we obtain different probability distributions as follows:

Let X be a signal with corresponding oscillation grain exponent $\alpha_{X, Osc}(I_\tau(k)) = \alpha_1, \alpha_2, \dots, \alpha_n$ and $n = \lfloor \frac{N}{\tau} \rfloor$. The sum of high-order of is defined by $\sum \alpha_i^t$. By calculating the roughness grain exponents and applying an arbitrary real number t , the escort distribution distributions $P = (p_1, p_2, \dots, p_i)$ is defined on events as for every α_i :

$$p_i = \frac{\alpha_i^t}{\sum \alpha_i^t}. \quad (12)$$

High order entropy with multiscale roughness grain exponents of X based on P is defined as:

$$E(P) = - \sum \left(\frac{\alpha_i^t}{\sum \alpha_i^t} \log \frac{\alpha_i^t}{\sum \alpha_i^t} \right). \quad (13)$$

More recently, fractional calculus and chaos theory of dynamic system are combined and non-additive entropy was proposed [30]:

$$S_q = - \sum p_i^q (\ln p_i) = - \sum \frac{\alpha_i^t}{\sum \alpha_i^t}^q (\ln \frac{\alpha_i^t}{\sum \alpha_i^t}), \quad 0 \leq q \leq 1. \quad (14)$$

In addition, with the use of optimization principles, another fractional order entropy has been proposed [30] in information theory:

$$S'_q = \sum p_i (-\ln p_i)^q = \sum \frac{\alpha_i^t}{\sum \alpha_i^t} (-\ln \frac{\alpha_i^t}{\sum \alpha_i^t})^q, \quad 0 \leq q \leq 1. \quad (15)$$

when the order is $q = 1$, it yields Eq.(12). The main goal of this section is to investigate relationships between the resulting entropies and q .

3 Numerical results for artificial time series

3.1 ARFIMA stochastic processes

The first example is a well-suited model to generate auto-correlated data with any power-law exponent, which is the Autoregressive Integrated Moving Average Model (ARFIMA) process [31]

$$z_t = \sum_{n=1}^{\infty} a_n(d) z_{t-n} + \omega_t, \quad (16)$$

where $\omega_t \sim N(0, 1)$, $d \in (0, 0.5)$ is a memory parameter related to Hurst exponent as $h_{zz} = 0.5 + d$ and $a_n(d)$ are the weights defined by $a_n(d) = d\Gamma(n-d)/\Gamma(1-d)\Gamma(n+1)$, Γ denotes the Gamma function and n is the time scale.

The two-component ARFIMA stochastic processes can be defined by two variables x_k and y_k :

$$\begin{aligned} X_t &= \sum_{n=1}^{\infty} a_n(d_1) X_{t-n}, \\ Y_t &= \sum_{n=1}^{\infty} a_n(d_2) Y_{t-n}, \\ x_t &= [W X_t + (1 - W) Y_t] + \omega_t, \\ y_t &= [(1 - W) X_t + W Y_t] + \nu_t, \end{aligned} \quad (17)$$

Fig. 2: Synthetic data sets: ARFIMA1 and ARFIMA2.

Table 1: The mean and variance of $\alpha_{X,Osc}(I_\tau(k))$ results about ARFIMA1 and ARFIMA2 series at different scales.

	scale 2	scale 3	scale 4	scale 5	scale 6	scale 7	scale 8	scale 9	scale 10
Avg-ARFIMA1	2.90	1.37	0.93	0.72	0.60	0.51	0.45	0.41	0.37
Avg-ARFIMA2	3.00	1.44	0.98	0.76	0.64	0.55	0.48	0.44	0.40
Var-ARFIMA1	2.5213	0.3274	0.1217	0.0635	0.0409	0.0266	0.0221	0.0149	0.0147
Var-ARFIMA2	2.5200	0.3254	0.1216	0.0631	0.0448	0.0263	0.0223	0.0148	0.0144

where $\omega_t, \nu_t \sim N(0, 1)$, $d_1, d_2 \in (0, 0.5)$. The parameter W represents the strength of the coupling and ranges in the interval $0.5 \leq W \leq 1$, where $W = 0.5$ gives the highest cross-correlation (maximum coupling strength) while the total absence of correlation is obtained for $W = 1$. In our simulations, we generate a set of cross correlated ARFIMA series with $N = 4096$. Fig.3 presents ARFIMA1 and ARFIMA2 respectively, which is for the coupling value $W = 0.4$. In this case, the d_1 parameter is fit to 0.1. And another d_1 parameter is set to 0.3. Fig.4 and Fig.5 show the corresponding fluctuation data points $OscX(I_\tau(k))$ (light blue dots) and $\alpha_{X,Osc}(I_\tau(k))$ (dark blue dots) of ARFIMA1 and ARFIMA2 at different scales, respectively.

The $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ results produced by ARFIMA series have been included in the supplementary materials section. On the whole, the $OscX(I_\tau(k))$ value is almost the same at different scales, keeping in the range of 0 to 0.8. Obviously, when the scale is 2, the $\alpha_{X,Osc}(I_\tau(k))$ is range from 0 to 10. With the increase of scale, the $\alpha_{X,Osc}(I_\tau(k))$ decreases gradually and tends to be stable. The fluctuation is similar when the scale is 8 and 12, which is significantly invariant. Especially, results at scale 8 and 12 show that the amplitude of oscillations naturally decreases with τ , and the range of the roughness exponents stabilizes across the scales. In particular, to illustrate that the $\alpha_{X,Osc}(I_\tau(k))$ have invariance properties at different scales when the scale is larger than 4, we show the mean values and variance at different scales in Tab.1 and Fig.3.

Fig.4 reflects that how the high order entropy works with the change of t and τ . With the increase of t , the high order entropy gradually decreases. When t goes larger than 15, the entropy value tends to be stable. The two series gradually become similar and almost overlap when scale is larger than 8. While the scale is smaller than 8, ARFIMA1 is much bigger than another. The difference between the mean and variance of the $\alpha_{X,Osc}(I_\tau(k))$ about ARFIMA1 and ARFIMA2 is gradually shortened, so the higher order entropies are gradually equal. In our simulations, we generate a set of cross correlated ARFIMA series with $N = 4096$. The following figure presents ARFIMA series, and the d parameter is fit to 0.1 to 0.4. It can be seen from the Fig.5 that the series affected by t is obvious. When, we can really capture the dynamic properties caused by different parameters. High order entropy can capture the asynchronous changes of information between systems, identify the differences of stochastic markets in different parameters.

Fig.5 interprets how entropy values change if t and q change at $\tau = 4$ correspondingly. These two series gradually grow up as q increases. We conclude that fractional order adjustment is highly sensitive to signal evolution, which helps to describe the complexity of dynamic systems. In order to illustrate more clearly, the effect of parameters on the t -entropy is shown in Fig.9.

Fig. 3: The mean and variance measures of ARFIMA1 and ARFIMA2 series at different scales. Curve Var-ARFIMA1 and curve Var-ARFIMA2 almost overlap.

Table 2: The mean and variance measures of Logistic-map series under different series under different parameters.

	scale 2	scale 3	scale 4	scale 5	scale 6	scale 7	scale 8	scale 9	scale 10
Avg-R=3.5	1.31	0.65	0.51	0.44	0.39	0.36	0.34	0.32	0.31
Avg-R=3.6	1.31	0.65	0.51	0.44	0.39	0.36	0.34	0.32	0.31
Avg-R=3.7	1.53	0.69	0.46	0.35	0.29	0.24	0.22	0.20	0.19
Avg-R=3.8	1.27	0.54	0.34	0.24	0.20	0.17	0.15	0.14	0.13
Avg-R=3.9	1.02	0.33	0.21	0.15	0.12	0.09	0.08	0.07	0.06
Var-R=3.5	10.09	0.01	5.69E-09	2.79E-09	7.16E-13	7.08E-13	7.09E-13	3.66E-13	6.49E-16
Var-R=3.6	10.09	0.01	5.69E-09	2.79E-09	7.16E-13	7.08E-13	7.09E-13	3.66E-13	6.49E-16
Var-R=3.7	2.5483	0.7474	0.3438	0.1736	0.0826	0.0448	0.0186	0.0071	0.0098
Var-R=3.8	1.5311	0.4583	0.1887	0.0758	0.0419	0.0195	0.0086	0.0044	0.0016
Var-R=3.9	1.0037	0.2669	0.0895	0.0441	0.0157	0.0062	0.0042	0.0037	0.0018

3.2 Logistic-map

To test the effectiveness of the proposed algorithm, we apply it to the Logistic map [32], a polynomial mapping that is widely utilized in the publications related to complex, chaotic and dynamical systems [33,34],

$$x_{n+1} = Rx_n(1 - x_n). \quad (18)$$

Here, x_n expresses the ratio of existing population to the maximum possible population that is a number limited between 0 and 1. The parameter R regulates the performance of the samples. For example, $R = 3.5$ is a particular case with periodic form. For $R > 3.5$, it is possible to evolve to chaotic dynamics

In our study, we use 4000 samples of data to acquire reliable test results with parameter values $R \in [3.5, 3.9]$ in Fig.8-10. Fig.8 assesses the performance of $\alpha_{X, Osc}(I_\tau(k))$ under the control of different R in the case of different scales, respectively. In order to prove a characteristic expression of invariance property, Fig.9 and Tab.2 display the mean values and variance of $\alpha_{X, Osc}(I_\tau(k))$ calculated by Logistic map.

The change of Fig.8(a) and Fig.8(b) is larger than that of Fig.8(c) and Fig.8(d), which shows that the fluctuation of the series is more intense in small scale. In Fig.9 and Tab.2, when the scale is greater than 4, the average of different series tends to be stable. Moreover, with the increase of R , the mean value gradually increases, which may be related to the chaotic degree of the series. Meanwhile, with the increasing scale, the

Fig. 4: t-Entropy calculated by ARFIMA1 and ARFIMA2 at different t and τ .

Fig. 5: t-Entropy calculated by ARFIMA series at different t and d .

Fig. 6: t-Entropy calculated by ARFIMA1 and ARFIMA2 at different t and q .

Fig. 7: t-Entropy calculated by ARFIMA1 and ARFIMA2 at different t and q .

curves of variance gradually overlap into a straight line. It shows that although the amplitude of oscillations naturally decreases with τ , the range of the roughness exponents stabilizes across the scales, which indicates a characteristic expression of scale-invariance property. Fig.10 reveals the influence of t on entropy at different scales. The entropy greatly varies for t ranging from 1 to 10, and the maximum value is obtained when t is 1. It is impossible to distinguish among different sequences when $t = 1$.

3.3 Results for financial time series

3.3.1 Data

In relatively mature stock markets, stock price can reflect the operation of macro-economy. In this paper, we analyze the stock indices from different countries and regions by $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$, MFRTE to explore the fluctuation patterns of the stock markets. We utilize the daily closing prices of three indices from different stock markets, including the Americas stock indices: Standard & Poors 500 index (GSPC), Asian-Pacific: Hang Seng Index (HSI) and European stock markets: Deutscher Aktien Index(DAX). Here, we choose the daily closing values of the above three stock markets from January 1, 2001, to December 31, 2018. We exclude the asynchronous datum and then reconnect the remaining parts of the original series to obtain the same length time series. The total records are 4400 days approximately. The log returns of the three indices are collected in Fig.14. We use the logarithmic rate of return to express the volatility of the stock market.

3.3.2 Results

Figs.12-14 depict the $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ at various scales. This scale invariance becomes clearly discernible when the scale is at four. To sum up, when the scale is small, the difference in the roughness grain exponents is large and fluctuates within the range of $[4,10]$. Moreover, the roughness grain exponents are observed to be the most confused at the scale of 2, fluctuating greatly when the scale is small, while gradually decreasing, aggregating within the range of $[1,2]$ and tending to be stable, when the scale becomes larger. Specifically, we find the scale invariance particularly is obvious for the three stock markets at scales 8 and 12. As can be seen from Fig.11, an economic crisis have tremendous consequences for our global security, resulting in sharp fluctuations in the global stock market in 2018. $\alpha_{X,Osc}(I_\tau(k))$ can capture the internal information of the stock market, so as to fully and effectively identify the fluctuations of the stock market, and provide effective information for the prediction of economic crisis.

Fig. 8: $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ of Logistic map series at different scales.

Fig. 9: The mean and variance measures of Logistic-map series at different scales. Curve $R=3.5$ and Curve $R=3.6$ almost overlap.

When $t = 1$, the high order entropy of the three stock markets all reach the maximum. Fig.15 represents the variations of high order entropy at $\tau = 4$. As it can be observed, the increase of t causes a gradual decrease of entropy. Obviously, when $t = 1$, three financial markets belonging to different regions are almost at the same point and cannot be distinguished. With the increase of t , the decrease of GSPC is more obvious than the other two stocks, and it tends to be stable when $t > 6$. DAX and HSI are always intersecting in the range of t from 1 to 9. When $t > 10$, we can distinguish them very well. Multiscale high-order entropy provides a very good idea for us to describe more complex dynamic systems. It can be seen from the figure that the change of GSPC affected by t is more obvious. The complexity of the US market is much greater than that of the stock markets of other countries. High order entropy can capture the asynchronous changes of information between systems, identify the differences of stock markets in different countries.

Fig.16 illustrate the t -Entropy and S_q for the HSI, GSPC and DAX at two different parameters q and t , respectively. Again, we are able to verify the behavior that is similar to the one indicated previously. From the above section, we can see that when the scale is 4, the roughness grain exponents tends to be stable.

We perceive that S_q demonstrated a high sensitivity to the fluctuation in the series, revealing apparent variations only for q values that were closer to one. With an increase in the value of q , and when q is valued between 0.1 and 0.5, the fluctuation in entropy became more obvious and the range of change was observed to be wider. Subsequently, with the increase in the value of q from 0.6 to 1, the range of change was perceived to gradually weaken.

4 Conclusion

In this paper, the fractional representation and the measurement of multiscale high-order entropy by roughness grain exponents are explored. The generalized fractional high order entropy is first adopted with escort distribution by roughness grain exponents definition. Using the algorithm described, we can estimate the fluctuation at each scale that describes the statistical repartition of the roughness exponents. The combination of multi-scale and the roughness grain exponents enables us to recognize that when the scale is greater than 4, the roughness grain exponents get stable. When $1 \leq t \leq 10$, the high order entropy decreases obviously, which is helpful for us to study the complexity of dynamic events. In addition, multiscale high-order entropy performs very well in distinguishing different time series. Particularly, the highly sensitive degree of fraction studied in this paper can also describe the complexity of dynamic systems.

Fig. 10: t -Entropy variation calculated by Logistic-map series at different t

Fig. 11: The log returns of the 3 indices from January 1, 2001 to June 5, 2018.

Fig. 12: $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ of HSI at different scales.

Fig. 13: $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ of GSPC at different scales.

Fig. 14: $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ of DAX at different scales.

Fig. 15: High order graph entropy for different parameters t at scale 4

Fig. 16: Fractional high order graph entropy for different parameters t and q at scale=4.

5 Statement

The data used in this paper are available.

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Conflict of interest

The authors declare that they have no conflict of interest. The data in the manuscript is available.

References

- [1] I. Ivanova, O. Strand, D. Kushnir, L. Leydesdorff, Economic and technological complexity: A model study of indicators of knowledge-based innovation systems, *Technological forecasting and social change* 120 (jul.) (2017) 77–89.
- [2] R. Siferd, P. Maybeck, Monte carlo approach for fatigue and damage calculation of nonlinear dynamical systems, *Chaos* 30 (7) (2020) 073135.
- [3] M. T. Martin, A. Plastino, O. A. Rosso, Generalized statistical complexity measures: Geometrical and analytical properties, *Physica A Statistical Mechanics and Its Applications* 369 (2) (2006) 439–462.
- [4] M. Xu, P. Shang, S. Zhang, Multiscale rényi cumulative residual distribution entropy: Reliability analysis of financial time series, *Chaos Solitons and Fractals* 143 (2021) 110410.
- [5] M. Costa, A. L. Goldberger, C. K. Peng, Multiscale entropy analysis of complex physiologic time series, *Physical Review Letters* 89 (6) (2007) 705 – 708.
- [6] M. Barreiro, A. C. Marti, C. Masoller, Inferring long memory processes in the climate network via ordinal pattern analysis, *Chaos* 21 (1) (2011) 93.
- [7] C. Steglich, S. M. Pearson, Dynamic networks and behavior: Separating selection from influence, *Sociological Methodology* 40 (2015) 329–393.
- [8] J. T. Machado, F. B. Duarte, G. M. Duarte, Analysis of stock market indices through multidimensional scaling, *Communications in Nonlinear Science and Numerical Simulation* 83 (11) (2011) 2015–2029.
- [9] T. Bian, Y. Deng, Identifying influential nodes in complex networks: A node information dimension approach, *Chaos* 28 (4) (2018) 043109.
- [10] A. M. Mathai, H. J. Haubold, Pathway model, superstatistics, tsallis statistics, and a generalized measure of entropy, *Physica A Statistical Mechanics and Its Applications* 375 (1) (2006) 110–122.
- [11] A. Mowshowitz, Entropy and the complexity of graphs. ii. the information content of digraphs and infinite graphs, *Bull Math Biophys* 30 (2) (1968) 225–240.
- [12] C. E. Shannon, W. Weaver, The mathematical theory of communication, *Bell Labs Technical Journal* 3 (9) (1950) 31–32.
- [13] Joseph, Bryngelson, Thermodynamics of chaotic systems: An introduction, *Journal of Statistical Physics*.

-
- [14] N. Rashevsky, Life, information theory, and topology, *Bulletin of Mathematical Biophysics* 17 (3) (1955) 229–235.
 - [15] E. Trucco, A note on the information content of graphs, *Bulletin of Mathematical Biophysics* 18 (2) (1956) 129–135.
 - [16] MatthiasDehmer, A novel method for measuring the structural information content of networks, *Journal of Cybernetics* 39 (8) (2008) 825–842.
 - [17] B. Zhang, P. Shang, Cumulative permuted fractional entropy and its applications, *IEEE Transactions on Neural Networks and Learning Systems* PP (99).
 - [18] J. Koerner, A novel method for measuring the structural information content of networks, *Transactions of the Sixth Prague Conference on Information Theory* 39 (8) (1973) 411–425.
 - [19] L. B. Kier, Information theoretic indices for characterization of chemical structures, *Journal of Pharmaceutical Sciences* 73 (9) (2010) 1336–1336.
 - [20] J. Li, P. Shang, Financial time series analysis using total-capen and avg-capen with cumulative histogram matrix, *Communications in Nonlinear Science and Numerical Simulation* 63 (2018) 239–252.
 - [21] J. Kim, G. Kim, S. An, Y. K. Kwon, S. Yoon, Entropy-based analysis and bioinformatics-inspired integration of global economic information transfer, *Plos One* 8 (1) (2013) e51986.
 - [22] L. Kristoufek, M. Vosvrda, Commodity futures and market efficiency, *Energy Economics* 42 (March 2014) (2014) 50–57.
 - [23] L. Zunino, M. Zanin, B. M. Tabak, D. G. Prez, O. A. Rosso, Complexity-entropy causality plane: A useful approach to quantify the stock market inefficiency, *Physica A Statistical Mechanics and Its Applications* 389 (9) (2010) 1891–1901.
 - [24] R. Gu, Multiscale shannon entropy and its application in the stock market, *Physica A Statistical Mechanics and Its Applications* 484 (2017) 215–224.
 - [25] O. Agrawal, Formulation of eulerclagrange equations for fractional variational problems, *Journal of Mathematical Analysis and Applications* 272 (1) (2002) 368–379.
 - [26] M. Xu, P. Shang, Multiscale fractional order generalized information of financial time series based on similarity distribution entropy, *Chaos* 29 (2019) 053108.
 - [27] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity - An Introduction to Mathematical Models*, 2010.
 - [28] C. M. Ionescu, The human respiratory system, *Series in Bioengineering* iii (1-4) (2013) 499–505.
 - [29] P. Loiseau, Médigue, Large deviations estimates for the multiscale analysis of heart rate variability, *Physica A Statistical Mechanics and Its Applications* 391 (22) (2012) 5658–5671.
 - [30] M. R. Ubriaco, Entropies based on fractional calculus, *Physics Letters A* 373 (30) (2009) 2516–2519.
 - [31] B. Podobnik, D. Horvati, A. Lam Ng, H. E. Stanley, P. C. Ivanov, Modeling long-range cross-correlations in two-component arfima and fiarch processes, *Physica A Statistical Mechanics and Its Applications* 387 (15) (2012) 3954–3959.
 - [32] I. Pan, A. Korre, S. Das, S. Durucan, Chaos suppression in a fractional order financial system using intelligent regrouping pso based fractional fuzzy control policy in the presence of fractional gaussian noise, *Nonlinear Dynamics* 70 (4) (2012) 2445–2461.
 - [33] H. Azami, J. Escudero, Improved multiscale permutation entropy for biomedical signal analysis: Interpretation and application to electroencephalogram recordings, *Biomedical Signal Processing and Control* 23 (2016) 28–41.
 - [34] K. Chon, C. G. Scully, S. Lu, Approximate entropy for all signals., *Engineering in Medicine and Biology Magazine IEEE* 28 (6) (2009) 18–23.

6 Supplementary material

Fig. 17: $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ of ARFIMA1 at different scales.

Fig. 18: $OscX(I_\tau(k))$ and $\alpha_{X,Osc}(I_\tau(k))$ of ARFIMA2 at different scales.