

# Estimating the critical points of an epidemic: An application of non-epidemiological logistic growth model for COVID-19

Ivan Bezerra Allaman (✉ [iballaman@uesc.br](mailto:iballaman@uesc.br))

State University of Santa Cruz <https://orcid.org/0000-0003-0883-0466>

Enio Galinkin Jelihovschi

State University of Santa Cruz <https://orcid.org/0000-0002-7286-1198>

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## Research Article

**Keywords:** point of asymptote, point of acceleration, point of deceleration, inflection point, accumulated cases, virus

**Posted Date:** October 15th, 2020

**DOI:** <https://doi.org/10.21203/rs.3.rs-93146/v1>

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**Version of Record:** A version of this preprint was published at Brazilian Journal of Biometrics on June 10th, 2022. See the published version at <https://doi.org/10.28951/bjb.v40i2.576>.

# Estimating the critical points of an epidemic: An application of non-epidemiological logistic growth model for COVID-19

Ivan Bezerra Allaman<sup>1,\*</sup> and Enio Galinkin Jelihovschi<sup>1,+</sup>

<sup>1</sup>State University of Santa Cruz, Department of exact and technological sciences, Ilhéus/BA, 45662-900, Brazil

\*iballaman@uesc.br

+these authors contributed equally to this work

## ABSTRACT

Epidemiological models have become a very important tool in understanding an epidemic development, mainly because they help researchers in finding good and new strategies in their fight against its spread. Several models have been proposed up to now, some are mathematical others apply models from other areas. The SIR and SEIR among others, mainly focus on the variable response and on epidemiological parameters as the basic reproduction number ( $R_0$ ) and infection rate per unit of time, nevertheless they do not focus on the variable 'time'. We propose the use of the variable time using the logistic model as it is generally used to describe the growth of animals. This model is important because it allows the estimation of the points of acceleration and deceleration, the point of maximum growth and the asymptotic point of the epidemic. This is only possible when the epidemic curve is stable and has an 'S' shape. In this work we use the variable 'accumulated cases' of China and Italy and point out the main socioeconomic facts that occurred in each period of the estimated critical points from the logistic growth model.

## Introduction

The first official statement by the World Health Organization (WHO) on COVID-19 pandemic acted as a trigger on the scientific community to start a real effort to study and publish papers in order to provide a way of understanding what has been happening and therefore help and guide the fight against the virus. In this way, new mathematical models like Sales<sup>1</sup> and Shaikh<sup>2</sup> added to the well-known epidemiological models SIR, SEIR, Generalized Logistic Model, Generalized Richards Model and Generalized Growth Model.

The logistic models, among the mathematical models proposed for epidemics, stand out because they present an 'S' format and, therefore, make it possible to obtain interesting information about the curve such as absolute and relative growth rate, points of acceleration and asymptote whose parameters may have biological interpretation, see Seber<sup>3</sup>. All logistic models proposed in the literature up to now are those derived from the Richards model.

In the case of epidemics, Chowell<sup>4</sup> used, in addition to the generalized growth model and the generalized Richards model, the logistic model to characterize the contagiousness, forecast patterns and final burden of Zika Epidemics. However, the authors were vague about the logistic model used. On the other hand, Wu<sup>5</sup>, put it explicitly the parameters of the logistic model used to describe and characterize an epidemic, nonetheless the authors expressed the logistic model as a solution of differential equations, as it is commonly used in epidemiological models. Only Zhou<sup>6</sup> presented the logistic growth model in the form of a function which is the closest to the model proposed in this work.

All models proposed so far studied the dynamics of virus transmission given some specific scenario, none of these studies presented a proposal to determine when acceleration or deceleration occur in the number of individuals infected or when the number of those infected will stabilize. Time is a very important factor. We do not try to predict the future given predetermined settings, we start from an already established scenario, and then adjust a logistic model in order to estimate the critical points of this model and thereafter try to understand how political, social or environmental factors are related to such critical points of the proposed curve.

Our proposal is to place the variable time as the main factor using the classic logistic model. Derived from Richards model, this model is commonly used to describe the growth of animals<sup>7</sup>. In order to avoid any confusion with the logistical models used in epidemiological studies also derived from the Richards model, we will call our model a 'non-epidemiological logistic growth model'.

In short, the aim of our work is to estimate critical points of the non-epidemiological logistical growth such as the point of maximum acceleration, inflection point, point of maximum deceleration and asymptotic deceleration point using the number of

accumulated cases as the dependent variable. Furthermore, we use the main stock exchange indexes and some main political facts in China and Italy to show possible relationships to the critical points of the curve. China and Italy were chosen due to the fact that they have shown a steady epidemic curve by the time of our study.

## Results

The values of the arima model for each country are shown in table 1.

Table 2 shows the point estimates and confidence intervals of parameters and critical points. The point of **map** has been estimated at 34.93 days for China and 50.92 days for Italy counting from the first official case reported for both countries. The estimated value of the point of **ip** was 41.68 and 65.53 days for China and Italy respectively. For the point of **dmp**, the estimates were 48.43 days for China and 80.14 days for Italy. Finally, the point estimated of **adp** for China and Italy were 57 and 94 days respectively.

Figure 2 places the stock exchange index and the percentage of deaths during the period studied with each period established by the estimated critical points through the logistic model for both China and Italy.

We highlight below the main interventions in each period comprising the critical points of the logistic model for China.

1. Period from the beginning of the pandemic to the **map** point (35 days).
  - (a) The Huanan Seafood Wholesale Market in Wuhan city was closed for environmental sanitation and disinfection<sup>8</sup>;
  - (b) Beginning of the Chinese festival<sup>9</sup>;
  - (c) Prohibition of Wuhan residents to leave the city and closing off the public transport in Wuhan and Hubei province<sup>10</sup>;
2. The facts mentioned between the period of the point of **map** and the point of **ip** (7 days), were taken from Times<sup>11</sup>.
  - (a) Wuhan began to classify patients and adopted centralized patient management, dedicated hospital built to fight against COVID-19;
  - (b) Huoshenshan makeshift hospital was put into use;
  - (c) The first 3 make shift hospitals were put into use;
3. A single fact stands out between period of the points of **ip** and **mdp** (6 days)
  - (a) Following a change in the diagnosis criteria of confirmed cases, the Chinese mainland reported 15, 152 new confirmed COVID-19 cases, including 13,332 in Hubei Province.
4. Among the points of **mdp** and **adp** period (9 days), the following facts stand out :
  - (a) The number of newly discharged patients began to surpass the new confirmed case;
  - (b) Provinces and regions across the Chinese mainland began to downgrade the COVID-19 emergency response;
  - (c) The China-WHO joint expert team held a press conference in Beijing;
5. Finally, from point of **adp** until 2020-06-30 (126 days), the following facts stand out:
  - (a) New local cases in the Chinese mainland declined to single digits;
  - (b) Hubei Province gradually revoked the outbound travel restrictions;
  - (c) National Health Commission began to daily report the asymptomatic cases;
  - (d) All inbound travelers to China were mandated to undergo a compulsory nucleic acid test;
  - (e) Leading group of the CPC (Central Committee for Novel Coronavirus Prevention and Control published guidelines for expediting work resumption with strict adherence to epidemic control;
  - (f) Wuhan lifted outbound travel restrictions;

- (g) Beijing and its nearby provinces downgraded the emergency response to the second-level;
- (h) Hubei Province downgraded its emergency response to the second-level;
- (i) The State Council released Guidelines on conducting COVID-19 prevention and control on an ongoing basis;

The Dow Jones Shanghai, which is the main index of the Chinese stock exchange, has shown a drop until to the point of **map** (figure 2). Then, the index rose until shortly after the point of **adp**, the stabilization point of the number of people infected, then a sharp drop for 27 days and a resumption of growth until the last evaluation day.

In Italy, the main facts in the periods as shown in figure 2, were selected from the wikipedia website<sup>12</sup>.

1. From the beginning until the point of **map** (51 days), the following facts stand out:
  - (a) The government announced a new decree imposing the quarantine of more than 50,000 people from 11 municipalities in Northern Italy;
  - (b) Schools were closed in ten municipalities in Lombardy, one in Veneto and one in Emilia Romagna;
  - (c) All sporting events in the regions of Lombardy and Veneto, which included three Serie A football matches in those regions, as well as one in Piedmont, that were to be played the following day;
  - (d) The Ministry of Health announced that it had engaged 31 laboratories in Italy to carry out the analysis of swabs from suspected COVID-19 cases;
  - (e) Multiple regions in Italy such as Liguria, Trentino Alto Adige, Friuli Venezia Giulia, Abruzzo and Marche decided to close all schools and universities for two days to a week;
  - (f) The Italian government imposed the shutdown of all schools and universities nationwide for two weeks as the country reached 100 deaths from the outbreak;
  - (g) The government announced that all sporting events in Italy would be cancelled until at least 3 April, but the ban does not include Italian clubs or national teams participating in international competitions;
  - (h) The Ministry of Health ordered tighter regulations on free movement;
  - (i) Conte announced further restrictions within the nationwide lockdown, by halting all non-essential production, industries and businesses in Italy, following the rise in the number of new cases and deaths in the previous days;
2. Between the point of **map** and point of **ip** (15 days) the following facts stand out:
  - (a) The government extended the period of lockdown until 13 April, with health minister Speranza saying that the restrictive measures had begun to yield the first positive results;
3. Among the point of **ip** and point of **mdp** points (14 days), the following facts stand out:
  - (a) A government's decree closed all Italian ports until 31 July, stating that they do not ensure the necessary requirements for the classification and definition of 'safe place', established by the Hamburg Rules on maritime search and rescue;
  - (b) Conte announced the prolongation of the lockdown until 3 May, as well as the reopening of some businesses like bookshops and forestry;
4. Between the point of **mdp** and the point of **adp** (13 days), a single fact stands out:
  - (a) The Prime Minister announced a starter plan for the so-called 'phase 2', that would start from 4 May;
5. Finally, from the point of **adp** until 2020-06-30 (59 days), the following facts stand out:
  - (a) Conte announced the government plan for the easing of restrictions - Starting from 18 May most businesses could reopen, and free movement was granted to all citizens within their Region;
  - (b) Movement across Regions was still banned for non-essential motives;

The main index of the Italian stock exchange (FTSE MIB) dropped until a few days before the first critical point of **map**, then remained stable until the last day of our study.

## Discussion

The critical points estimated through the classic non-epidemiological logistic model are the only ones which support the researcher to speak in terms of ‘acceleration’ and ‘asymptote’ of a given epidemic. Although many studies mention the terms transmission speed and ‘curve stabilization’, none of them make in a suitable statistical way.

The model hitherto proposed, was intended to describe the outbreak of the epidemic, that is, how the variables such as number of infected people, number of people recovered should behave as a function of epidemiological parameters. Some researchers applied new models, as in the case of Sales<sup>1</sup>, who makes an application of the prey-predator model proposed by Loka and Volterra for COVID-19, and Giordano<sup>13</sup> that uses the model of SIDARTHE to discriminate between detected and undetected cases of infection and between different severity of illness (SOI), non-life-threatening cases (asymptomatic and paucisymptomatic; minor and moderate infection) and potentially life threatening cases (major and extreme) that require ICU admission. Shaikh<sup>2</sup>, proposed a mathematical model using fractional derivative. Established epidemiological models were also applied like SIR<sup>14–16</sup>, SEIR<sup>6,17,18</sup>, Generalized Logistic Model<sup>5</sup>, Generalized Richards Model<sup>5</sup> and Generalized Growth Model<sup>5</sup>. Our approach is unique because it estimates when (time) the main critical points of an epidemic occur using a simple model, that is, without differential equations that requires deep mathematical skills for the user. The model proposed in this paper allows us to answer questions like: ‘How many days are necessary to stop the epidemic?’,<sup>19</sup>. In China, it lasted 57 days until the point of **adp**, in Italy, 93 days.

Some other questions may arise and any researcher interested in the field could use the tool proposed above to try to answer them. For instance, why the time between the beginning of the outbreak and the point of **map** in China was shorter (35 days) than in Italy (51 days)? As already pointed out by other studies, the lockdown was indeed an effective measure in the battle against the spread of an epidemic<sup>10</sup>. In China, the lockdown was implemented 23 days, in Wuhan, epicenter of the pandemic at that time, after the first case officially reported. In Italy, the lockdown was implemented 37 days after the first occurrence, in Lombardy together with 14 additional northern and central provinces like Piedmont, Emilia-Romagna, Veneto and Marche. In addition to the lockdown, air pollution may have been another major factor in Italy as showed in Coccia<sup>19</sup>. Another interesting result was the proximity between the lockdown and the point of **map**. In China, this time was 12 days while in Italy it was 14 days. Does this behavior remain the same in other countries?

The logistics growth model complements all the studies already proposed for the understanding of a given epidemic, as it puts the variable time as the most important one. Our proposal has the limitation of being applied only after the epidemic has already been stabilized. On the other hand, it allows the estimation of more precise parameters that have not yet been addressed in the literature and, does not make conjectures about the future and its various scenarios as is the case with epidemiological models.

For instance, after estimating the critical points of the logistic growth model, the researchers could try to understand why the time until the peak of infection or the time until the stabilization were different among countries; what health, social and economics measures or which environmental factors may contribute to decrease or increase the time among the critical points.

## Methods

### Data collection

The database used in this paper is available on the website [githubusercontent](#).

Since the data are updated daily, we considered the day 2020-06-30 as deadline. The countries included in the analyzes are China and Italy, because those countries managed to stabilize contamination by SARS COVID-19 by the time of our deadline. We must point out that the new cases obtained on 2020-02-13 (China) and 2020-06-20 (Italy) were considered very atypical compared to the values of the previous day and the day after. Thereat, these values were replaced by the average of the previous day with the day after.

As for the data on the indexes of the Chinese stock exchanges at Dow Jones in Shanghai and at FTSE Mib in Italy, those indexes were obtained from the website [www.investing.com](#). Since the stock exchange doesn't work on weekends and holidays, the indexes on these days have been replaced by the immediately preceding days so that the evaluated period of the COVID 19 cases coincided with the stock indexes.

### Estimation of the Autoregressive Integrated Moving Average (ARIMA) model

As the accumulated variable (object of this study) is a consequence of the daily cases and this in turn is characterized as a temporal series whose values and residues are correlated over time, we adjusted an arima model for all countries using the *auto.arima* function **forecast** package version 8.12<sup>20</sup>. However, since this model requires an assumption of normality, homoscedasticity and independence, we made a boxcox transformation.

## Estimation of the logistic growth model

We first adjusted the arima model in order to determine the estimated accumulated values and then used them to adjust the logistic model. The reparametrization was:

$$f(x) = \frac{A}{1 + Be^{-kx}} \quad (1)$$

where,

A: is the parameter that represents the model's asymptote as  $x \rightarrow \infty$ ;

B: it is a parameter without biological interpretation;

k: is the parameter that represents the maturity index of an organism.

The parameters of the logistic model were determined using the function *nls* of the **stats** package version 4.0.0 using the Gauss-Newton algorithm. The maximum acceleration points (**map**), inflection point (**ip**), maximum deceleration point (**mdp**) and asymptotic deceleration point (**adp**) were determined from model 1. The points of **map** and **mdp** were determined using the third derivative, the point of **ip** was by using the second derivative and the point of **adp** by using the method of non-significant difference<sup>21</sup>. The equations of each critical point are:

1. For **map** and **mdp** we first use the following equation,

$$\frac{d^3}{dx^3} = \frac{ABk^3 e^{kx} (e^{2kx} - 4Be^{kx} + B^2)}{(e^{kx} + B)^4} \quad (2)$$

then finding the points where the derivative is equal to zero.

$$\text{map} = \frac{\ln(2B - \sqrt{3}B)}{k} \quad (3)$$

and **mdp**

$$\text{mdp} = \frac{\ln(\sqrt{3}b + 2b)}{k} \quad (4)$$

2. For the **ip** we use,

$$\frac{d^2}{dx^2} = -\frac{ABk^2 e^{kx} (e^{kx} - B)}{(e^{kx} + B)^3} \quad (5)$$

then finding the points where the derivative is equal to zero.

$$\text{ip} = \frac{\ln(B)}{k} \quad (6)$$

3. The **adp** is found by using the non-significant difference method. We use the algorithm shown below to explain how it works.

(a) Calculate  $\Delta_{x_i} = p \cdot A - f(x_i)$  where A is the estimated parameter of the logistic model asymptote, p is the proportion of the asymptote, a fixed value between 0 and 1 and  $f(x_i)$  is the value of the logistical function (eq. 1) at point  $x_i$ .

(b) Estimate the variance of  $\Delta$  that is  $S^2(\Delta) = \text{diagonal}(f(w)' \cdot \Sigma \cdot f(w))$  where,

$$f(w) = [\partial\Delta/\partial A \quad \partial\Delta/\partial B \quad \partial\Delta/\partial k] \quad (7)$$

and  $\Sigma$  is the of variance-covariance matrix obtained from the estimated logistics regression model.

(c) For each  $x_i$  find the statistic  $T = \Delta_{x_i} / \sqrt{S^2(\Delta_{x_i})}$ .

(d) Set a level of significance  $\alpha$  and evaluate the quantile  $t_\alpha$  of the student's distribution with  $n - p$  degrees of freedom where  $n$  is the sample size and  $p$  is the number of parameters of the model.

(e) Compare until  $T_{x_i} < t_\alpha$ . The value of  $x_i$  where this occurs for the first time is the asymptote point.

Figure 1 identifies the critical points of the first and second derivative from the logistic model.

The 95% confidence intervals were calculated for all parameters and critical points<sup>3,22</sup>.

All analyzes were performed using the R software<sup>23</sup>.

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## Author contributions statement

Ivan B. Allaman conceived the idea, performed the data analysis and wrote the article. Enio G. Jeliovski improved and corrected the wording and translated the article.

## Additional information

The author(s) declare no competing interests.

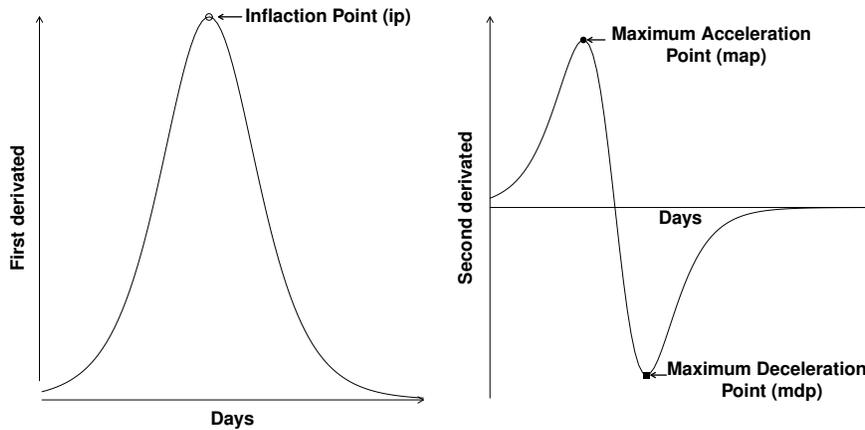
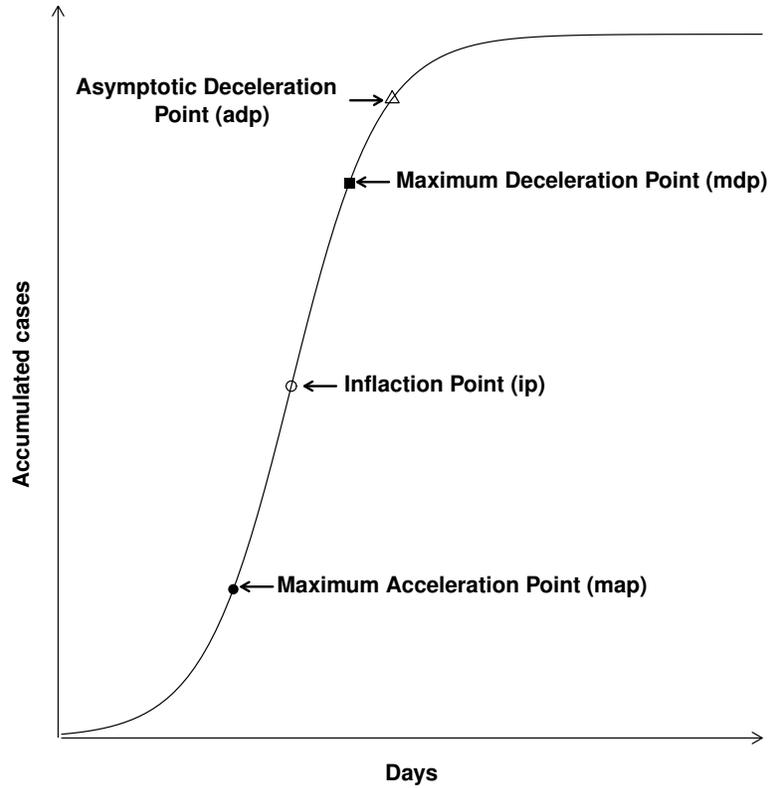
p	d	q	Country
4	1	3	China
1	2	2	Italy

**Table 1.** Values of p, d and q from the arima model of the daily cases for China and Italy

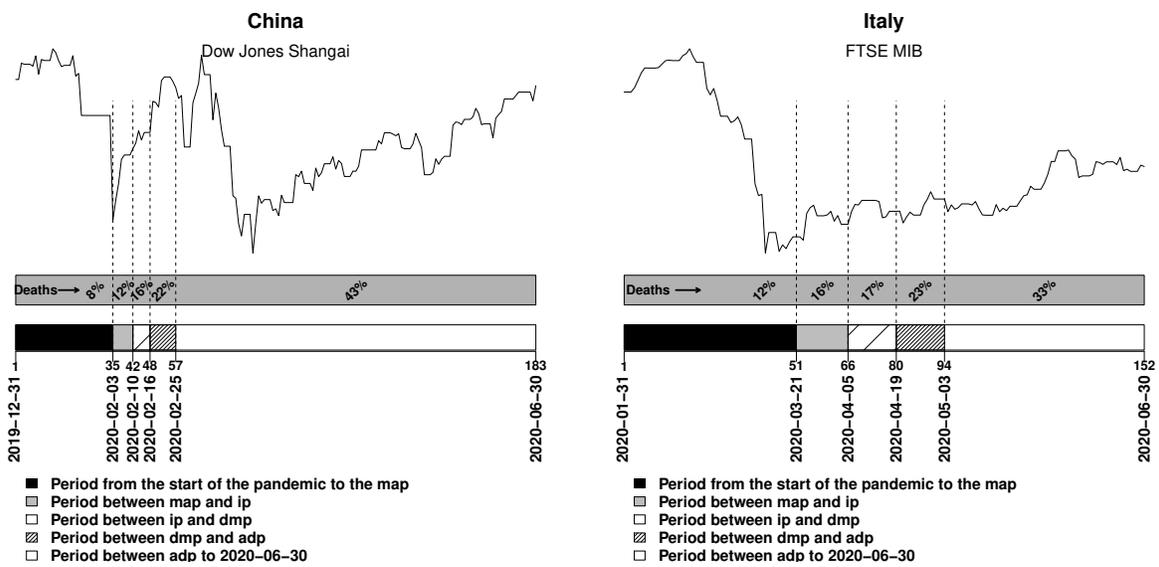
Country	Parameters	Fitted	Confidence interval (95%)	
			Lower	Upper
China	A	73273.75	73048.35	73499.35
	B	3406.8	2614.1	4498.81
	k	0.20	0.19	0.20
	amp	34.93	34.69	35.17
	ip	41.68	41.52	41.84
	dmp	48.43	48.18	48.68
	adp	57.00		
Italy	A	240775.4	238842.12	242731.6
	B	368.04	297.6	460.66
	k	0.09	0.09	0.09
	amp	50.92	50.36	51.48
	ip	65.53	65.11	65.96
	dmp	80.14	79.43	80.85
	adp	94.00		

map-maximum acceleration point,mdp-maximum deceleration point,ip-inflection point,adp-asymptotic deceleration point

**Table 2.** Critical points and 95 % confidence interval estimates adjusted for China and Italy



**Figure 1.** Curves of the logistic model, the first derivative, the second derivative and their respective critical points



**Figure 2.** Stock exchange index (line graph), percentage of deaths and periods comprising the critical points of the logistic growth model: maximum acceleration point (map), inflation point (ip), maximum deceleration point (mdp) and asymptotic deceleration point (adp)

# Figures

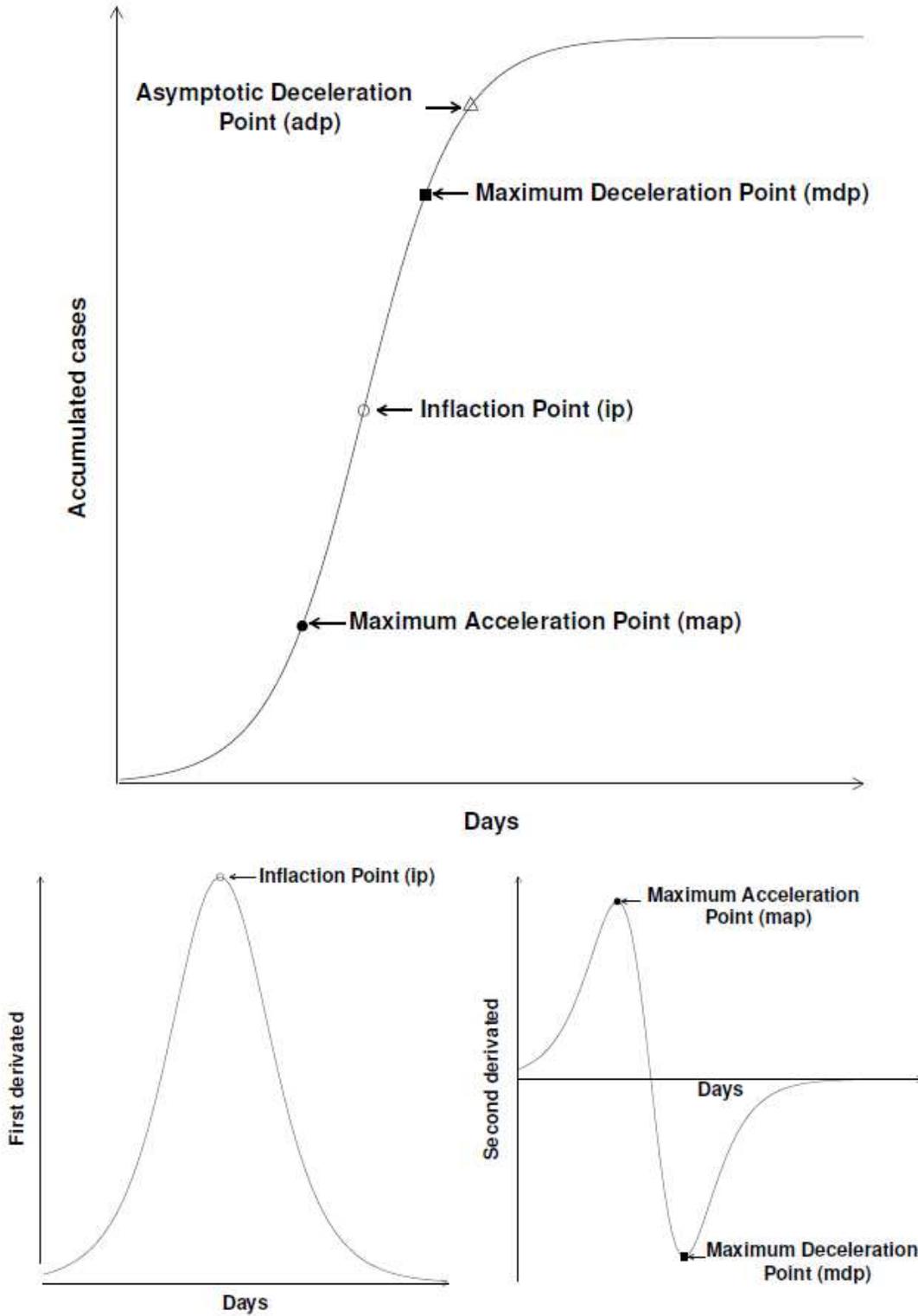
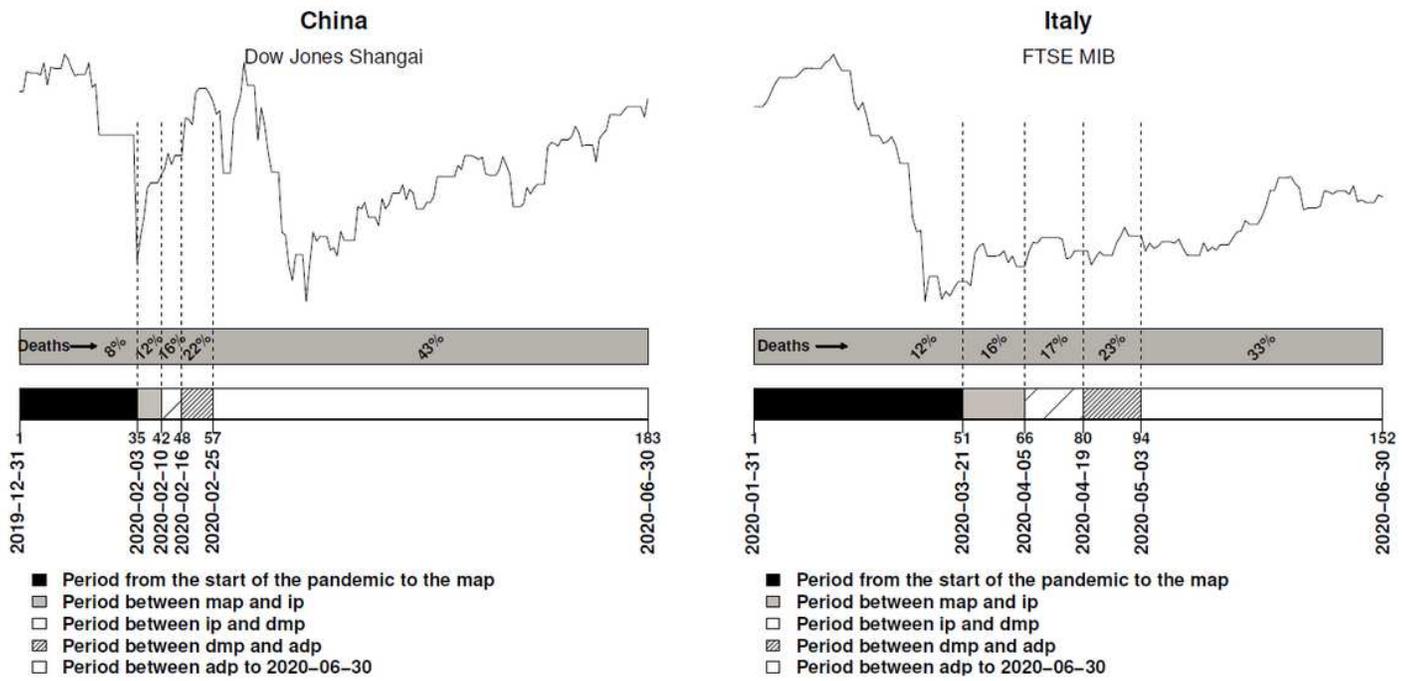


Figure 1

Curves of the logistic model, the first derivative, the second derivative and their respective critical points



**Figure 2**

Stock exchange index (line graph), percentage of deaths and periods comprising the critical points of the logistic growth model: maximum acceleration point (map), inflection point (ip), maximum deceleration point (mdp) and asymptotic deceleration point (adp)