

Closed form solution in buckling optimization problem of twisted rod

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Abstract

The applications of this method for stability problems are illustrated in this manuscript. In the context of twisted rods, the counterpart for Euler's buckling problem is Greenhill's problem, which studies the forming of a loop in an elastic bar under torsion (Greenhill, 1883). We search the optimal shape of the rod along its axis. A priori form of the cross-section remains unknown. For the solution of the actual problem the stability equations take into account all possible convex, simply connected shapes of the cross-section. Thus, we drop the assumption about the equality of principle moments of inertia for the cross-section. The cross-sections are similar geometric figures related by a homothetic transformation with respect to a homothetic center on the axis of the rod and vary along its axis. The distribution of material along the length of a twisted rod is optimized so that the rod is of the constant volume V and will support the maximal moment without spatial buckling. The cross section that delivers the maximum or the minimum for the critical eigenvalue must be determined among all convex, simply connected domains. We demonstrate at the beginning the validity of static Euler's approach for simply supported rod (hinged), twisted by the conservative moment. The applied method for integration of the optimization criteria delivers different length and volumes of the optimal twisted rods. Instead of the seeking for the twisted rods of the fixed length and volume, we directly compare the twisted rods with the different lengths and cross-sections using the invariant factors. The solution of optimization problem for twisted rod is stated in closed form in terms of the higher transcendental functions. In the torsion stability problem, the optimal shape of cross-section is the equilateral triangle.

1. Isoperimetric inequality for a twisted rod with an arbitrary convex simply-connected cross-section

Consider a thin elastic rod with isotropic cross-section, twisted by couples applied at its ends alone (Greenhill's problem). The problems of optimizing stability of the simply supported twisted rod were studied numerically in (Banichuk, Barsuk, 1982a). The rod possessed the similarly shaped cross-sections with the varying cross-sectional area. More advanced problem of twisted and compressed rod was studied numerically in (Banichuk, Barsuk, 1982b). The optimization problem consisted in finding the distribution of cross-sectional areas that assigns the largest value to the critical moment causing a loss of stability and such that the constraint on the volume of material and the constraints on the admissible thickness of the rod were satisfied.

The problem of determining the compressed and twisted column of maximal efficiency was studied in (Spasic, 1993). Following the structural optimization theory, for a given load, the shape of the column of minimal weight was determined numerically.

The problem of the optimal design of columns under combined compression and torsion is investigated (Kruzelecki, Ortwein, 2012). A cross-sectional area varying along the axis of the column which leads to the maximal critical loading is sought. The varying cross-section was approximated by a function with free parameters. Alternatively, the varying cross-section was determined numerically using Pontriagin's maximum principle.

Generally speaking, for non-conservative problems the dynamic approach for stability investigation must be applied (Kim, Choi, 2011), (Kirillov, 2013) (Banichuk et al, 2020). However, Euler's static approach is proved to be valid for the considered boundary value problem with conditions of fixed axes of rotation, Cases 1 and 2 (Ziegler, 1977, §5.4, Table 5.1, Section 1.2, page 127). The twisting couple A retains its initial direction during buckling and the boundary conditions of fixed axes of rotation are satisfied (Ziegler, 1977, Cases 1 and 2 in Table 1.1, Section 1.2, page 5). Euler's static approach is shown to be valid also for an infinite, periodically simply supported rod. The corresponding boundary-value problems are neither self-adjoint nor conservative in the classical sense. However, it is a conservative system of the second kind (Leipholtz, 1974). Consider a thin elastic rod with isotropic cross-section, twisted by couples applied at its ends alone (Greenhill's problem) (Pastrone, 1992). According to Euler's theory, the magnitude of the critical moment is determined by the smallest positive eigenvalue Λ .

We search the closed-form analytical solutions for the optimal shape of the rod along its axis with the application of the variational methods. According to Euler's theory, the magnitude of the critical moment is determined by the smallest positive eigenvalue Λ . We demonstrate at the beginning the validity of static Euler's approach for simply supported rod (hinged), twisted by the conservative moment. The distribution of material along the length of a twisted rod is optimized so that the rod is of the constant volume V and will support the maximal moment without spatial buckling. The cross section that delivers the maximum or the minimum for the critical eigenvalue must be determined among all convex, simply connected domains. A priori form of the cross-section remains unknown. For the solution of the actual problem the stability equations take into account all possible convex,

simply connected shapes of the cross-section. Thus, we drop the assumption about the equality of principle moments of inertia for the cross-section. The cross-sections are similar geometric figures related by a homothetic transformation with respect to a homothetic center on the axis of the rod and vary along its axis. The area of some reference cross-section Ω is A_0 . The area of the cross-section with the coordinate x is $A(x)$, $-L \leq x \leq L$. The scaling dimensionless function is positive

$$a(x) = \frac{A(x)}{A_0} > 0, \quad 0 < a < 1. \quad (1)$$

The function $a(x)$ is the locally integrable function over the length of the rod ($-L \leq x \leq L$)

$$\int_{-L}^L a(x) dx = V. \quad (2)$$

For an arbitrary cross-section, the moments of inertia J_{yy}, J_{yz}, J_{zz} of the transverse cross-sectional area are the second power of (x) , such that $\alpha = 2$. The moments of inertia with respect to lines passing through a point on the neutral axis of bending and parallel to the axes y and z read:

$$J_{yy} = I_{yy}a^2(x), J_{yz} = I_{yz}a^2(x), J_{zz} = I_{zz}a^2(x), -L \leq x \leq L. \quad (3)$$

The constants I_{yy}, I_{yz} and I_{zz} denote the moments of inertia of the reference cross-section Ω .

The moments of inertia I_{yy}, I_{yz} and I_{zz} are the components of tensor of the second rank. For an arbitrary angle θ between the principal axis of inertia and y axis the moments of inertia are

$$\begin{aligned} I_{yy} &= I_1 \cos^2 \theta + I_2 \sin^2 \theta, \\ I_{yz} &= (I_1 - I_2) \sin \theta \cos \theta, \\ I_{zz} &= I_1 \sin^2 \theta + I_2 \cos^2 \theta. \end{aligned} \quad (4)$$

The angle θ remains constant along the axis of the rod.

The rod is twisted by conservative couples, such that the torque is constant over the axis of the rod. The bending moments in terms of curvatures of the rod are

$$M_z = EI_{yy}y'' + EI_{yz}z'', M_y = EI_{yz}y'' + EI_{zz}z''. \quad (5)$$

We chose the axes y and z in the direction of principal axes of inertia through the center of gravity ($\theta = 0$). The curvatures of the axis κ_z, κ_y in course of buckling are assumed to be small, such that the geometrically linear equations could be applied for the solutions. The bending moments in terms of curvatures κ_z, κ_y of the rod in the principal axes are:

$$\begin{aligned} M_z &= EI_1 \kappa_z, \\ M_y &= EI_2 \kappa_y, \end{aligned} \quad (6)$$

$$\kappa_z = y'',$$

$$\kappa_y = z''.$$

At first, we assume, that both moments of inertia are proportional to one function $j(x)$:

$$I_1(x) = \eta_1 j(x), \quad I_2(x) = \eta_2 j(x), \quad j(x) = k_\alpha a^\alpha.$$

The constants are positive values $\eta_1 > 0, \eta_2 > 0$.

The quantity a^α is proportional to the flexural rigidity, and the exponent α takes the values of 1, 2 and 3 (Banichuk, 1990). The case $\alpha = 2$ corresponds to a congruent change in the form of the cross-section.

The optimal convex shape of the simply connected cross-section with topological genus null was determined (Ting 1963). The topological genus of a surface (or Euler characteristic) is in essence the number of its "holes." The solution grounds on the following isoperimetric equation. Of all convex domains with the area A_Δ , the equilateral triangle yields the maximum of $I_1 I_2$:

$$\sqrt{I_1 I_2} \leq \frac{\sqrt{3}}{18} A_\Delta^2.$$

Thus, the rod with the cross-section in form of an equilateral triangle delivers the maximum for critical eigenvalue for all convex domains of the same cross-sectional area A_Δ . For the cross-section in form of the equilateral triangle $k_2 = \sqrt{3}/18 \approx 0.9622$.

For the circular cross-section the constant is $k_2 = (4\pi)^{-1} \approx 0.07957$... The rod with the circular, simply connected cross-section delivers correspondingly the minimum for critical eigenvalue. In these both cases $\eta_1 = \eta_2 = 1$.

Two other cases describe the situations in which the form of transverse cross-section undergoes the transformation such that one of the geometrical dimensions of the cross-section changes. For the technically important case of the thin-walled tubes with the variable thickness of wall $t(x)$ and the mean diameter of tube $D(x)$, the second moments of

$$I_1(x) = I_2(x) = \frac{\pi}{64} [(D + t/2)^4 - (D - t/2)^4] = \frac{\pi}{16} t D^3 + \frac{\pi}{64} D t^3 \approx \frac{\pi}{16} t D^3, \text{ for } t \ll D.$$

The case $\alpha = 1$ corresponds the adjustable wall thickness (as the design variable) and constant mean diameter of the tube. The case $\alpha = 3$ corresponds the inconstant mean diameter of the tube (as the design variable) and constant wall thickness.

Consider the beam hinged on two supports $x = L$ and $x = -L$:

$$\begin{aligned} y(L) = 0, y(-L) = 0, \quad z(L) = 0, z(-L) = 0, \\ E j \eta_1 y''(L) = 0, E j \eta_1 y''(-L) = 0, \quad E j \eta_2 z''(L) = 0, E j \eta_2 z''(-L) = 0. \end{aligned} \quad (7)$$

The buckling equations the beam read

$$(E j \eta_1 y'')'' = -\tilde{\Lambda} z''', \quad (E j \eta_2 z'')'' = \tilde{\Lambda} y'''. \quad (8)$$

For the boundary value problem (7), (8), the actual curvatures and displacements y^*, z^* minimize the quotient:

$$\tilde{\Lambda}[j, \tilde{y}, \tilde{z}] = \frac{\int_{-L}^L E j (\eta_1 \tilde{y}''^2 + \eta_2 \tilde{z}''^2) dx}{\int_{-L}^L (\tilde{z}'' \tilde{y}' - \tilde{z}' \tilde{y}'') dx}, \quad (9)$$

$$\tilde{\Lambda}[j, y, z] \equiv \tilde{\Lambda}[k_\alpha a^\alpha, y, z] = \min_{\tilde{y}, \tilde{z}} \tilde{\Lambda}[j, \tilde{y}, \tilde{z}].$$

2. Optimization problem and isoperimetric inequality for stability

Based on the above solution, the distribution of material along the length of a twisted rod will be optimized. We search the rod is of constant volume V that support the maximal moment without spatial buckling. The mass \mathfrak{M} of one section the rod is given by

$$\mathfrak{M} = \rho V, \quad V[a] = \int_{-L}^L a(x) dx, \quad (10)$$

We introduce the new variables:

$$Y = \sqrt{\eta_1} y'' , Z = \sqrt{\eta_2} z'' , \tilde{\Lambda} = \sqrt{\eta_1 \eta_2} \Lambda.$$

The formal formulation of the optimization problem is the following:

$$\Lambda[k_\alpha a^\alpha, y, z] = \frac{\tilde{\Lambda}}{\sqrt{\eta_1 \eta_2}} \rightarrow \max_{a(x)} , \quad \int_{-L}^L a(x) dx = \mathfrak{B}_0, a(x) > 0. \quad (11)$$

The distribution of material along the length of a twisted rod is optimized so that the rod is of constant volume and provides the maximal moment without spatial buckling. The augmented functional for the optimization problem (10)-(11) reads:

$$\mathcal{L}[A] = V[a] + \lambda \Lambda[k_\alpha a^\alpha, y, z]. \quad (12)$$

The first variation of the augmented Lagrangian (12) reads with an auxiliary constant $c = (Ek_\alpha)^{-1} \lambda$ reads:

$$-c + a^{\alpha-1} K^2 = 0, \quad K^2 = \kappa_y^2 + \kappa_z^2 \quad (13)$$

The spatial curvature is signed in Eq. (13) as K , where $\kappa_y = z''$, $\kappa_z = y''$. Let optimal distribution of the cross-sectional area along the span of the rod is $A(x)$. The application of the Fermat's principle for the optimization problems leads to the necessary optimality condition:

$$A = (K/c)^{2/(1-\alpha)}. \quad (14)$$

The necessary optimality condition (14) is the requirement that the augmented Lagrangian has a stationary value. The optimal second moment of the cross-section follows from (14) as:

$$J = Ek_\alpha (K/c)^{2\alpha/(1-\alpha)} = (y''^2 + z''^2)^{\alpha/(1-\alpha)}. \quad (15)$$

The applied method of scaling allows the arbitrary selection of the constant c . For briefness of the governing equations, we set the constant c as the positive solution of the equation:

$$E\alpha k_\alpha (1/c)^{2\alpha/(1-\alpha)} = 1.$$

With this choice the bending stiffness reduces to:

$$EJ = (y''^2 + z''^2)^{\alpha/(1-\alpha)} \equiv K^{2\alpha/(1-\alpha)}. \quad (16)$$

3. Closed-form solution of the governing equations

The buckling equations (8) transform with Eq. (16) to:

$$\begin{aligned} (y''^2 + z''^2)^{\alpha/(1-\alpha)} y'' &= -\Lambda z', \\ (y''^2 + z''^2)^{\alpha/(1-\alpha)} z'' &= \Lambda y'. \end{aligned} \quad (17)$$

For the solution of two simultaneous equations $e_1[\kappa_y, \kappa_z] = 0$, $e_2[\kappa_y, \kappa_z] = 0$ we use the representation of the unknowns with two new functions $K(x)$, $\theta(x)$:

$$\kappa_y = K \cos \theta, \kappa_z = K \sin \theta, \quad (18)$$

The substitution of () in Eq. (20) reduces it to two simultaneous equations $\tilde{e}_1[K, \theta] = 0, \tilde{e}_2[K, \theta] = 0$ for the new unknowns:

$$\begin{aligned} \tilde{e}_1[K, \theta] &= e_1[\kappa_y, \kappa_z] \sin(\theta) - e_2[\kappa_y, \kappa_z] \cos(\theta) = 0, \\ \tilde{e}_2[K, \theta] &= e_1[\kappa_y, \kappa_z] \cos(\theta) + e_2[\kappa_y, \kappa_z] \sin(\theta) = 0, \end{aligned} \quad (19)$$

$$K(L) = K(-L) = 0, \theta(0) = 0, A(0) = 1.$$

The equations $\tilde{e}_1[K, \theta]$ and $\tilde{e}_2[K, \theta]$ in (22) reduce to:

$$\begin{aligned} -(\alpha - 1)K^2\theta'' + K'[2(\alpha + 1)K \theta' + (\alpha - 1)\Lambda K^{(3\alpha-1)/(\alpha-1)}] &= 0, \\ (1 - \alpha^2)K^4K'' - K^5(1 - \alpha)^2\theta'^2 + \alpha(1 + \alpha)K^3K'^2 + (1 - \alpha)^2\Lambda\theta'K^{(7\alpha-5)/(\alpha-1)} &= 0, \end{aligned} \quad (20)$$

It is possible to solve Eq. (23) for K, θ , but we prefer to find the solution for the functions A, θ , using Eq. (14) to replace K by A .

$$\begin{aligned} A^{\alpha+1}\theta'' + \frac{1}{2}A'[2(\alpha + 1)A^\alpha\theta' + (\alpha - 1)\Lambda A'] &= 0, \\ 2(1 + \alpha)A'' - (1 - \alpha^2)A^{-1}A'^2 - 4A \theta'^2 + 4A^{1-\alpha} \Lambda\theta' &= 0. \end{aligned} \quad (21)$$

The integration constant could be put to zero from the symmetry considerations: $\vartheta(0) = 0$. The solution $\vartheta(x)$ of the second equation (21) with respect to $\theta(x)$ reads:

$$\vartheta = \frac{\alpha-1}{2} \Lambda \int_0^x A^{-\alpha}(z) dz. \quad (22)$$

Substitution of $\vartheta(x)$ instead of $\theta(x)$ into the first equation (21) leads to an equation in terms of A only:

$$(1 - \alpha)A'^2 - 2A A'' + \Lambda^2(\alpha - 1)A^{2-2\alpha} = 0. \quad (23)$$

For solution the dependent and independent variables in (25) must be exchanged:

$$\frac{d^2x}{dA^2} = \frac{1-\alpha}{2A} \left(\Lambda^2 A^{2-2\alpha} \left(\frac{dx}{dA} \right)^2 - 1 \right) \frac{dx}{dA}. \quad (24)$$

The above equation (26) is the equation of the second order with missing $x(A)$ and allows the order reduction to an equation of the first order.

$$x = \int_0^A \frac{a^\alpha}{\sqrt{C_1 a^{\alpha+1} - \Lambda^2 a^2}} da + C_2 \text{ for } 0 < A < 1, \text{ right half} \quad (25)$$

$$x = - \int_0^A \frac{a^\alpha}{\sqrt{C_1 a^{\alpha+1} - \Lambda^2 a^2}} da + C_2 \text{ for } 0 < A < 1, \text{ left half}$$

The solution $A(x)$ is an even function of x . According to the symmetry of the solution, dA/dx must vanish in the point $x = 0$. The integral Eq. (25) is summable, if $C_1 = \Lambda^2$. The resulting equation of the first order is solvable in quadrature:

$$x = \int_0^A \frac{a^{\alpha-1}}{\Lambda \sqrt{1-z^{\alpha-1}}} da \text{ for } 0 < A < 1, \text{ right half} \quad (26)$$

$$x = - \int_0^A \frac{a^{\alpha-1}}{\Lambda \sqrt{1-a^{\alpha-1}}} da \text{ for } 0 < A < 1, \text{ left half}$$

The function $A(x)$, as parametrically specified by Eq. (26), is an even function of variable $-L \leq x \leq L$. Remarkably, that the singular integral (26) expresses in terms of the higher functions (Gradstein, Ryzhik, 2014):

$$x = \frac{A^\alpha}{\Lambda \alpha} {}_2F_2 \left(\left[\frac{1}{2}, \frac{\alpha}{\alpha-1} \right], \left[\frac{2\alpha-1}{\alpha-1} \right], A^{\alpha-1} \right), \quad \alpha > 1. \quad (27)$$

$$x = \frac{1}{\Lambda} \sqrt{\frac{\pi}{\alpha-1}} \left(1 - \operatorname{erf} \left(\sqrt{-\ln(A)} \right) \right) + o(\alpha-1), \quad \alpha \rightarrow 1, \alpha > 1. \quad (28)$$

$$x = \frac{2}{3\Lambda} (2 - 2\sqrt{1-A} - A\sqrt{1-A}) + o(\alpha-2), \quad \alpha \rightarrow 2. \quad (29)$$

$$x = \frac{1}{2\Lambda} \left(\arcsin(A) - A\sqrt{(A+1)(A-1)} \right) + o(\alpha-3), \quad \alpha \rightarrow 3. \quad (30)$$

The functions in Eqs. (27)-(30) are the real functions of $0 < A < 1$. The limit in Eq. (28) is the right-hand limit $\alpha \rightarrow 1+$.

The area of the optimal cross-sections vanishes in the end points of the rod, where the bending moment disappears:

$$x(A=0) = L.$$

Substitution of the condition $A(0) = 1$ into (26) leads to:

$$L = \int_0^1 \frac{a^{\alpha-1}}{\Lambda \sqrt{1-a^{\alpha-1}}} da. \quad (31)$$

The integral (31) another time evaluates in terms of beta-function (Gradstein, Ryzhik, 2014), (Abramovitz, Stegun, 1983):

$$L = \frac{\sqrt{\pi}}{\Lambda (\alpha-1)} \frac{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}{\Gamma\left(\frac{3\alpha-1}{2\alpha-2}\right)}. \quad (32)$$

For volume evaluation the integrand in Eq. (31) must be multiplied by area a (Gradstein, Ryzhik, 2014):

$$v = \int_0^1 \frac{a^\alpha}{\Lambda \sqrt{1-z^{\alpha-1}}} da = \frac{\sqrt{\pi}}{\Lambda (\alpha-1)} \frac{\Gamma\left(\frac{\alpha+1}{\alpha-1}\right)}{\Gamma\left(\frac{3\alpha+3}{2\alpha-2}\right)}. \quad (33)$$

For estimation of elastic energy \mathcal{E} the integrand in Eq. (28) must be multiplied by area a^α (Gradstein, Ryzhik, 2014):

$$\varepsilon = \int_0^1 \frac{a^{\alpha+1}}{\Lambda \sqrt{1-z^{\alpha-1}}} da = \frac{(2^{\alpha/\alpha-1})^2 \Gamma(\frac{\alpha}{\alpha-1}) \Gamma(\frac{3\alpha-1}{2\alpha-2})}{2\Lambda(\alpha-1) \Gamma(\frac{5\alpha-1}{2\alpha-2})}. \quad (1)$$

We the normalize coordinate: $X = x/L$. Finally, the shape of the right side is given by the implicit function of the normalized coordinate $-1 \leq X \leq 1$:

$$X = \frac{(\alpha-1)\Gamma(\frac{3\alpha-1}{2\alpha-2})}{\alpha\sqrt{\pi}\Gamma(\frac{\alpha}{\alpha-1})} \cdot {}_2F_2\left(\left[\frac{1}{2}, \frac{\alpha}{\alpha-1}\right], \left[\frac{2\alpha-1}{\alpha-1}\right], A^{\alpha-1}\right) A^\alpha. \quad (34)$$

For volume of the half of the rod with the unit length we get the expression:

$$V = \frac{v}{L} = \frac{\Gamma(\frac{\alpha-1}{\alpha+1})\Gamma(\frac{3\alpha-1}{2\alpha-2})}{\Gamma(\frac{3\alpha+1}{2\alpha-3})\Gamma(\frac{\alpha}{\alpha-1})}. \quad (35)$$

For assessment of elastic energy \mathcal{E} in the normalized coordinates the formula is valid:

$$\mathcal{E} = \frac{\varepsilon}{L} = (2^{\alpha/\alpha-1})^2 \frac{1}{2\sqrt{\pi}} \frac{\Gamma^2(\frac{3\alpha-1}{2\alpha-2})}{\Gamma(\frac{5\alpha-1}{2\alpha-2})}. \quad (36)$$

For practically interesting cases the shape reduces to the elementary functions, as shown in Table 1.

Table 1 Normalized cross-sectional area along the span of the twisted rod, the normalized volume and normalized elastic energy

	Cross-sectional area	Normalized volume	Normalized elastic energy
	$-1 \leq X \leq 1, 0 \leq A \leq 1$	$V = \frac{v}{L}$	$\mathcal{E} = \frac{\varepsilon}{L}$
$\alpha = 1$	$X = 1 - \operatorname{erf}(-\ln(A))$	$V = \frac{1}{\sqrt{2}}$	$\mathcal{E} = \frac{1}{\sqrt{3}}$
$\alpha = 2$	$X = 1 - \sqrt{1-A} - \frac{A\sqrt{1-A}}{2}$	$V = \frac{4}{5}$	$\mathcal{E} = \frac{24}{35}$
$\alpha = 3$	$X = \frac{2}{\pi} \left(\arcsin(A) - A\sqrt{1-A^2} \right)$	$V = \frac{8}{3\pi}$	$\mathcal{E} = \frac{3}{4}$

The expression $\operatorname{erf} x$ is the Gauss error function.

The buckling shape of the twisted column with the simply supported (hinged) ends is demonstrated on Figure 1 for $\alpha = 2$. For the other values of parameter α the buckling shapes are similar. Figure 2 displays the areas of cross-sections of the optimal twisted columns for $\alpha = 1, 2, 3$. Second moments of inertia of the cross-sections of the optimal twisted columns are shown for $\alpha = 1, 2, 3$ on Figure 3.

4. Effectiveness

The influence of the exponent α influences the estimations for the optimization effects. Another argument for the introduction of the invariant optimization factors is methodical. In the variational calculus is common to get one factor as the optimization objective and the others as the a-priori given constraints. To convert it into an unconstrained problem the method of Lagrange multipliers are commonly used. For the twisted rod the optimality expresses with the isoperimetric inequalities. The resulting unconstrained problem with Lagrange multipliers increases number of variables. The new number of unknown variables is the original number of variables plus the original number of constraints. The constraints are usually solved for some of the variables in terms of the others, and the former can be substituted out of the objective function, leaving an unconstrained problem in a smaller number of variables. This method of solution leads to the nonlinear algebraic equations for Lagrange multipliers. These nonlinear equations in the most cases do not possess the closed analytical solutions and are solvable only numerically.

Instead of dealing with the Lagrange multipliers, we introduce the certain invariant factors. Consider the twisted rods with the same form of cross-sections, fixing the exponent α . For each fixed value of α we introduce the two dimensionless factors:

$$F_V = \Lambda \frac{L^{p_1}}{v^{p_2}}, F_\varepsilon = 2\Lambda \frac{v^{p_3}}{\varepsilon^{p_4}}. \quad (1)$$

For some arbitrary powers p_1, p_2, p_3, p_4 , the factors alter for any affine transformation of the rod. The affine transformation of the rod is the product of two elementary transformations, namely homothety and scaling. The homothety of ratio ζ multiplies lengths by ζ . Thus, ζ is the ratio of magnification or dilation factor or scale factor or similitude ratio. The cross-section function $A(\xi)$ scales by another factor ρ , such that for the affine transformed twisted rod the cross-section function will be $\rho A(\xi)$. Apparently, the eigenvalue Λ alters in course of the affine transformation of the twisted rod.

We use the factors F_V, F_ε for the comparisons of different designs. The critical buckling moment $\widetilde{M} = k_\alpha E\Lambda$ inherits the factor k_α and is proportional to this value. Evidently, that the ratios of the buckling loads for different designs with the same form of the cross-sections do not depend on the constants k_α . For different cross-sections the actual value of k_α have to be used.

With the methods of dimensional analysis, we can immediately determine the distinctive choice of powers

$$p_1 = 1 + \alpha, p_2 = \alpha, p_3 = 2 + \alpha, p_4 = 1 + \alpha. \quad (2)$$

Using the expressions (32) and (33) the factors result to:

$$F_V = \frac{L^{1+\alpha}}{v^\alpha}, F_\varepsilon = 2 \frac{v^{2+\alpha}}{\varepsilon^{1+\alpha}}. \quad (3)$$

The factors F_V, F_ε do not alter for any affine transformation of the twisted rod. Thus, the factors F_V, F_ε are the invariants to the affine transformation of the twisted rod and provide a natural basis for the comparison of different designs.

With the above factor, the estimation of the effect of mass optimization turns out to be trivial. For this purpose, we consider the reference design with the constant cross-section along the span. The invariant factors for the reference design is \widetilde{F}_V . The factor equals for all exponents α and for the boundary conditions with both hinged ends reads:

$$\widetilde{F}_V = \frac{1}{2}. \quad (4)$$

The greater the factor is, the higher the buckling moment for the given length and volume of the twisted rod. For example, the buckling force of the reference clamped twisted rod is four times the buckling force of the reference twisted rod with the hinged ends.

The dual formulations are typical the optimization of buckling twisted rod as well. For the dual formulations, the masses of the twisted rods for the fixed lengths and fixed buckling forces are compared. The volumes and masses of the optimal and reference twisted rods relate to each other as the inverse roots of the order α of the factors F_V :

$$\frac{V}{\tilde{V}_{ref}} = \alpha \sqrt{\frac{\tilde{F}_V}{F_V}}, \quad (5)$$

Specifically, the twisted rod with the higher value of the factor F_V possesses the lower mass. We use systematically the method of dimensionless factors for the optimization analysis. The applied method for integration of the optimization criteria delivers different length and volumes of the optimal twisted rods. Instead of the seeking for the twisted rods of the fixed length and volume, we directly compare the twisted rods with the different lengths and cross-sections using the invariant factors. The results of the evaluation of the dimensionless volume factors and volumes for the fixed critical eigenvalue $\Lambda = \pi/2$ and the half-length $L = 1$ are presented in Table 2. The half-volumes of the optimal columns V and the volume optimization factor F_V are shown as the functions α of on Figure 4. Figure 5 demonstrates the half-volumes V and elastic energy \mathcal{E} of the optimal columns.

Table 2 Dimensionless volume factor and volume for the fixed critical eigenvalue $\Lambda = \pi/2$ and the half-length $L = 1$

	F_V	$V(\Lambda = \pi/2, L = 1)$
Constant cross section, reference $\alpha = 0$	1	1
$\alpha = 1$	$\sqrt{2}$	$\sqrt{2}/2$
$\alpha = 2$	25/16	4/5
$\alpha = 3$	$27\pi^3/512$	$8/3\pi$

5. Conclusions

In this [article](#) the solution of optimization problem for twisted rod is stated in closed form, involving the length of the rod, its volume and critical torque. The solution expresses in terms of the higher transcendental functions. Remarkable, that in the torsion stability problem the optimal shape of the rod is roughly parabolic along its length and the optimal shape of cross-section is the equilateral triangle.

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Replication of results: No results are presented

Data availability: the research data associated with a paper is available in form of closed form mathematical solutions

Figures

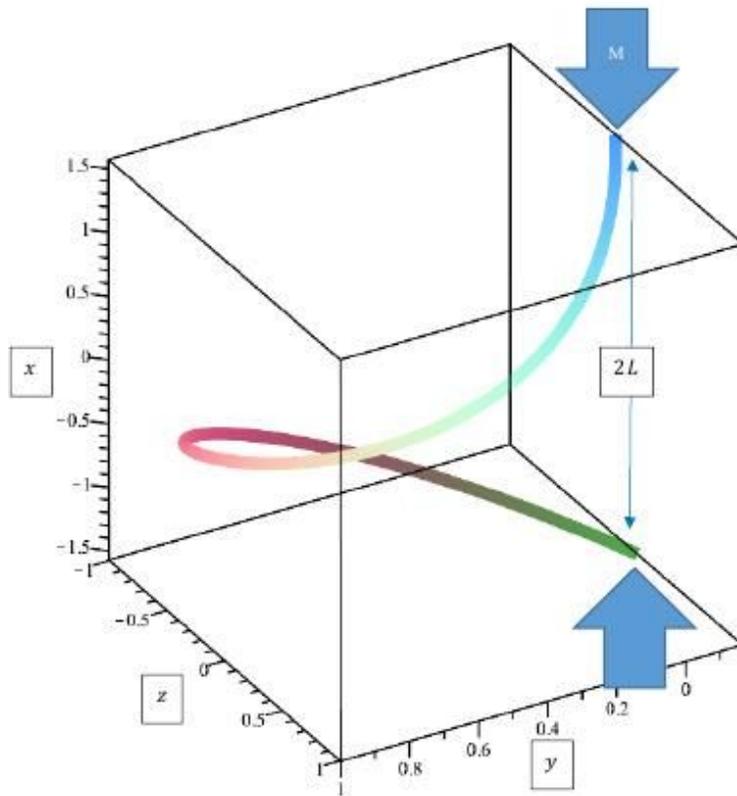


Figure 1 The buckling shape of the twisted column with the simply supported (hinged) ends

Figure 1

The buckling shape of the twisted column with the simply supported (hinged) ends

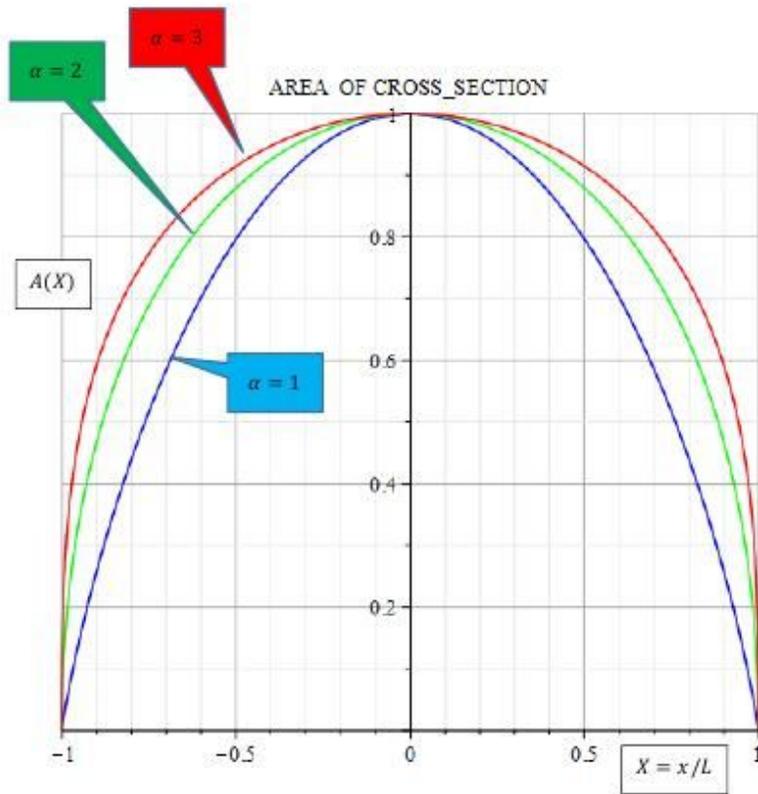


Figure 2 Areas of cross-sections of the optimal twisted columns for $\alpha = 1, 2, 3$

Figure 2

Areas of cross-sections of the optimal twisted columns for $\alpha=1,2,3$

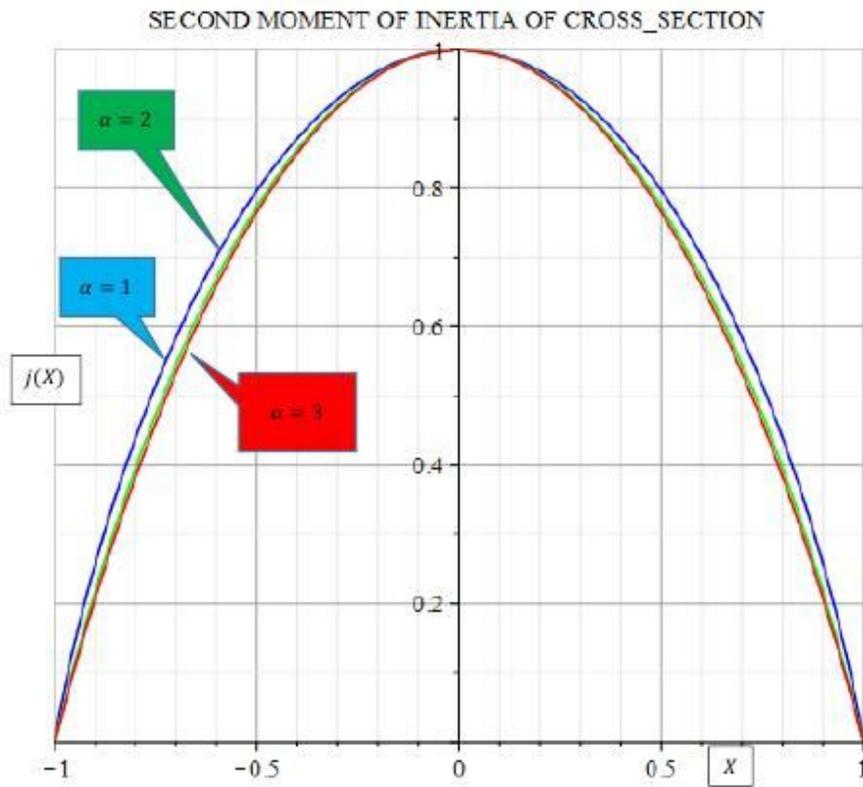


Figure 3 Second moments of inertia of the cross-sections of the optimal twisted columns for $\alpha = 1, 2, 3$

Figure 3

Second moments of inertia of the cross-sections of the optimal twisted columns for $\alpha=1,2,3$

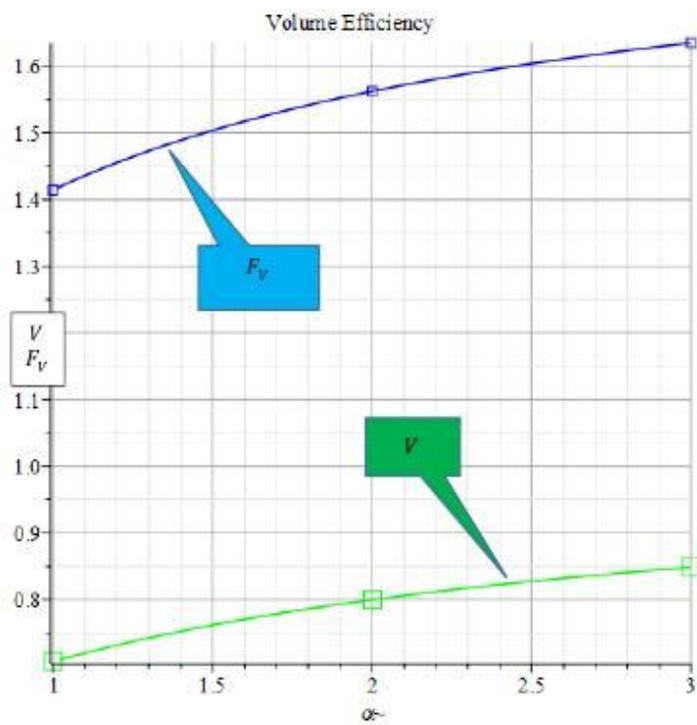


Figure 4 The half-volumes of the optimal columns V and the volume optimization factor F_V

Figure 4

The half-volumes of the optimal columns V and the volume optimization factor F_V

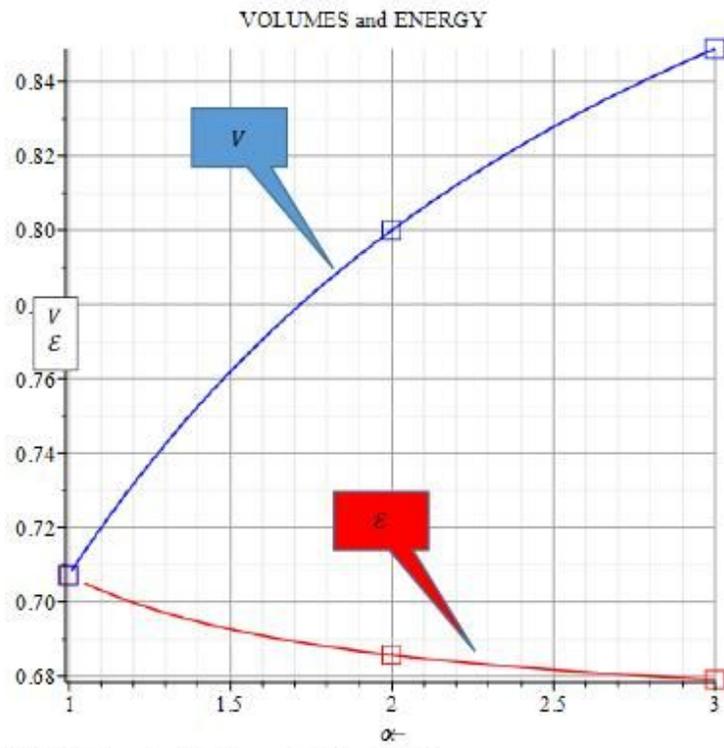


Figure 5 The half-volumes V and elastic energy E of the optimal columns

Figure 5

The half-volumes V and elastic energy E of the optimal columns