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Exponentially mean stability analysis of positive Markov jump neural networks with time delay

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Abstract A new result for the analysis of positive Markov jump neural networks (PMJNNs) with time delay is described in this paper. By rewriting the PMJNNs with time delay in both continues-time and discrete-time domains into equivalent positive neural networks(PNNs) and analyzing their stability issues, two delay-dependent sufficient conditions are presented to ensure that the continuous-time and the discrete-time PMJNNs with time delay are exponentially mean stable(EMS) through using the inequality technique. All conditions obtained in the paper are in terms of standard linear programming, which reduces the conservatism. Finally, two numerical examples are provided to verify the validity of our results.

Keywords Positive Markov jump neural networks · Linear programming · Inequality technique · Exponentially mean stability

1 Introduction

Research on neural networks(NNs) has attracted substantial attention of scholars over that last few decades by the fact that they can be applied far and wide in many prac-

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tical fields, such as pattern recognition [1], signal processing [2], finance [3] and so forth. It is public knowledge that many practical systems only experience non-negative variables(see, for example, [4–6]), which gives rise to exploration of positive neural networks(PNNs). By using some novel comparison techniques, the work in [7] has analysed the global exponential stability of PNNs with time-varying delay and a testable condition has been derived to guarantee the uniqueness of positive equilibrium point. The authors in [8] have finished off the work where the exponential stability issue of PNNs in bidirectional associative memory(BAM) model with multiple time-varying delays was solved. The filter design with l_1 -gain disturbance attenuation performance has been implemented for discrete-time PNNs in [9].

On another research front, Markov jump systems(MJSs) [10–13] that can be used to explore the practical system with random changes in structure and parameters were a special class of stochastic hybrid systems. In fact, there exist random mutations in structure for NNs due to component failures, sudden environmental disturbances or changing subsystem interconnections, which can be modeled as a Markov model and brings about the discussion of Markov jump neural networks(MJNNs). Up to date, some substantial results of MJNNs have been explored in a large body of literature on stability analysis [14, 15], synchronization control [16–18], state estimation [19–21], filter design [22–24] and so forth. Very recently, some initial efforts have been committed to the research of positive Markov jump neural networks(PMJNNs) [25]. The work in [25] has investigated the finite-time stabilization of uncertain PMJNNs and a finite-time stabilizable controller has also been designed, where all obtained conditions exist in form of linear matrix inequality(LMI). As far as we know, only a few limited works were devoted to PMJNNs and the existing results are highly conservatism. How to reduce the conservatism of the results may make the stability analysis of the PMJNNs with time delay more complex and the corresponding work remains to be studied, which is more challenging and meaningful.

In accordance with the analysis of the aforementioned results, the main purpose of this paper is to study the exponentially mean stability of the PMJNNs with time delay and to obtain less conservative conditions. The major contributions of this paper are as below:

(1)For the first time, the linear programming(LP) method is applied to explore the stability issues of PMJNNs with time delay in both continues-time and discrete-time domains. Compared with the LMI method in [25], it is on reducing computational complexity that the LP method is obviously better than the LMI method and the relation between the LP method and the LMI method has been studied in detail in [26].

(2)By virtue of the augmentation system approach, the PMJNNs with time delay in this paper are rewritten as the augmented systems and the equivalence between the PMJNNs with time delay and the augmented systems are further discussed. And by means of the inequality technique, two delay-dependent sufficient conditions that make PMJNNs with time delay exponentially mean stable(EMS) are provided. Besides, numerical examples further verify that the stability of the PMJNNs is related to the size of time-delay.

The rest of the paper is as follows. In Section II, some essential Lemmas, Definitions, Assumption and the system formulae are given. The main results are discussed

in Section III and verified by numerical examples in Section IV. Finally, the conclusion of the paper is given in Section V.

Notations: The set of real numbers and n-dimensional (positive) Euclidean space are expressed as R , R^n (R_+^n), respectively. $R^{n \times m}$ denotes the space of $n \times m$ matrix. $A \succeq 0$ and $A \preceq 0$ indicate that all elements of matrix A are nonnegative and non-positive. $A \succ 0$ ($A \prec 0$) means that every element of matrix A is positive (negative). a_{ij} represents the element of the i -th row and j -th column of the matrix $A = \{a_{ij}\}$. $E\{\cdot\}$ stands for the mathematical expectation and \otimes denotes Kronecker product. The 1-norm and the transposition of the matrix A are described as $\|A\|_1$ and A^T , respectively. I_n denote the identity matrix of $n \times n$.

2 Problem statements and preliminaries

For a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we consider a class of continuous-time PMJNNs with time delay as follows

$$\Sigma_1 : \begin{cases} \dot{x}(t) = -D_{r(t)}x(t) + A_{r(t)}f(x(t)) + B_{r(t)}g(x(t - \tau)) + J_{r(t)} \\ x(\theta) = \phi(\theta), \theta \in [-\tau, 0] \end{cases}$$

where $x(t) \in R_+^n$ is the system state associated with n-neurons. $\{r(t), t \geq 0\}$ represent a Markov process, taking values in the set $\mathbb{M} = \{1, 2, \dots, M\}$. $Y = \{\lambda_{ij}\}$ is transition rate matrix of Markov process $\{r(t), t \geq 0\}$ and the rates is characterized as

$$\mathcal{P}\{r_{(k+\Delta)} = j | r_t = i\} = \begin{cases} \lambda_{ij}\Delta + o(\Delta), & i \neq j \\ 1 + \lambda_{ij}\Delta + o(\Delta), & i = j \end{cases} \quad (1)$$

where $\Delta > 0$, $\lim_{\Delta \rightarrow 0} (o(\Delta)/\Delta) = 0$, $\lambda_{ij} \geq 0 (i, j \in \mathbb{M}, i \neq j)$, and $\lambda_{ii} = -\sum_{j=1, j \neq i}^M \lambda_{ij}$.

For $r(t) = i \in \mathbb{M}$, the i th mode of the system Σ_1 are described by D_i, A_i, B_i, J_i . $D_i = \text{diag}\{d_z^i\} \in R^{n \times n}$, $A_i \in R^{n \times n}$ and $B_i \in R^{n \times n}$ are constant matrices, where $d_z^i > 0$ is the self-feedback term, $z \in [1, 2, \dots, n]$. $f(y(\cdot)) = [f_1(x_1(\cdot)), f_2(x_2(\cdot)), \dots, f_n(x_n(\cdot))]^T$ and $g(x(\cdot)) = [g_1(x_1(\cdot)), g_2(x_2(\cdot)), \dots, g_n(x_n(\cdot))]^T$ are two neuron activation functions. J_i is the external positive input vector. $\phi(\theta)$ is the initial value with $\phi(\theta) = \phi(0)$ for $\theta \in [-\tau, 0]$ and τ is a given constant time delay.

Concerning the discrete-case, we examine a kind of discrete-time PMJNNs with time delay as below

$$\Sigma_2 : \begin{cases} x(k+1) = D_{r(k)}x(k) + A_{r(k)}f(x(k)) + B_{r(k)}g(x(k - \beta)) + J_{r(k)} \\ x(\theta) = \varphi(\theta), \theta \in [-\beta, 0] \end{cases}$$

where the system state $x(k) \in R_+^n$ is controlled by n-neurons. Markov sequence $\{r(k), k \geq 0\}$ take values in set $\mathbb{N} = \{1, 2, \dots, N\}$ with transition probability matrix $\Pi = \{\pi_{pq}\}$ defined by

$$\mathcal{P}_r\{r(k+1) = q | r(k) = p\} = \pi_{pq} \quad (2)$$

Clearly, for all $p, q \in \mathbb{N}$, $\pi_{pq} \in [0, 1]$, and for all $p \in \mathbb{N}$, $\sum_{q=1}^M \pi_{pq} = 1$.

For $r(k) = p \in \mathbb{N}$, D_p, A_p, B_p and J_p indicate the corresponding matrixes of the p th mode of the system Σ_2 . $\varphi(\theta)$ is the initial value with $\varphi(\theta) = \varphi(0)$ for $\theta \in [-\beta, 0]$

and β is a given constant time delay.

Assumption 1. we assume that the activation function $f_u(\cdot)$ and $g_u(\cdot)$ are continuous and bounded on R_+ for $u \in [1, 2, \dots, n]$ such that the following conditions

$$0 \leq \frac{f_u(x) - f_u(y)}{x - y} \leq L_{uf}, 0 \leq \frac{g_u(x) - g_u(y)}{x - y} \leq L_{ug}$$

holds for $x, y \in R^+, x \neq y$. We describe

$$L_f = \text{diag}\{L_{uf}\}, L_g = \text{diag}\{L_{ug}\}. \quad (3)$$

Assume that x_e is equilibrium point of system Σ_1 , let $y(t) = x(t) - x_e$, Σ_1 can be rewritten as:

$$\Sigma_3 : \begin{cases} \dot{y}(t) = -D_{r(t)}y(t) + A_{r(t)}F(y(t)) + B_{r(t)}G(y(t - \tau)) \\ y(\theta) = \phi_y(\theta), \theta \in [-\tau, 0] \end{cases}$$

where $F(y(\cdot)) = [h_1(y_1(\cdot)), h_2(y_2(\cdot)), \dots, h_n(y_n(\cdot))]^T$ and $G(y(\cdot)) = [\kappa_1(y_1(\cdot)), \kappa_2(y_2(\cdot)), \dots, \kappa_n(y_n(\cdot))]^T$ with $h_u(y_u(\cdot)) = f_u(y_u(\cdot) + x_e) - f_u(x_e)$ and $\kappa_u(y_u(\cdot)) = g_u(y_u(\cdot) + x_e) - g_u(x_e)$ for $u \in [1, 2, \dots, n]$, $\phi_y(\theta) = \phi(\theta) - x_e$. Obviously, $h_u(\cdot)$ and $\kappa_u(\cdot)$ also satisfy the Assumption 1.

Assume that x_{1e} is equilibrium point of system Σ_2 , let $y(k) = x(k) - x_{1e}$, Σ_2 can be rewritten as:

$$\Sigma_4 : \begin{cases} y(k+1) = D_{r(k)}y(k) + A_{r(k)}F(y(k)) + B_{r(k)}G(y(k - \beta)) \\ y(\theta) = \varphi_y(\theta), \theta \in [-\beta, 0] \end{cases}$$

where $\varphi_y(\theta) = \varphi(\theta) - x_{1e}$.

Remark 1. For systems Σ_3 and Σ_4 , according to assumption 1 and the analysis in [25], we know that the external input vector J_i and J_p are removed by $y(t) = x(t) - x_e$ and $y(k) = x(k) - x_{1e}$.

Lemma 1. [7] If $D_i \geq 0, A_i \geq 0, B_i \geq 0, J_i \geq 0$ and Assumption 1 hold, the system Σ_1 and Σ_3 are positive for nonnegative initial state.

Lemma 2. If $D_p \geq 0, A_p \geq 0, B_p \geq 0, J_p \geq 0$ and Assumption 1 hold, the system Σ_2 and Σ_4 are positive for nonnegative initial state.

Definition 1. The system Σ_1 and Σ_3 are exponentially mean stable(EMS) if there exist two positive constants ε and γ such that

$$E\{\|y(t)\|_1\} < \varepsilon E\{\|y(0)\|_1\} e^{-\gamma(t-t_0)}$$

for any nonnegative initial condition.

Definition 2. The system Σ_2 and Σ_4 are EMS if there exist two positive constants ε_1 and $0 < \xi < 1$ such that

$$E\{\|y(k)\|_1\} < \varepsilon_1 E\{\|y(0)\|_1\} \xi^{(k-k_0)}$$

for any nonnegative initial condition.

Lemma 3. [11] Consider a stochastic process $\{f(t), r(t), t \leq 0\}$ such that the jumping process $r(t)$ is a homogeneous Markov chain with right-continuous trajectories

and takes in set \mathbb{M} . Assuming $E[f(t)\mathbf{1}_{r_t=i}] := \mathbf{f}_i(t)$ exists, then $E[f(t)d(\mathbf{1}_{r_t=i})] = \sum_{j=1}^M \lambda_{ji} \mathbf{f}_j(t) dt$, where

$$\mathbf{1}_{\{r(t)=i\}} \begin{cases} 1 & r(t) = i, \\ 0 & \text{otherwise.} \end{cases}$$

When $\mathbb{M}=\{1\}$ and $\mathbb{N}=\{1\}$, the Σ_3 and Σ_4 are transformed into the continuous-time PNNs Γ_1 and the discrete-time PNNs Γ_2 respectively, as shown below

$$\Gamma_1 : \begin{cases} \dot{y}(t) = -Dy(t) + AF(y(t)) + BG(y(t - \tau)) \\ y(\theta) = \phi_y(\theta), \theta \in [-\tau, 0] \end{cases}$$

$$\Gamma_2 : \begin{cases} y(k+1) = Dy(k) + AF(y(k)) + BG(y(k - \beta)) \\ y(\theta) = \phi_y(\theta), \theta \in [-\beta, 0] \end{cases}$$

Definition 3. The system Γ_1 is exponentially stable(ES) if there exist two positive constants ε and γ such that

$$\|y(t)\|_1 < \varepsilon \|y(0)\|_1 e^{-\gamma(t-t_0)}$$

for any nonnegative initial condition.

Definition 4. The system Γ_2 are ES if there exist two positive constants ε_1 and $0 < \xi < 1$ such that

$$\|y(k)\|_1 < \varepsilon_1 \|y(0)\|_1 \xi^{(k-k_0)}$$

for any nonnegative initial condition.

3 Main Results

3.1 EMS of continuous-time PMJNNs

In this subsection, the continuous-time PMJNNs is rewritten as an equivalent continuous-time PNNs. By means of analyzing the exponential stability issue of the continuous-time PNNs, a sufficient condition is derived to ensure that the system Σ_3 is EMS.

For simplicity of presentation, according to Lemma 3, introduce the following notations:

$$\begin{aligned} \mathbb{Y} &= [\mathbf{y}_1^T(t) \ \mathbf{y}_2^T(t) \ \cdots \ \mathbf{y}_M^T(t)] \\ \mathbf{y}_i(t) &= E\{y(t)\mathbf{1}_{\{r(t)=i\}}\} \\ F(\mathbf{y}_i(t)) &= E\{F(y(t))\mathbf{1}_{\{r(t)=i\}}\} \end{aligned} \quad (4)$$

$$\begin{aligned} \mathbb{D} &= \text{diag}\{D_i\} - Y^T \otimes I_n, \\ \mathbb{A} &= \text{diag}\{A_i\}, \\ \mathbb{B} &= \text{diag}\{B_i\}(\Omega^T(\tau) \otimes I_n) \end{aligned} \quad (5)$$

where $\Omega(\tau) = \{\rho_{ij}(\tau)\}$ and $\rho_{ij}(\tau) = \mathcal{P}_r\{r(t+\tau) = j | r(t) = i\}$ for $i, j \in \mathbb{M}, \tau \geq 0, t \geq 0$. And $\Omega(\tau)$ satisfies the forward Kolmogorov differential equation: $\dot{\Omega}(\tau) = \Omega(\tau)Y, \Omega(\tau) = I_M, \tau \geq 0$. It can obtain that $\Omega(\tau) = e^{Y\tau}, \tau \geq 0$.

Theorem 1. Under Lemma 1, if there exists a vector $\mathbb{P} = [P_1, P_2, \dots, P_M]^T \in \mathbb{R}_+^{Mn}$ and a constant $\gamma > 0$ for $i, j \in \mathbb{M}$, such that

$$\sum_{j=1}^M \lambda_{ji} P_j + \sum_{j=1}^M \rho_{ji}(\tau) e^{\gamma\tau} B_i L_g P_j + (A_i L_f + \gamma - D_i) P_i < 0 \quad (6)$$

then, the system Σ_3 is positive and EMS for a given τ .

where $P_i = [o_1^i, o_2^i, \dots, o_c^i]^T$ for $i \in \mathbb{M}$ and $c \in [1, 2, \dots, n]$.

Proof. From (4), for each $i, j \in \mathbb{M}$, we have

$$\begin{aligned} dy_i(t) &= dE[y(t)\mathbf{1}_{r_t=i}] = E[dy(t)\mathbf{1}_{r_t=i} + y(t)d\mathbf{1}_{r_t=i}] \\ &= -D_i y_i(t) dt + A_i F(y_i(t)) dt + B_i E\{G(y(t-\tau))\mathbf{1}_{r_{t-\tau}=i}\} dt + \sum_{j=1}^M \lambda_{ji} y_j(t) dt \\ &= -D_i y_i(t) dt + B_i \sum_{j=1}^M \mathcal{P}_r\{r_t = i | r_{t-\tau} = j\} E\{G(y(t-\tau))\mathbf{1}_{r_{t-\tau}=j}\} dt \\ &\quad + A_i F(y_i(t)) dt + \sum_{j=1}^M \lambda_{ji} y_j(t) dt \quad (7) \\ &= -D_i y_i(t) dt + A_i F(y_i(t)) dt + B_i \sum_{j=1}^M \rho_{ji}(\tau) G(y_j(t-\tau)) dt \\ &\quad + \sum_{j=1}^M \lambda_{ji} y_j(t) dt \end{aligned}$$

that is

$$\dot{y}_i(t) = -D_i y_i(t) + A_i F(y_i(t)) + B_i \sum_{j=1}^M \rho_{ji}(\beta) G(y_j(t-\beta)) + \sum_{j=1}^M \lambda_{ji} y_j(t) \quad (8)$$

On the basis of (5), (8) is written in vector form as follows

$$\Sigma_5 : \begin{cases} \dot{\mathbb{Y}}(t) = -\mathbb{D}\mathbb{Y}(t) + \mathbb{A}\mathbb{F}(\mathbb{Y}(t)) + \mathbb{B}\mathbb{G}(\mathbb{Y}(t-\tau)) \\ \mathbb{Y}(\theta) = \phi_y(\theta), \theta \in [-\tau, 0] \end{cases}$$

where $\mathbb{F}(\mathbb{Y}(t)) = [F(\mathbf{y}_1(t)), F(\mathbf{y}_2(t)), \dots, F(\mathbf{y}_M(t))]$, $\mathbb{G}(\mathbb{Y}(t-\tau)) = [G(\mathbf{y}_1(t-\tau)), G(\mathbf{y}_2(t-\tau)), \dots, G(\mathbf{y}_M(t-\tau))]$. Clearly, the activation functions $\mathbb{F}(\cdot)$ and $\mathbb{G}(\cdot)$ are continuous on R^{Mn} and satisfy Assumption 1. We define

$$\mathbb{L}_f = \text{diag}\{L_f\}, \mathbb{L}_g = \text{diag}\{L_g\} \quad (9)$$

Then

$$\|\mathbb{Y}(t)\|_1 = \mathbf{1}_{Mn}^T(\mathbb{Y}(t)) = \mathbf{1}_n^T \left(\sum_{i=1}^M (\mathbf{y}_i(t)) \right) = E\{\|y(t)\|_1\} \quad (10)$$

From (10) and Definition 3, we know that there exist positive constants ε and γ such that

$$E[\|y(t)\|_1] = \|\mathbb{Y}(t)\|_1 < \varepsilon \|\mathbb{Y}(0)\|_1 e^{-\gamma(t-t_0)} = \varepsilon E[\|y(0)\|_1] e^{-\gamma(t-t_0)} \quad (11)$$

So, from Definition 1, we know that the system Σ_3 is EMS if the system Σ_5 is ES.

Next, we will discuss the exponential stability of the system Σ_5 . Under Lemma 1, let $\mathbb{Z}(t) = e^{\gamma(t-t_0)} \mathbb{Y}(t)$, taking the upper right derivative of $\mathbb{Z}(t)$, we obtain

$$\begin{aligned} D^+ \mathbb{Z}(t) &= e^{\gamma(t-t_0)} \{-\mathbb{D}\mathbb{Y}(t) + \mathbb{A}\mathbb{F}(\mathbb{Y}(t)) + \mathbb{B}\mathbb{G}(\mathbb{Y}(t-\tau))\} + \gamma e^{\gamma(t-t_0)} \mathbb{Y}(t) \\ &\preceq e^{\gamma(t-t_0)} \{-\mathbb{D}\mathbb{Y}(t) + \mathbb{A}\mathbb{L}_f \mathbb{Y}(t) + \mathbb{B}\mathbb{L}_g \mathbb{Y}(t-\tau)\} + \gamma e^{\gamma(t-t_0)} \mathbb{Y}(t) \\ &= e^{\gamma(t-t_0)} \{(-\mathbb{D} + \boldsymbol{\omega})\mathbb{Y}(t) + \mathbb{A}\mathbb{L}_f \mathbb{Y}(t) + \mathbb{B}\mathbb{L}_g \mathbb{Y}(t-\tau)\} \end{aligned} \quad (12)$$

where $\boldsymbol{\omega} = \gamma I_{Mn}$. After substituting $\mathbb{Z}(t)$ for $e^{\gamma(t-t_0)} \mathbb{Y}(t)$ in Equ.(12), the following formula is obtained

$$D^+ \mathbb{Z}(t) = (-\mathbb{D} + \boldsymbol{\omega})\mathbb{Z}(t) + \mathbb{A}\mathbb{L}_f \mathbb{Z}(t) + e^{\gamma\tau} \mathbb{B}\mathbb{L}_g \mathbb{Z}(t-\tau) \quad (13)$$

Defining a curve in Mn -dimensional space $\chi = \{\mathbb{H}(l) : H_i = o_c^i l, l > 0, i = 1, 2, \dots, M, c = 1, 2, \dots, n\}$ and a set $\Lambda(\mathbb{H}) = \{\mathbb{Z} : 0 \preceq \mathbb{Z} \preceq \mathbb{H}, \mathbb{H} \in \chi\}$. Let $l_0 = \|\mathbb{Y}(0)\|_1 / o_{min}$, where $o_{min} = \min_{1 \leq i \leq M; 1 \leq c \leq n} \{o_c^i\} > 0$. It is clear that $\Lambda(\mathbb{H}(l')) \subset \Lambda(\mathbb{H}(l))$ as $l > l'$. Defining the $\mathbb{Z}(0) = e^{-\gamma_0} \mathbb{Y}(0)$, we know $\mathbb{Z}(0) \in \Lambda(\mathbb{H}(l_0))$. That is $\mathbb{Z}(0) = e^{-\gamma_0} \mathbb{Y}(0) \prec \mathbb{P}l_0$.

For $t > 0$, we assume that $\mathbb{Z}(t) \prec \mathbb{P}l_0$. If this is not true, then there are corresponding $t' > 0$, which make $\mathbb{Z}(t') = \mathbb{P}l_0$, $D^+ \mathbb{Z}(t') \succeq 0$. According to (13), we get

$$\begin{aligned} D^+ \mathbb{Z}(t') &\preceq (-\mathbb{D} + \boldsymbol{\omega})\mathbb{P}l_0 + \mathbb{A}\mathbb{L}_f \mathbb{P}l_0 + e^{\gamma\tau} \mathbb{B}\mathbb{L}_g \mathbb{P}l_0 \\ &\preceq (-\mathbb{D} + \boldsymbol{\omega} + \mathbb{A}\mathbb{L}_f + e^{\gamma\tau} \mathbb{B}\mathbb{L}_g) \mathbb{P}l_0 \end{aligned} \quad (14)$$

let

$$W = (-\mathbb{D} + \boldsymbol{\omega} + \mathbb{A}\mathbb{L}_f + e^{\gamma\tau} \mathbb{B}\mathbb{L}_g) \mathbb{P} \quad (15)$$

Substituting (5) and (9) into (15) yields

$$W = \begin{pmatrix} A_{11} & \lambda_{21} I_n + B_1 \rho_{21}(\tau) L_g & \cdots & \lambda_{M1} I_n + B_1 \rho_{M1}(\tau) L_g \\ \lambda_{12} I_n + B_2 \rho_{12}(\tau) L_g & A_{22} & \cdots & \lambda_{M2} I_n + B_2 \rho_{M2}(\tau) L_g \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1M} I_n + B_3 \rho_{1M}(\tau) L_g & \lambda_{2M} I_n + B_M \rho_{2M}(\tau) L_g & \cdots & A_{MM} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_M \end{pmatrix} \quad (16)$$

where $A_{ii} = A_i L_f + \gamma I_n + \lambda_{ii} I_n - D_i + e^{\gamma\tau} B_i \rho_{ii}(\tau) L_g$, $i \in \mathbb{M}$. Then,

$$W = \begin{pmatrix} \sum_{j=1}^M \lambda_{j1} P_j + \sum_{j=1}^M \rho_{j1}(\tau) e^{\gamma\tau} B_1 L_g P_j + (A_1 L_f + \gamma - D_1) P_1 \\ \sum_{j=1}^M \lambda_{j2} P_j + \sum_{j=1}^M \rho_{j2}(\tau) e^{\gamma\tau} B_2 L_g P_j + (A_2 L_f + \gamma - D_2) P_2 \\ \vdots \\ \sum_{j=1}^M \lambda_{j3} P_j + \sum_{j=1}^M \rho_{j3}(\tau) e^{\gamma\tau} B_3 L_g P_j + (A_3 L_f + \gamma - D_3) P_3 \end{pmatrix} \quad (17)$$

From (6) and (17), it is easy to get $W = (-\mathbb{D} + \mathbf{A} + \mathbb{L}_f + e^{\gamma\tau} \mathbb{B} \mathbb{L}_g) \mathbb{P} \prec 0$, which means $D^+ \mathbb{Z}(t') \prec 0$. Obviously, this is a contradiction. Thus $\mathbb{Z}(t) \prec \mathbb{P} l_0$ for $t \geq 0$. That is

$$\|\mathbb{Y}(t)\|_1 \prec e^{-\gamma(t-t_0)} \|\mathbb{P}\|_1 l_0 = \frac{\|\mathbb{P}\|_1}{o_{min}} E\{\|y(0)\|_1\} e^{-\gamma(t-t_0)} \quad (18)$$

Then, we have

$$E\{\|y(t)\|_1\} \prec \varepsilon E\{\|y(0)\|_1\} e^{-\gamma(t-t_0)} \quad (19)$$

where $\varepsilon = Mn \cdot \max_{1 \leq i \leq M, 1 \leq c \leq n} \{o_c^i\} / o_{min}$. The proof is completed.

3.2 EMS of discrete-time PMJNNs

In this subsection, the discrete-time PMJNNs is rewritten as an equivalent discrete-time PNNs. Through the discussion on the exponential stability issue of the discrete-time PNNs, a sufficient condition for the EMS of the system Σ_4 is provided.

First, define the indicator function [10]

$$\mathbf{1}_{\{r(k)=i\}}(\omega) \begin{cases} 1 & \text{if } r(k)(\omega) = i, i \in \mathbb{M} \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

and introduce the following notations

$$\begin{aligned} \mathbb{Y}(k) &= [\mathbf{y}_1^T(k) \ \mathbf{y}_2^T(k) \ \cdots \ \mathbf{y}_N^T(k)], \\ \mathbf{y}_p(k) &= E\{y(k) \mathbf{1}_{\{r(k)=p\}}\} \\ F(\mathbf{y}_p(k)) &= E\{F(y(k)) \mathbf{1}_{\{r(k)=p\}}\} \end{aligned} \quad (21)$$

$$\begin{aligned} \mathbb{D} &= (\mathbf{\Pi}^T \otimes I_n) \text{diag}\{D_p\}, \\ \mathbb{A} &= (\mathbf{\Pi}^T \otimes I_n) \text{diag}\{A_p\} \\ \mathbb{B} &= (\mathbf{\Pi}^T \otimes I_n) \text{diag}\{B_p\} (\mathbf{\Psi}^T(\beta) \otimes I_n) \end{aligned} \quad (22)$$

where $\mathbf{\Psi}(\beta) = \{\delta_{pq}(\beta)\}$ and $\delta_{pq}(\beta) = \mathcal{P}_r\{\{r(k+\beta) = q | r(k) = p\}\}$ for $p, q \in \mathbb{N}, \beta \geq 0, k \geq 0$. And $\mathbf{\Psi}(\beta)$ satisfies the equation $\mathbf{\Psi}(\beta) = \mathbf{\Pi}^\beta$.

Theorem 2. Under Lemma 2, if there exists a vector $\mathbb{P} = [P_1, P_2, \dots, P_N]^T \in \mathbb{R}_+^{Nn}$ and a positive constant $0 < \xi < 1$ for $p, q, \eta \in \mathbb{N}$, such that

$$\xi^{-1} \left\{ \sum_{p=1}^N \pi_{pq} D_p P_p + \sum_{p=1}^N \pi_{pq} A_p L_f P_p + \xi^{-\beta} \sum_{p=1}^N \pi_{pq} B_p \sum_{\eta=1}^N \delta_{\eta p} L_g P_\eta \right\} - P_q \prec 0 \quad (23)$$

then, the system Σ_4 is positive and EMS for a given β .

where $P_p = [o_1^p, o_2^p, \dots, o_c^p]^T$ for $p \in \mathbb{N}$ and $c \in [1, 2, \dots, n]$.

Proof. For each $p, q \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbf{y}_q(k+1) &= E\{y(k+1)\mathbf{1}_{r_{k+1}=q}\} \\
&= \sum_{p=1}^N \pi_{pq} D_p E\{y(k)\mathbf{1}_{r_k=p}\} + \sum_{p=1}^M \pi_{pq} A_p E\{F(y(k)\mathbf{1}_{r_k=p})\} \\
&\quad + \sum_{p=1}^N \pi_{pq} B_p E\{G(y(k-\beta)\mathbf{1}_{r_k=p})\} \\
&= \sum_{p=1}^N \pi_{pq} D_p \mathbf{y}_p(k) + \sum_{p=1}^N \pi_{pq} A_p F(\mathbf{y}_p(k)) \\
&\quad + \sum_{p=1}^N \pi_{pq} B_p \sum_{\eta=1}^N \mathcal{P}_r\{r(k)=p|r(k-\beta)=\eta\} E\{G(y(k-\beta))\mathbf{1}_{r_{(k-\beta)}=\eta}\} \\
&= \sum_{p=1}^N \pi_{pq} D_p \mathbf{y}_p(k) + \sum_{p=1}^N \pi_{pq} A_p F(\mathbf{y}_p(k)) \\
&\quad + \sum_{p=1}^N \pi_{pq} B_p \sum_{\eta=1}^N \delta_{\eta p} G(\mathbf{y}_\eta(k-\beta))
\end{aligned} \tag{24}$$

Write it in vector form as follows

$$\Sigma_6 : \begin{cases} \mathbb{Y}(k+1) = \mathbb{D}\mathbb{Y}(k) + \mathbb{A}\mathbb{F}(\mathbb{Y}(k)) + \mathbb{B}\mathbb{G}(\mathbb{Y}(k-\beta)) \\ \mathbb{Y}(\theta) = \boldsymbol{\varphi}_y(\theta), \theta \in [-\beta, 0] \end{cases}$$

where $\mathbb{F}(\mathbb{Y}(k)) = [F(\mathbf{y}_1(k)), F(\mathbf{y}_2(k)), \dots, F(\mathbf{y}_N(k))]$, $\mathbb{G}(\mathbb{Y}(k-\beta)) = [G(\mathbf{y}_1(k-\beta)), G(\mathbf{y}_2(k-\beta)), \dots, G(\mathbf{y}_N(k-\beta))]$. According to (10), Definition 2 and Definition 4, we know that the systems Σ_4 is EMS if the system Σ_6 is ES.

Next, we will discuss exponential stability issue of the system Σ_6 . Under Lemma 2, let $\mathbb{Z}(k) = \xi^{-(k-k_0)}\mathbb{Y}(k)$, we have

$$\begin{aligned}
\mathbb{Z}(k+1) &= \xi^{-(k+1-k_0)}\mathbb{Y}(k+1) \\
&\preceq \xi^{-(k+1-k_0)}\{\mathbb{D}\mathbb{Y}(k) + \mathbb{A}\mathbb{F}(\mathbb{Y}(k)) + \mathbb{B}\mathbb{G}(\mathbb{Y}(k-\beta))\} \\
&\preceq \xi^{-1}\xi^{-(k-k_0)}\{\mathbb{D}\mathbb{Y}(k) + \mathbb{A}\mathbb{L}_f\mathbb{Y}(k) + \mathbb{B}\mathbb{L}_g\mathbb{Y}(k-\beta)\}
\end{aligned} \tag{25}$$

Substituting $\mathbb{Z}(k) = \xi^{-(k-k_0)}\mathbb{Y}(k)$ into (25) gives

$$\mathbb{Z}(k+1) \preceq \xi^{-1}\{\mathbb{D}\mathbb{Z}(k) + \mathbb{A}\mathbb{L}_f\mathbb{Z}(k) + \xi^{-\beta}\mathbb{B}\mathbb{L}_g\mathbb{Z}(k-\beta)\} \tag{26}$$

Defining a curve in Nn -dimensional space $\vartheta = \{\mathbb{E}(l) : E_p = o_p^c l, l > 0, p = 1, 2, \dots, N, c = 1, 2, \dots, n\}$ and a set $\Lambda(\mathbb{E}) = \{\mathbb{Z} : \mathbf{0} \preceq \mathbb{Z} \preceq \mathbb{E}, \mathbb{E} \in \vartheta\}$. Let $l_0 = \|\mathbb{Y}(0)\|/o_{min}$, where $o_{min} = \min_{1 \leq p \leq N, 1 \leq c \leq n} \{o_c^p\}$. It is clear that $\Lambda(\mathbb{E}(l')) \subset \Lambda(\mathbb{E}(l))$ as $l > l'$. Defining the $\mathbb{Z}(0) = \xi^{k_0}\mathbb{Y}(0)$, we know $\mathbb{Z}(0) \subset \Lambda(\mathbb{E}(l_0))$. That is $\mathbb{Z}(0) = \xi^{k_0}\mathbb{Y}(0) \prec \mathbb{P}l_0$.

For $k > 0$, we assert that $\mathbb{Z}(k) = \xi^{-(k-k_0)}\mathbb{Y}(k) \prec \mathbb{P}l_0$. If this assertion is invalid, then there must be a corresponding $k' > 0$ which makes $\mathbb{Z}(k') = \mathbb{P}l_0$ and $\Delta\mathbb{Z}(k') \succeq 0$ hold. Nevertheless, from (26), we know that

$$\begin{aligned} \Delta\mathbb{Z}(k') &= \mathbb{Z}(k'+1) - \mathbb{Z}(k') \\ &\preceq \xi^{-1}\{\mathbb{D}\mathbb{Z}(k') + \mathbb{A}\mathbb{L}_f\mathbb{Z}(k') + \xi^{-\beta}\mathbb{B}\mathbb{L}_g\mathbb{Z}(k' - \beta)\} - \mathbb{Z}(k') \\ &\preceq \{\xi^{-1}\{\mathbb{D} + \mathbb{A}\mathbb{L}_f + \xi^{-\beta}\mathbb{B}\mathbb{L}_g\}\mathbb{P} - \mathbb{P}\}l_0 \end{aligned} \quad (27)$$

Let

$$W_1 = \xi^{-1}\{\mathbb{D} + \mathbb{A}\mathbb{L}_f + \xi^{-\beta}\mathbb{B}\mathbb{L}_g\}\mathbb{P} - \mathbb{P} \quad (28)$$

Substituting (9) and (22) into (28) yields

$$w_1 = \xi^{-1} \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1N} \\ Q_{21} & Q_{22} & \cdots & Q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{N1} & Q_{N2} & \cdots & Q_{NN} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} - \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} \quad (29)$$

where

$$\begin{aligned} Q_{11} &= \pi_{11}D_1 + \pi_{11}A_1L_f + \xi^{-\beta}(\pi_{11}B_1\delta_{11}L_g + \pi_{21}B_2\delta_{12}L_g + \pi_{N1}B_N\delta_{1N}L_g) \\ Q_{12} &= \pi_{21}D_2 + \pi_{21}A_2L_f + \xi^{-\beta}(\pi_{11}B_1\delta_{21}L_g + \pi_{21}B_2\delta_{22}L_g + \pi_{N1}B_N\delta_{2N}L_g) \\ Q_{1N} &= \pi_{N1}D_N + \pi_{N1}A_NL_f + \xi^{-\beta}(\pi_{11}B_1\delta_{N1}L_g + \pi_{21}B_2\delta_{N2}L_g + \pi_{N1}B_N\delta_{NN}L_g) \\ Q_{21} &= \pi_{12}D_1 + \pi_{12}A_1L_f + \xi^{-\beta}(\pi_{12}B_1\delta_{11}L_g + \pi_{22}B_2\delta_{12}L_g + \pi_{N2}B_N\delta_{1N}L_g) \\ Q_{22} &= \pi_{22}D_2 + \pi_{22}A_2L_f + \xi^{-\beta}(\pi_{12}B_1\delta_{21}L_g + \pi_{22}B_2\delta_{22}L_g + \pi_{N2}B_N\delta_{2N}L_g) \\ Q_{2N} &= \pi_{N2}D_N + \pi_{N2}A_NL_f + \xi^{-\beta}(\pi_{12}B_1\delta_{N1}L_g + \pi_{22}B_2\delta_{N2}L_g + \pi_{N2}B_N\delta_{NN}L_g) \\ Q_{N1} &= \pi_{1N}D_1 + \pi_{1N}A_1L_f + \xi^{-\beta}(\pi_{1N}B_1\delta_{11}L_g + \pi_{2N}B_2\delta_{12}L_g + \pi_{NN}B_N\delta_{1N}L_g) \\ Q_{N2} &= \pi_{2N}D_2 + \pi_{2N}A_2L_f + \xi^{-\beta}(\pi_{1N}B_1\delta_{21}L_g + \pi_{2N}B_2\delta_{22}L_g + \pi_{NN}B_N\delta_{2N}L_g) \\ Q_{NN} &= \pi_{NN}D_N + \pi_{NN}A_NL_f + \xi^{-\beta}(\pi_{1N}B_1\delta_{N1}L_g + \pi_{2N}B_2\delta_{N2}L_g + \pi_{NN}B_N\delta_{NN}L_g) \end{aligned}$$

Then

$$w_1 = \begin{pmatrix} \xi^{-1}\{\Xi_1 + \sum_{p=1}^N \pi_{p1}A_pL_fP_p + \xi^{-\beta}\sum_{p=1}^N \pi_{p1}B_p\sum_{\eta=1}^N \delta_{\eta p}L_gP_\eta\} - P_1 \\ \xi^{-1}\{\Xi_2 + \sum_{p=1}^N \pi_{p2}A_pL_fP_p + \xi^{-\beta}\sum_{p=1}^N \pi_{p2}B_p\sum_{\eta=1}^N \delta_{\eta p}L_gP_\eta\} - P_2 \\ \vdots \\ \xi^{-1}\{\Xi_N + \sum_{p=1}^N \pi_{pN}A_pL_fP_p + \xi^{-\beta}\sum_{p=1}^N \pi_{pN}B_p\sum_{\eta=1}^N \delta_{\eta p}L_gP_\eta\} - P_N \end{pmatrix} \quad (30)$$

where $\Xi_q = \sum_{p=1}^N \pi_{pq}D_pP_p$, $q \in \mathbb{N}$.

From (23) and (30), it's obvious that $W_1 = \xi^{-1}\{\mathbb{D} + \mathbb{A}\mathbb{L}_f + \xi^{-\beta}\mathbb{B}\mathbb{L}_g\}\mathbb{P} - \mathbb{P} \prec 0$ is established, which means $\Delta\mathbb{Z}(k') \prec 0$. It's easy to see that this is a contradiction to the previous assertion. So, $\mathbb{Z}(k) = \xi^{-(k-k_0)}\mathbb{Y}(k) \prec \mathbb{P}l_0$ for $k \geq 0$ holds.

Consequently

$$\|\mathbb{Y}(k)\|_1 \prec \xi^{(k-k_0)}\|\mathbb{P}\|_1l_0 = \frac{\|\mathbb{P}\|_1}{o_{min}}\|\mathbb{Y}(0)\|_1\xi^{(k-k_0)} \quad (31)$$

Let $\varepsilon_1 = Nn \cdot \max_{1 \leq p \leq N, 1 \leq c \leq n} \{o_c^p\} / o_{min}$, we get

$$E\{\|y(k)\|_1\} \prec \varepsilon_1 E\{\|y(0)\|_1\}\xi^{(k-k_0)} \quad (32)$$

The proof is complete.

Remark 2. In this paper, by utilizing the special properties of the PMJNNs itself, the augmentation system approach is applied to solve the stability problem of PMJNNs with time-delay, which can reduce the complexity of system analysis. The EMS conditions in terms of the standard linear programming are derived in Theorem 1 and Theorem 2. Although stability conditions for PMJSSs have been discussed in [25], there are only sufficient conditions in the form of LMI. Thus, the conservatism of the results in [25] is greatly reduced by the proposed method in the paper.

Remark 3. Delay-dependent EMS conditions for continuous time PMJNNs and discrete time PMJNNs are given in Theorem 1 and Theorem 2. In this paper, the stability of the PMJNNs are directly affected by time delays, parameters γ and ξ , which also can be verified by the example at numerical example section.

4 Numerical example

Example 1. Consider a class of the system Σ_3 with three operation modes described as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.30 & 0.12 & 0.45 \\ 0.56 & 0.25 & 0.30 \\ 0.60 & 0.20 & 0.80 \end{bmatrix}, B_1 = \begin{bmatrix} 0.05 & 0.10 & 0.03 \\ 0.10 & 0.08 & 0.04 \\ 0.15 & 0.06 & 0.10 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.55 & 0.60 & 0.30 \\ 1.20 & 0.50 & 0.28 \\ 1.00 & 0.55 & 0.35 \end{bmatrix}, B_2 = \begin{bmatrix} 0.06 & 0.02 & 0.15 \\ 0.05 & 0.04 & 0.17 \\ 0.15 & 0.27 & 0.36 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 1.30 & 0.65 & 0.80 \\ 0.40 & 0.75 & 1.00 \\ 0.26 & 0.68 & 1.10 \end{bmatrix}, B_3 = \begin{bmatrix} 0.12 & 0.25 & 0.09 \\ 0.39 & 0.33 & 0.22 \\ 0.12 & 0.30 & 0.18 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1.6 \end{bmatrix}, D_2 = \begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}, \\ D_3 &= \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}. \end{aligned}$$

And the transition probability matrix Υ is given as below

$$\Upsilon = \begin{bmatrix} -0.6 & 0.2 & 0.4 \\ 0.8 & -2 & 1.2 \\ 0.7 & 1.5 & -2.2 \end{bmatrix}$$

In this paper, boltzmann sigmoid activation functions are selected as: $h_u(y_u) = \frac{1-e^{-\frac{\gamma y_u}{\theta_{uf}}}}{1+e^{-\frac{\gamma y_u}{\theta_{uf}}}}$, $\kappa_u(y_u) = \frac{1-e^{-\frac{\gamma y_u}{\theta_{ug}}}}{1+e^{-\frac{\gamma y_u}{\theta_{ug}}}}$. Parameter γ and time delay τ are given as 0.5 and 0.2. Then, from Assumption 1, we can obtain: $L_{uf} = \frac{1}{2\theta_{uf}}$, $L_{ug} = \frac{1}{2\theta_{ug}}$, $\theta_{uf} > 0$, $\theta_{ug} > 0$, $i \in \mathbb{M}$. Choosing the corresponding θ_{uf} and θ_{ug} , we get

$$\begin{aligned} L_f &= \text{diag}\{0.5, 0.12, 0.2\}, \\ L_g &= \text{diag}\{0.2, 0.10, 0.4\}. \end{aligned}$$

Solving the linear programming problem in Theorem 1, we get:

$$P_1 = [0.0016; 0.0043; 0.0018]^T;$$

$$P_2 = [0.0011; 0.0020; 0.0014]^T;$$

$$P_3 = [0.0010; 0.0021; 0.0012]^T.$$

The initial states for the system Σ_3 is chosen as: $\phi_y(0) = [5; 2.5; 3.5]$. Fig. 1 shows mode evolution of the system Σ_3 . Fig. 2 depicts the state trajectories of the system Σ_3 , which can draw the conclusion that the systems Σ_3 is positive and EMS.

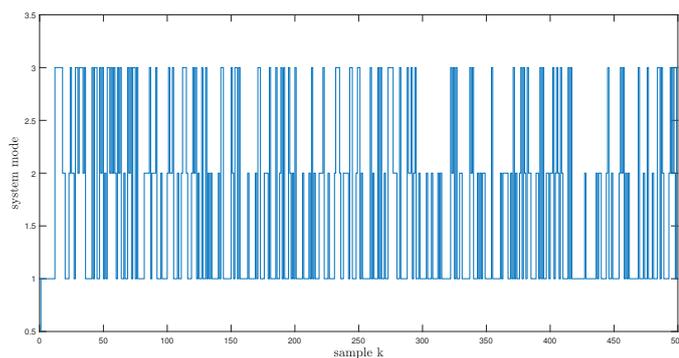


Fig. 1: System mode evolution in Example 1.

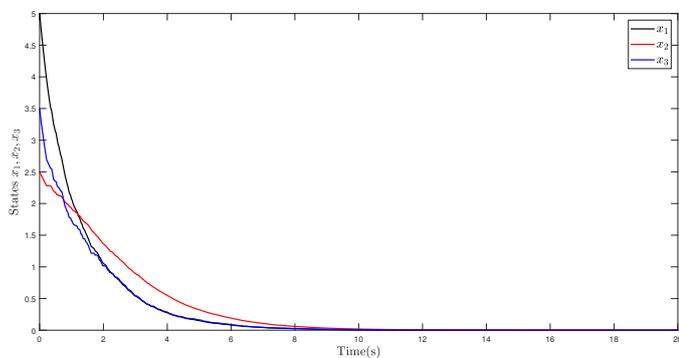


Fig. 2: System state evolution in Example 1.

Table 1 and Table 2 show the effects of different time delays τ and different system parameters γ on the stability of the system Σ_3 .

Table 1: Comparison results for different time delay τ ($\gamma = 0.5$)

τ	0	0.1	0.2	0.3
Theorem 1	Feasible	Feasible	Feasible	Feasible
τ	0.4	0.5	0.6	0.7
Theorem 1	Feasible	Infeasible	Infeasible	Infeasible

Table 2: Comparison results for different system parameter γ ($\tau = 0.2$)

γ	0	0.1	0.2	0.3
Theorem 1	Feasible	Feasible	Feasible	Feasible
γ	0.4	0.5	0.6	0.7
Theorem 1	Feasible	Feasible	Infeasible	Infeasible

Example 2. Consider a class of the system Σ_4 with three operation modes described as follows:

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 0.35 & 0.12 & 0.15 \\ 0.26 & 0.35 & 0.50 \\ 0.60 & 0.20 & 0.80 \end{bmatrix}, B_1 = \begin{bmatrix} 0.08 & 0.06 & 0.03 \\ 0.12 & 0.07 & 0.04 \\ 0.15 & 0.06 & 0.20 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} 0.80 & 0.70 & 0.30 \\ 0.40 & 0.60 & 0.28 \\ 0.45 & 0.60 & 0.20 \end{bmatrix}, B_2 = \begin{bmatrix} 0.10 & 0.02 & 0.15 \\ 0.25 & 0.04 & 0.17 \\ 0.15 & 0.09 & 0.06 \end{bmatrix}, \\
 A_3 &= \begin{bmatrix} 0.25 & 0.35 & 0.42 \\ 0.15 & 0.30 & 0.45 \\ 0.26 & 0.68 & 0.30 \end{bmatrix}, B_3 = \begin{bmatrix} 0.05 & 0.15 & 0.09 \\ 0.30 & 0.13 & 0.10 \\ 0.08 & 0.20 & 0.18 \end{bmatrix}, \\
 D_1 &= \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.85 \end{bmatrix}, D_2 = \begin{bmatrix} 0.35 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}, \\
 D_3 &= \begin{bmatrix} 0.45 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}.
 \end{aligned}$$

And the transition probability matrix Π is shown as

$$\Pi = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

Similarly, by selecting the corresponding θ_{uf} and θ_{ug} , we obtain

$$\begin{aligned}
 L_f &= \text{diag}\{0.10, 0.12, 0.06\}, \\
 L_g &= \text{diag}\{0.20, 0.10, 0.15\}.
 \end{aligned}$$

The system parameter ξ and time delay β are selected as 0.82 and 2. Solving the linear programming problem in Theorem 2, we get:

$$P_1 = [0.0010; 0.0014; 0.0031]^T;$$

$$P_2 = [0.0013; 0.0020; 0.0041]^T;$$

$$P_3 = [0.0010; 0.0011; 0.0023]^T.$$

The initial state is initialized to $\varphi_y(0) = [5; 2.5; 3.5]$. Fig. 3-4 demonstrates the simulation results of the system Σ_4 , which can illustrate that the systems Σ_4 is positive and EMS.

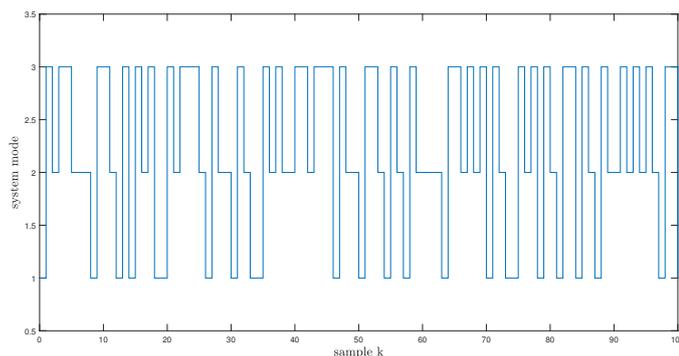


Fig. 3: System mode evolution in Example 2.

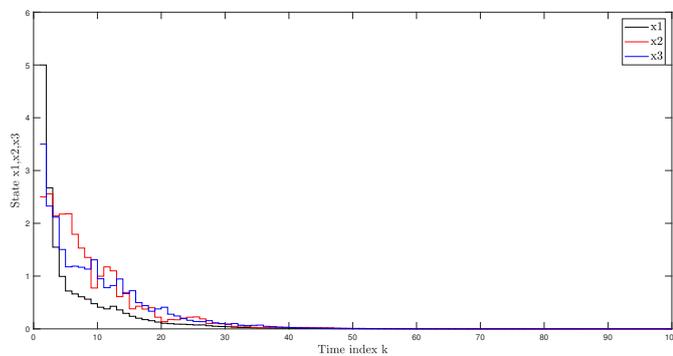


Fig. 4: System state evolution in Example 2.

Table 3 and Table 4 show the effects of different time delays β and different system parameters ξ on the stability of the system Σ_4 .

Table 3: Comparison results for different time delay β ($\xi = 0.82$)

β	0	1	2	3
Theorem 1	Feasible	Feasible	Feasible	Infeasible
β	4	5	6	7
Theorem 1	Infeasible	Infeasible	Infeasible	Infeasible

Table 4: Comparison results for different system parameter ξ ($\beta = 2$)

γ	0.2	0.4	0.8	0.82
Theorem 1	Infeasible	Infeasible	Infeasible	Feasible
γ	0.86	0.90	0.94	0.98
Theorem 1	Feasible	Feasible	Feasible	Feasible

5 Conclusion

In this paper, the linear programming method, the augmentation system method and inequality technique are used to investigate the EMS issues of PMJNNs with time delay. Sufficient criteria in the form of linear programming are put forward to guarantee the EMS on PMJNNs with time delay in both continuous-time domain and discrete-time domain. Further more, this paper reveals that the EMS of PMJNNs with time delay is influenced by the size of time-delay. Two numerical examples is provided to prove the validity of our theoretical discovery. Finally, the effect of time-varying delay for positive Markov jump neural networks will be discussed in future work.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declaration of Interest Statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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