Discrete Approximate Solution of Differential Inclusion with Normal Cone and Prox-Regular Set.

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An approximate solution of differential inclusion with prox-regular set

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Abstract

The aim of this paper is to calculate the approximate solution of differential inclusion with prox-regular set, the question is how to calculate this solution? We use the discrete approximation property of a new variant of nonconvex sweeping processes involving normal cone and a finite element method. Knowing that the majority of mathematicians have proved only the existence and uniqueness of the solution for this type of inclusions, like: Mordukhovich, Thibault, Aubin, Messaoud,...etc.

Keywords: Differential inclusion, Discret solution, Prox-regular, Normal cone, Nonsmooth analysis.

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1 Introduction

The domain of differential inclusions plays a major role in finding solutions to some physical problems see, [1],[2],[21],[23],[8],[10]. In our work, we will present a numerical method for calculating the approximate solution of a differential inclusion with normal cone for prox-regular sets, this problem has been studied extensively see, [18],[19],[17], for existence and uniqueness of the solutions see, [11],[12],[3],[9] and also gave some properties and theories that help us reach the proof of the proposed convergence on our side. We use also the same method (Discrete approximation solution), see, [14],[15], [7],[22],[20],[4], knowing that these methods were applied to the differential inclusion $\dot{x}(t) \in F(x(t))$ a.e $t \in [0, T]$, $x(0) = x_0$ that was studied by Mordukhovich, see, [13] and also applied to the differential inclusion $\dot{x}(t) \in -N_C(x(t))$ a.e $t \in [0, T]$, $x(0) = x_0$ see, [16]. In this paper we will study the following problem.

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)) & \text{a.e } t \in [0, 1] \\ x(0) = x_0 \in C \end{cases} \quad (1.1)$$

for all absolutely continuous $x : [0, 1] \rightarrow \mathbb{R}^N$. Where C is a nonempty closed r -prox-regular subset of $\mathbb{R}^n$, ($r > 0$), $N_C(x)$ is the normal cone to C at $x \in C$, and $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a Lipschitz function and bounded in C . Then, there is $M > 0$ such that

$$\|f(x)\| \leq M, \forall x \in C + r\mathbb{B}. \quad (1.2)$$

The existence and uniqueness of solutions of (1.1) is proved in [11] by L. Thibault and J. F. Edmond, but he had not calculate this solution.

So, for this raison we consider the discrete approximations

$$\begin{cases} \dot{x}_n(t_j) \in f(x_n(t_j)) + n(x_n(t_j) - P_C(x_n(t_j))), & j = 0, \dots, n \\ x_n(0) = x_0 \in C, \end{cases} \quad (1.3)$$

where:

- 1) $P_C(s) = \left\{ h \in C : \|s - h\| = \inf_{y \in C} \|s - y\| \right\}$,
- 2) $t_j := jh_n, j = 0, \dots, n, h_n = \frac{1}{n}$,
- 3) $\dot{x}_n(t_j) = \frac{x_n(t_{j+1}) - x_n(t_j)}{h_n}, j = 0, \dots, n$,
- 4) $\dot{x}_n(t) = \dot{x}_n(t_j), \forall t \in [t_j, t_{j+1}[, j = 0, \dots, n$
- 5) $x_n(t) = x_n(t_j) - (t - t_j) \dot{x}_n(t_j), \forall t \in [t_j, t_{j+1}[, j = 0, \dots, n$.

Similarly observe that the discrete-time inclusions(1.3) is equivalent to

$$\begin{cases} x_n(t_{j+1}) \in \frac{1}{n}f(x_n(t_j)) + P_C(x_n(t_j)), & j = 0, \dots, n-1, \\ x_n(0) = x_0. \end{cases} \quad (1.4)$$

Our idea is to prove that, if $x_n(\cdot)$ being the piecewise extension of the unique discrete trajectory $x_n(t_j), j = 0, \dots, n$ is a solution of (1.3) and if $x(\cdot)$ is a

solution of (1.1), we have $x_n(\cdot)$ converges uniformly to $x(\cdot)$ as $n \rightarrow +\infty$, and there exist a subsequence $\dot{x}_{n_k}(\cdot) \rightharpoonup \dot{x}(\cdot)$ as $n \rightarrow +\infty$ weakly in $L^2([0, 1], \mathbb{R}^N)$, and for each $j = 0, \dots, n$, $x_n(t_j) \in C + \frac{M}{n}\mathbb{B}$, has a unique projection on C . Moreover, the piecewise extension $x_n(\cdot)$ of the discrete trajectory $x_n(t_j)$, $j = 0, \dots, n$, also satisfies $x_n(t) \in C + \frac{M}{n}\mathbb{B}$ a.e $t \in [0, 1]$, where $\mathbb{B}$ the unit ball in $\mathbb{R}^N$.

2 Definitions and Notations

The following definitions are quite standard. We can refer to [8],[17] for further details. First, we will give some notations used in this paper. The distance function to a closed set S is given by $d_S(x) := \inf_{y \in S} \|x - y\|$.

Let C be a nonempty subset of $\mathbb{R}^N$, $\overline{C}$ the closure of C , $\overline{co}(C)$ the closed convex cone of C .

2.1 Clarke tangent cone

1) Let $x \in \overline{C}$. We say that $h \in \mathbb{R}^N$ is a tangent vector to C at x , if,
 $\forall x_n \in C, \forall \lambda_n \in \mathbb{R}^+, \exists h_n \in \mathbb{R}^N, \forall n \in \mathbb{N}$ with $x = \lim_{n \rightarrow +\infty} x_n, \lim_{n \rightarrow +\infty} \lambda_n = 0$,
 $h = \lim_{n \rightarrow +\infty} h_n$ and $x_n - \lambda_n h_n \in C, \forall n \in \mathbb{N}$.

2) $T_{Cl}(C, x) = \left\{ h \in \mathbb{R}^N / \forall x_n \in C, \forall \lambda_n \in \mathbb{R}^+, \exists h_n \in \mathbb{R}^N \text{ with } \lim_{n \rightarrow +\infty} x_n = x, \right.$
 $\left. \lim_{n \rightarrow +\infty} \lambda_n = 0, \lim_{n \rightarrow +\infty} h_n = h \text{ and } x_n - \lambda_n h_n \in C, \forall n \in \mathbb{N} \right\}$

The set $T_{Cl}(C, x)$ is called (sequential) **Clarke tangent** cone to C at x .

3) We define by $N_{Cl}(C, x) = \{h \in \mathbb{R}^N / \langle h, v \rangle \leq 0, \forall v \in T_{Cl}(C, x)\}$
the normal cone of **Clarke** to C at x .

2.2 Bouligand tangent cone

1) Let $x \in \overline{C}$. We say that $h \in \mathbb{R}^N$ is a tangent vector to C at x , if, $\exists x_n \in C, \exists \lambda_n \in \mathbb{R}^+, \forall n \in \mathbb{N}$ with $x = \lim_{n \rightarrow +\infty} x_n, \lim_{n \rightarrow +\infty} \lambda_n = 0$ and $h = \lim_{n \rightarrow +\infty} \frac{x_n - x}{\lambda_n}$.

2) $T(C, x) = \left\{ h \in \mathbb{R}^N / \exists x_n \in C, \exists \lambda_n \in \mathbb{R}^+ \text{ with } x = \lim_{n \rightarrow +\infty} x_n, \right.$
 $\left. \lim_{n \rightarrow +\infty} \lambda_n = 0 \text{ and } h = \lim_{n \rightarrow +\infty} \frac{x_n - x}{\lambda_n} \right\}$.

The set $T(C, x)$ ou $T_C(x)$ is called **Bouligand tangent** cone to C at x .

3) We define by $N_C(x) = \{h \in \mathbb{R}^N / \langle h, v \rangle \leq 0, \forall v \in T(C, x)\}$ the normal cone to C at x .

Proposition 2.1. $T_{Cl}(C, x) \subset T_C(x)$ and $N_C(x) \subset N_{Cl}(C, x)$.

2.3 Prox-regular set

Let $C \subset \mathbb{R}^N$ be a closed subset, $x \in C$ and $r \in]0, +\infty]$. We define by $N^P(C, x) := \{v \in X : \exists r > 0, x \in P_C(x + rv)\}$ the proximal normal cone to C at x .

Definition 2.2. The set C is a r -prox-regular ($r > 0$) if: $\forall s \in U_r(C) = \{x \in \mathbb{R}^N / d_C(x) < r\}$ has a unique $P_C(s)$ in C and the mapping $P_C(\cdot)$ is continuous on $U_r(C)$.

Definition 2.3. We say that C is r -prox-regular if and only if, for all $x \in C$, for all $v \in N^P(C, x) \cap \mathbb{B}_{\mathbb{R}^N}$ and for every real $t \in]0, r]$, we have $x \in P_C(x + tv)$.

Proposition 2.4. [17] The set C is r -prox-regular if and only if the function $d_C^2(\cdot)$ is differentiable on $U_r(C)$ with a locally Lipschitz derivative and $\nabla d_C^2(x) = 2(x - P_C(x))$ for all $x \in U_r(C)$.

Proposition 2.5. [17] Let C be a prox-regular at x . Then

- 1) $P_C(x)$ is single-valued around x .
- 2) d_C^2 is C^1 around x .

Lemma 2.6. [17] Let C be a closed set in $\mathbb{R}^N$ and take $r > 0$. Then the following are equivalent:

- (1) C is r -prox-regular.
- (2) For all $x \in C$ and $\xi \in N_C(x)$ such that $\|\xi\| < r$, we have

$$x = P_C(x + \xi). \quad (2.1)$$

- (3) For all $x \in C$ and $\xi \in N_C(x)$, we have

$$\langle \xi, x - y \rangle \leq \frac{\|\xi\|}{2r} \|x - y\|^2, \forall y \in C. \quad (2.2)$$

- (4) For all $x, x' \in C$, $\xi \in N_C(x)$, $\xi' \in N_C(x')$ and $\zeta, \xi' \in B$ we have

$$\langle \zeta - \xi', x - x' \rangle \geq -\frac{1}{r} \|x - x'\|^2. \quad (2.3)$$

One can refer to [19],[4], for further details.

Lemma 2.7. [4] Let b be a positive real number and $f(\cdot), g(\cdot)$ two functions in $L^1([0, b], \mathbb{R})$ such that the function $g(\cdot)$ is positive on $[0, b]$, and let w be an absolutely continuous function from $[0, b]$ into $\mathbb{R}^+$ such that

$$(1 - \lambda)w'(t) \leq f(t)w(t) + g(t)w^\lambda(t) \text{ a.e. } t \in [0, b]$$

with $0 \leq \lambda < 1$. Then the following inequality is satisfied for all $0 \leq t \leq b$

$$w^{1-\lambda}(t) \leq w^{1-\lambda}(0) \exp \left(\int_0^t f(\alpha) d\alpha \right) + \int_0^t \exp \left(\int_k^t f(\alpha) d\alpha \right) g(s) ds.$$

Lemma 2.8. [19] If $C \subset \mathbb{R}^N$ is a prox-regular subset that the two systems,

$$\begin{cases} \dot{y}(t) \in f(y(t)) - N_C(y(t)) \text{ for almost all } t \in [0, 1], \\ y(t) \in C, \forall t \in [0, 1], \\ y(0) = x_0 \in C, \end{cases} \quad (2.4)$$

and

$$\begin{cases} \dot{y}(t) = P_{\overline{\text{co}}T_C(y(t))}(f(y(t))) \text{ for almost all } t \in [0, 1], \\ y(t) \in C, \forall t \in [0, 1], \\ y(0) = x_0 \in C, \end{cases} \quad (2.5)$$

have the same set of solutions.

3 Main result

We give the main result of this section:

3.1 Assumptions

Let $C \subset \mathbb{R}^N$ be a closed r -prox-regular set. We shall also assume, without loss of generality, that f is bounded on C , namely, taking into account the Lipschitzianity of f , that

$$\|f(x)\| \leq M, \forall x \in C + r\mathbb{B}. \quad (3.1)$$

Theorem 3.1. Let $x(\cdot)$ be a solution of (1.1). Then $\dot{x}(\cdot)$ and $x(\cdot)$ are bounded.

Proof : As $x(\cdot)$ solution of (1.1) we find

$$\dot{x}(t) \in P_{\overline{\text{co}}T_C(x(t))}^{f(x(t))} \text{ a.e } t \in [0, 1], \text{ (see, [19])},$$

from (1), $\|\dot{x}(t) - f(x(t))\| \leq \|f(x(t))\|$, a.e $t \in [0, 1]$ because $0 \in \overline{\text{co}}T_C(x(t))$.

Since for all $t \in [0, 1]$ we have $x(t) \in C$, hence

$$\|\dot{x}(t)\| \leq 2\|f(x(t))\| \leq 2\|f(x(t))\| \leq 2M. \quad (3.2)$$

Writing $x(t) = x(0) + \int_0^t \dot{x}(s)ds$, we get

$$\|x(t)\| \leq 2M + \|x_0\|,$$

□

Theorem 3.2. For every $n \in \mathbb{N}$, there is a unique set of elements $\{x_n(t_j), t_0 = 0, t_{j+1} = t_j + \frac{1}{n}, j = 1, \dots, n\}$ satisfies

$$\begin{cases} x_n(t_{j+1}) \in \frac{1}{n}f(x_n(t_j)) + P_C(x_n(t_j)), j = 0, \dots, n-1, \\ x_n(0) = x_0. \end{cases} \quad (3.3)$$

Such that for each $j = 0, \dots, n$, $x_n(t_j) \in U_{\frac{M}{n}}(C)$, has a unique projection on C . Moreover, the piecewise extension $x_n(\cdot)$ of the discrete trajectory $x_n(t_j)$, $j = 0, \dots, n$, also satisfies $x_n(t) \in U_{\frac{M}{n}}(C)$ a.e $t \in [0, 1]$ and $\sup_{t \in [0, 1]} \max\{\|x_n(t)\|, \|\dot{x}_n(t)\|\} \leq 3M + \|x_0\|$ for all $t \in [0, 1]$.

Proof : Let $x_n(t_j)$, $j = 0, \dots, n$, be a solution of (3.3), there exists some $z_n(t_j) \in P_C(x_n(t_j))$ such that

$$x_n(t_{j+1}) - z_n(t_j) = \frac{1}{n} (f(x_n(t_j))). \quad (3.4)$$

For sufficiently large n it holds

$$x_n(t_j) \in U_{\frac{M}{n}}(C), \text{ for all } j = 0, \dots, n. \quad (3.5)$$

Because, (3.4) gives us

$$d(x_n(t_1), C) \leq \|x_n(t_1) - x_0\| \leq \frac{\|f(x_0)\|}{n} \leq \frac{M}{n},$$

then $x_n(t_1) \in U_{\frac{M}{n}}(C)$, since $\|f(x)\| \leq M, \forall x \in U_r(C)$ we can take $\frac{M}{n} \leq r$, and

$$x_n(t_2) - z_n(t_1) = \frac{1}{n} (f(x_n(t_1))).$$

So,

$$d(x_n(t_2), C) \leq \|x_n(t_2) - z_n(t_1)\| \leq \frac{\|f(x_n(t_1))\|}{n} \leq \frac{M}{n},$$

then $x_n(t_{j-1}) \in U_{\frac{M}{n}}(C)$, since $\|f(x)\| \leq M, \forall x \in U_r(C)$ we can take $\frac{M}{n} \leq r$, and

$$x_n(t_j) - z_n(t_{j-1}) = \frac{1}{n} (f(x_n(t_{j-1}))).$$

So,

$$d(x_n(t_j), C) \leq \|x_n(t_j) - z_n(t_{j-1})\| \leq \frac{\|f(x_n(t_{j-1}))\|}{n} \leq \frac{M}{n}, \quad (3.6)$$

which proves that

$$x_n(t_j) \in U_{\frac{M}{n}}(C), \quad (3.7)$$

from (3.4) we obtain that

$$x_n(t_{j+1}) = \frac{1}{n}f(x_n(t_j)) + P_C(x_n(t_j)),$$

hence, $(x_n(t_j))_{j=1, \dots, n}$ is uniquely defined.

To prove the last assertion of the lemma, we observe that, for every $t \in [t_j, t_{j+1})$,

$$\dot{x}_n(t) = n(x_n(t_{j+1}) - x_n(t_j)) = f(x_n(t_j)) - n(x_n(t_j) - P_C(x_n(t_j))), \quad (3.8)$$

and by using (3.6) we get

$$\|\dot{x}_n(t)\| \leq 2M. \quad (3.9)$$

Thus, writing $x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s)ds$ and $x_n(t) = x_n(t_j) + \int_{t_j}^t \dot{x}_n(s)ds$, we get

$$\|x_n(t)\| \leq 2M + \|x_0\|,$$

and

$$d(x_n(t), C) \leq d(x_n(t_j), C) + \frac{2M}{n} \leq \frac{3M}{n},$$

which proves that $x_n(t) \in U_{\frac{1}{n}}(C)$, again by an scaling of n . $\square$

Theorem 3.3. Let $x_n(\cdot)$, $n \geq 1$, be as in Theorem (3.2). Then $(x_n(\cdot))_n$ converges uniformly to the unique solution $x(\cdot)$ of (1.1).

Proof : By [11] The system (1.1) has a unique absolute continuous solution $x(\cdot)$. Then, for all n , we write for a.e $t \in [t_j, t_{j+1})$

$$\langle \dot{x}_n(t) - \dot{x}(t), x_n(t) - x(t) \rangle = \langle \dot{x}_n(t_j) - \dot{x}(t), x_n(t) - x(t) \rangle, \quad j = 0, \dots, n-1,$$

hence,

$$\begin{aligned} \langle \dot{x}_n(t) - \dot{x}(t), x_n(t) - x(t) \rangle &= \langle \dot{x}_n(t_j) - \dot{x}(t), x_n(t) - x(t) \rangle \\ &= \langle \dot{x}_n(t_j) - \dot{x}(t_j), x_n(t_j) - x(t_j) \rangle \\ &\quad + \langle \dot{x}_n(t_j) - \dot{x}(t_j), x(t_j) - x(t) - x_n(t_j) + x_n(t) \rangle \\ &\quad + \langle \dot{x}(t_j) - \dot{x}(t), x_n(t) - x(t) \rangle. \end{aligned} \quad (3.10)$$

To estimate this last inequality, we first observe that there is $(w_n(\cdot))_n$ of simple/step functions defined on $[0, 1]$ and converging to $\dot{x}(\cdot)$ in $L^\infty([0, 1], \mathbb{R}^N)$ ($w_n(t)$ is constant on $[t_j, t_{j+1})$ for every $j = 0, \dots, n-1$), and since $x(\cdot)$ is bounded by $2M + \|x_0\|$ (as a consequence of (3.2)) and $\sup_{t \in [0, 1]} \|x(t)\| \leq 2M + \|x_0\|$ (by Lemma 3.2), we get for a.e $t \in [t_j, t_{j+1})$, $j = 0, \dots, n-1$,

$$\begin{aligned} \langle \dot{x}(t_j) - \dot{x}(t), x_n(t) - x(t) \rangle &= \langle \dot{x}(t_j) - w_n(t_j), x_n(t) - x(t) \rangle + \langle w_n(t) - \dot{x}(t), x_n(t) - x(t) \rangle \\ &\leq 2(3M + 2\|x_0\|) \|\dot{x} - w_n\|_\infty = \mathcal{O}\left(\frac{1}{n}\right), \end{aligned} \quad (3.11)$$

where $\mathcal{O}(\frac{1}{n})$ is a value such that $n\mathcal{O}(\frac{1}{n})$ is bounded independently of n . Also, using (3.2) and (3.9), we have that $\|x(t) - x(t_j)\| \leq \frac{M}{n}$ and $\|x_n(t) - x_n(t_j)\| \leq \frac{M}{n}$ for a.e $t \in [t_j, t_{j+1})$ and, so,

$$\langle \dot{x}_n(t_j) - \dot{x}(t_j), x(t_j) - x(t) - x_n(t_j) + x_n(t) \rangle \leq \mathcal{O}\left(\frac{1}{n}\right). \quad (3.12)$$

On the other hand, using the definition of $\dot{x}_n(t)$ (see (3.8) and $\dot{x}(t)$), namely, that $\dot{x}(t) = f(x(t)) - y(t)$ for some $y(t) \in N_C(x(t))$, we have for $j = 0, \dots, n-1$

$$\begin{aligned} \langle \dot{x}_n(t_j) - \dot{x}(t_j), x_n(t_j) - x(t_j) \rangle &= \langle f(x_n(t_j)) - n(x_n(t_j) - P_C(x_n(t_j))), x_n(t_j) - x(t_j) \rangle \\ &\quad - \langle f(x_\mu(t_j)) - y(t_j), x_n(t_j) - x(t_j) \rangle \\ &= \langle f(x_n(t_j)) - f(x(t_j)), x_n(t_j) - x(t_j) \rangle \\ &\quad - \langle n(x_n(t_j) - P_C(x_n(t_j))) - y(t_j), x_n(t_j) - x(t_j) \rangle. \end{aligned} \quad (3.13)$$

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Since, $n(x_n(t_j) - P_C(x_n(t_j))) \in N_C(P_C(x_n(t_j)))$, and $y(t_j) \in N_C(x(t_j))$, the monotonicity of the operator $N_C \cap \mathbb{B} + \frac{1}{r} \lim id$, resulting by Proposition 2.7(4), leads us to (recall that $x_n(t_j) \in U_{\frac{3M}{n}}(C)$, by (3.7)),

$$\begin{aligned}
& - \langle n(x_n(t_j) - P_C(x_n(t_j))) - y(t), x_n(t_j) - x(t_j) \rangle \\
& = -M \left\langle \frac{n}{M}(x_n(t_j) - P_C(x_n(t_j))) + \frac{1}{r}P_C(x_n(t_j)), P_C(x_n(t_j)) - x(t_j) \right\rangle \\
& \quad - M \left\langle -\frac{1}{M}y(t_j) - \frac{1}{r}x_\mu(t_j), P_C(x_n(t_j)) - x(t_j) \right\rangle + \frac{M}{r} \|x(t_j) - P_C(x_n(t_j))\|^2 \\
& \quad - \langle n(x_n(t_j) - P_C(x_n(t_j))) - y(t_j), x_n(t_j) - P_C(x_n(t_j)) \rangle \\
& \leq \frac{M}{r} \|x(t_j) - P_C(x_n(t_j))\|^2 \\
& \quad - \langle n(x_n(t_j) - P_C(x_n(t_j))) - y(t_j), x_n(t_j) - P_C(x_n(t_j)) \rangle \\
& \leq \frac{M}{r} (2 \|x(t_j) - x_n(t_j)\|^2 + 2 \|x_n(t_j) - P_C(x_n(t_j))\|^2) + O(\frac{1}{n}) \\
& \leq \frac{2M}{r} \|x(t_j) - x_n(t_j)\|^2 + O(\frac{1}{n}).
\end{aligned} \tag{3.14}$$

Finally, the lipschitzianity of f (with a Lipschitz constant $L \geq 0$) implies that

$$\langle f(x_n(t_j)) - f(x(t_j)), x_n(t_j) - x(t_j) \rangle \leq L \|x(t_j) - x_n(t_j)\|^2. \tag{3.15}$$

Consequently, by (3.1), (3.14), and (3.15), inequality (3.13) gives us,

$$\langle \dot{x}_n(t_j) - \dot{x}(t_j), x_n(t_j) - x(t_j) \rangle \leq (\frac{2M}{r} + L) \|x(t_j) - x_n(t_j)\|^2 + \mathcal{O}(\frac{1}{n}). \tag{3.16}$$

Hence, by (3.11), (3.12), and (3.16), inequality (3.10) gives us, for a.e. $t \in [t_j, t_{j+1})$ and $j = 0, \dots, n-1$,

$$\langle \dot{x}_n(t) - \dot{x}(t), x_n(t) - x(t) \rangle \leq (\frac{2M}{r} + L) \|x(t_j) - x_n(t_j)\|^2 + \mathcal{O}(\frac{1}{n}). \tag{3.17}$$

But, using (3.2) and (3.9), we have

$$\begin{aligned}
\|x(t_j) - x_n(t_j)\| & \leq \|x(t) - x_n(t)\| + \|x(t) - x(t_j)\| + \|x_n(t) - x_n(t_j)\| \\
& \leq \|x(t) - x_n(t)\| + \mathcal{O}(\frac{1}{n}),
\end{aligned}$$

and, so, (3.17) gives us

$$\langle \dot{x}_n(t) - \dot{x}(t), x_n(t) - x(t) \rangle \leq 2(\frac{2M}{r} + L) \|x(t) - x_n(t)\|^2 + \mathcal{O}(\frac{1}{n}).$$

At this stage, we apply Gronwall's Lemma to obtain

$$\sup_{t \in [0,1]} \|x_n(t) - x(t)\|^2 \leq \mathcal{O}(\frac{1}{n}), \tag{3.18}$$

which completes the proof. $\square$

3.2 The basic Theorem

Theorem 3.4. The differential inclusion (1.1) has a unique absolute continuous solution $x(\cdot)$, which is the uniform limit of a sequence $(x_n(\cdot))_n$, with $x_n(t) \in U_{\frac{M}{n}}(C)$ and has a unique projection on C , a.e. $t \in [0, 1]$. Each $x_n(\cdot)$

being the piecewise extension of the unique discrete trajectory $x_n(t_j)$, $j = 0, \dots, n$ solution of

$$\begin{cases} x_n(t_{j+1}) = \frac{1}{n}f(x_n(t_j)) + P_C(x_n(t_j)), j = 0, \dots, n-1, \\ x_n(0) = x_0. \end{cases}$$

Moreover, $\dot{x}_n(\cdot) \rightharpoonup \dot{x}(\cdot)$ weakly in $L^2([0, 1], \mathbb{R}^N)$ as $n \rightarrow +\infty$.

Proof : By [11] The system (1.1) has a unique absolute continuous solution, and by Theorem(3.3), $(x_n(\cdot))_n$ converges uniformly to $x(\cdot)$, the unique solution of (1.1). Since $(\dot{x}_n(\cdot))_{n \geq 1}$ is bounded, we may assume that it weakly converges to some $v(\cdot) \in L^2([0, 1])$. Classical arguments permit us to show that $v(\cdot) = \dot{x}(\cdot)$. $\square$

3.3 Numerical application

We can find an approximate solution of the differential inclusion(1.1) by taking n large enough, we have $x_n(t_j)$, $j = 0, \dots, n$ is a solution of:

$$x_n(t_{j+1}) = \frac{1}{n}f(x_n(t_j)) + P_C(x_n(t_j)), j = 0, \dots, n-1, \text{ with } x_n(0) = x_0,$$

and $t_j := jh_n \in [0, 1]$ as $j = 0, \dots, n, h_n = \frac{1}{n}$.

So, $x_n(0) = x_0$, $x_1 = x_n(\frac{1}{n}) = \frac{1}{n}f(x_0) + P_C(x_0)$, $x_2 = x_n(\frac{2}{n}) = \frac{1}{n}f(x_1) + P_C(x_1)$, ..., $x_{j+1} = x_n(\frac{j}{n}) = \frac{1}{n}f(x_j) + P_C(x_j)$, ..., $x_n(1) = \frac{1}{n}f(x_{n-1}) + P_C(x_{n-1})$.

And $(x_n(\cdot))_n$ converges uniformly to $x(\cdot)$, the unique solution of (1.1), then for n large enough, the discrete trajectory $x_n(t_j)$, $j = 0, \dots, n$ is approximate value of solution to (1.1) in t_j , $j = 0, \dots, n$.

We obtain $x_n(t) = x_n(t_j) - (t - t_j)\dot{x}_n(t_j)$, $\forall t \in [t_j, t_{j+1}[$, as $j = 0, \dots, n$.

So, (3.18) which gives the below equality

$$\|x_n(t) - x_n(t_j)\| \leq \mathcal{O}(\frac{1}{n}), \forall t \in [t_j, t_{j+1}[, \text{ as } j = 0, \dots, n.$$

Then $x_n(\cdot)$ is an approximate solution to (1.1) for n large enough.

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