A New Model of Fuzzy Logic: Monadic Monoidal T-Norm Based Logic

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A new model of fuzzy logic: Monadic Monoidal t-norm based logic

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Abstract

In this article, we introduce the variety of monadic MTL-algebras as MTL-algebras equipped with two monadic operators. After a study of the basic properties of this variety, we define and investigate monadic filters in monadic MTL-algebras. By using the notion of monadic filters, we prove the subdirect representation theorem of monadic MTL-algebras and characterize simple and subdirectly irreducible monadic MTL-algebras. Moreover, present monadic monoidal t-norm based logic ($\mathcal{MMTL}$), a system of many valued logic capturing the tautologies of monadic MTL-algebras and prove a completeness theorem.

Keywords: (Monadic) MTL-algebras, Monadic filter, Monadic monoidal t-norm logic.

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1. introduction

Fuzzy logic has become a subject of increasing interest as the basis for reasoning with vague knowledge. In recent decades, different kinds of nonclassical logic systems have been proposed and researched. Among various non-classical logic systems, triangular norm and so monoidal t-norm based logic (MTL for short) is one of the most significant. The history of triangular norms started with the paper "Statistical metrics" (Menger 1942). The main idea of Karl Menger was to construct metric spaces where probability distributions rather than numbers are used in order to describe the distance between two elements of the space in question. Many results concerning t-norms were obtained in the course of this development, most of which are summarized in the monograph.

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Building on these ideas, logicians like Hájek , Nývák, Gottwald and others consider the core of fuzzy logic as a family of residuated many-valued logical calculi with truth values on the real unit interval $[0, 1]$. Hájek introduced a very general many-valued logic, called Basic Logic [5]. It is a well-known result that a t-norm has a residuum if and only if the t-norm is left-continuous, so this shows that Basic Logic is not the most general t-norm based logic. Actually, a logic weaker than Basic Logic, called Monoidal t-norm based logic (MTL for short), was defined by Esteva and Godo in [4] and proved to be the logic of left-continuous t-norms and their residua. Thus, the MTL is indeed the logic of left-continuous t-norms, and MTL algebras are the algebraic counterpart of this logic. The point of departure in defining MTL and the corresponding MTL-algebras is the structure introduced on $[0, 1]$ by a left continuous t-norm [4].

Halmos introduced monadic Boolean algebra: a variety of Boolean algebra equipped with a closure operator $\exists$, which abstracts algebraic properties of the standard existential quantifier for some [6]. The name monadic comes from the connection with predicate logics for languages having one placed predicates and a single quantifier. Similar algebraic structures were considered for various logics in [7, 8]. Monadic Heyting algebras, monadic BL-algebras and monadic bounded residuated lattices were introduced and investigated in [10, 9]. J. T.Wang et al. introduced and considered monadic NM-algebras and their monadic filters and defined the relations between quantifiers and modal operators on NM-algebras [11].

The goal of this manuscript is to chart the landscape of formal fuzzy logics more specifically, to develop a logic that formally characterizes tautologies (true formulas) in similarity monadic MTL-algebras. To this purpose, in the present paper, we define and study monadic MTL-algebras endowed with unary operators $\forall$ and $\exists$, namely monadic MTL-algebras. In addition, monadic filters are defined. Definition of monadic filters in monadic MTL-algebras is different from that of filter in other algebraic structures such as MTL-algebras. Since operator $\forall$ and $\exists$ in this structure are important and are used in the definition of monadic filters, these types of filters play a basic role and thus the extended filters behave differently. And so, we give some properties of their factor algebras. In Section 2, we define the notion of monadic MTL-algebras and give their basic properties that will be used in the extant of the article. Therefore, the concept of monadic filters of monadic MTL-algebras have been introduced then, we classification these algebraic structure.

In this article, we will introduce the variety of monadic MTL-algebras and give a standard study of their basic properties, which includes the characterization of their congruences and subdirectly irreducible algebras. The notion of monadic filters on MTL-algebras have been defined and we

consider them in detail. Since monadic filters enable the classification of monadic MTL-algebras, they are of particular importance. In particular, the paper [1] was a first attempt to define two analytic calculi for $\text{MTL}_\forall$ and gave some properties of this logic. However, there is no research about monadic MTL-algebras so far. It would be interesting to study monadic MTL-algebras in order to define and discuss monadic monoidal t-norm based logic. These are motivations for us to investigate monadic MTL-algebras and their logic.

In the first part of this paper, we overview recent results by the authors and others about MTL-algebras. In Section 2, we define the notion of monadic MTL-algebras and give their basic properties that will be used in the extant of the article. Therefore, the concept of monadic filters of monadic MTL-algebras have been introduced then, by means of this notion we classification these algebraic structure. We define and characterize simple and subdirectly irreducible monadic MTL-algebras by monadic filters. In the last section, the monadic monoidal t-norm based logic ($\mathcal{MMTL}$) is defined, the Lindenbaum algebra of such logic, is a monadic MTL-algebra and completeness is proved.

2. Preliminaries

We first recollect some definitions and results that are used later.

Definition 2.1. [4] *A residuated lattice is an algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, 0, 1)$ with four binary operations and two constant 0,1 such that:*

(RL_1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(RL_2) $*$ is commutative and associative, with 1 as neutral element, and

(RL_3) $x * y \leq z$ iff $x \leq y \rightarrow z$, for all x, y and z in A (residuation principle).

• A residuated lattice A is called an MTL-algebra if it satisfies the following equation for all $a, b \in A$:

(MTL) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

• An MTL-algebra A is called a BL-algebra if it satisfies the following equation for all $a, b \in A$:

(BL) $x \wedge y = x * (x \rightarrow y)$ (divisibility).

The ordering $\leq$ and negation $\neg$ in a residuated lattice $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, 0, 1)$ are defined as follows, for all x and y in A : $x \leq y$ iff $x \wedge y = x$ (or equivalently, iff $x \vee y = y$; or, also equivalently,

iff $x \rightarrow y = 1$) and $\neg x = x \rightarrow 0$.

Lemma 2.1. [4] *Let $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, 0, 1)$ be an MTL-algebra. Then the following properties are valid, for all x, y and z in L :*

$$(1) \neg x \vee y \leq x \rightarrow y,$$

$$(2) x * (x \rightarrow y) \leq x \wedge y \text{ (in particular } x * \neg x = 0),$$

$$(3) x \vee y \leq (x \rightarrow y) \rightarrow y \text{ (in particular } x \leq \neg \neg x),$$

$$(4) x \rightarrow y = ((x \rightarrow y) \rightarrow y) \rightarrow y,$$

$$(5) \neg \neg \neg x = \neg x,$$

$$(6) (x \rightarrow y) * z \leq x \rightarrow (y * z),$$

$$(7) (x \rightarrow y) * (y \rightarrow z) \leq (x \rightarrow z),$$

$$(8) (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z) \text{ (in particular } \neg(x \vee y) = \neg x \wedge \neg y),$$

$$(9) (x * y) \rightarrow z = x \rightarrow (y \rightarrow z) \text{ (in particular } \neg(x * y) = x \rightarrow \neg y),$$

$$(10) \text{ If } x \leq y, \text{ then } x * z \leq y * z, z \rightarrow x \leq z \rightarrow y \text{ and } y \rightarrow z \leq x \rightarrow z,$$

$$(11) (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z),$$

$$(12) x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$$

$$(13) ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y,$$

$$(14) x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z),$$

$$(15) \neg \neg x * \neg \neg y \leq \neg \neg (x * y).$$

From now on $(A, *, \rightarrow, \vee, \wedge, 0, 1)$ or simply A is an MTL-algebra unless otherwise specified.

Definition 2.2. [5] *A deductive system of an MTL-algebra A is a non empty subset F containing 1 such that $x \rightarrow y \in F$ and $x \in F$ imply that $y \in F$.*

A filter of an MTL-algebra A is a non-void subset F of A such that, for all $a, b \in A$,

$$(i) \text{ if } x, y \in F, \text{ then } x * y \in F,$$

$$(ii) \text{ if } x \in F \text{ and } x \leq y, \text{ then } y \in F.$$

Note that a subset F of an MTL-algebra A is a deductive system of A iff F is a filter of A .

3. Monadic MTL-algebras and monadic filters

In this section, we introduce monadic MTL-algebras and monadic filters on monadic MTL-algebras and investigate some of their algebraic properties. The relationship between monadic filters and some related structures will be discussed.

Definition 3.1. An algebra $(A, *, \rightarrow, \vee, \wedge, \exists, \forall, 0, 1)$ is called a monadic MTL-algebra if $(A, *, \rightarrow, \vee, \wedge, 0, 1)$ is an MTL-algebra and the following identities are satisfied:

$$(M_1) \forall x \rightarrow x = 1,$$

$$(M_2) \forall(x \rightarrow \forall y) = \exists x \rightarrow \forall y,$$

$$(M_3) \forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y,$$

$$(M_4) \forall(\exists x \vee y) = \exists x \vee \forall y,$$

$$(M_5) \exists(x * x) = \exists x * \exists x.$$

In the following we give some examples of monadic MTL-algebras.

Example 3.1. Let $A = \{0, a, b, c, 1\}$ and the operations $*, \rightarrow$ defined as follows:

$*$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Clearly, $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is an MTL-algebra.

Consider the maps $\forall_i, \exists_j : A \longrightarrow A$, $i = 1, \dots, 3$, $j = 1, 2, 3$, given in the tables below:

x	0	c	a	b	1
$\forall_1 x$	0	0	0	0	1
$\forall_2 x$	0	c	c	c	1
$\forall_3 x$	0	c	a	b	1

x	0	c	a	b	1
$\exists_1 x$	0	1	1	1	1
$\exists_2 x$	0	c	1	1	1
$\exists_3 x$	0	c	a	b	1

Then the structures $(A, \forall_i, \exists_i)$, $i = 1, 2, 3$ are monadic MTL-algebras.

Lemma 3.1. [3] Let $(A, \forall)$ be a monadic MTL-algebra. Then the following properties hold, for all $x, y \in A$:

	(1) $\forall \exists x = \exists x$.	(2) $x \leq \exists x$.
115	(3) $\forall 0 = 0$ and $\forall 1 = 1$.	(4) $\exists 0 = 0$ and $\exists 1 = 1$.
	(5) $\forall(x \rightarrow \exists y) = \exists x \rightarrow \exists y$.	(6) $\exists \forall x = \forall x$.
	(7) $\exists \exists x = \exists x$ and $\forall \forall x = \forall x$.	(8) $\forall(\exists x \rightarrow \exists y) = \exists x \rightarrow \exists y$.
	(9) $\forall(\exists x \rightarrow \exists y) = \exists x \rightarrow \exists y$.	(10) $\exists(x \rightarrow \exists y) \rightarrow (\forall x \rightarrow \exists y) = 1$.
	(11) $\exists(\exists x * \exists y) = \exists x * \exists y$.	(12) $\exists(x \vee y) = \exists x \vee \exists y$.
120	(13) $\forall x = x$ iff $\exists x = x$.	(14) $\forall(x \wedge y) = \forall x \wedge \forall y$.
	(15) $\forall(\exists x \rightarrow b) = \exists x \rightarrow \forall y$.	(16) $\forall(\forall x \vee y) = \forall x \vee \forall y$.
	(17) $\exists(\exists x \rightarrow b) \rightarrow (\exists x \rightarrow \exists y) = 1$.	(18) $\forall(\exists x \vee \exists y) = \exists a \vee \exists y$.
	(19) $\forall(x \rightarrow y) \rightarrow (\forall x \rightarrow \forall y) = 1$.	(20) $\forall(x \rightarrow y) \rightarrow (\exists x \rightarrow \exists y) = 1$.
	(21) $(\forall x * \exists y) \rightarrow \exists(x * y) = 1$.	(21) $(\forall x * \forall y) \rightarrow \exists(x * y) = 1$.
125	(22) $\exists(a * \exists y) = \exists x * \exists y$.	(23) $\exists(x * \forall y) = \exists x * \forall y$.
	(24) $\exists(\exists x \rightarrow \exists y) = \exists x \rightarrow \exists y$.	(25) $\exists(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.
	(26) $\exists(\exists x \wedge \exists y) = \exists x \wedge \exists y$.	(27) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$.
	(28) $\forall(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.	(29) $\exists(\forall x * \forall y) = \forall x * \forall y$.
	(30) $\forall(\forall x * \forall y) = \forall x * \forall y$.	(31) $\forall(\forall x \wedge \forall y) = \forall x \wedge \forall y$.

130 **Remark 3.1.** The operations $\forall$ and $\exists$ are order preserving, i.e. if $x \leq y$, then $\forall x \leq \forall y$ and $\exists x \leq \exists y$.

Proof. Let $x \leq y$. Then $\forall x \leq x \leq y \leq \exists y$. By (M_3) , we have $\forall x \rightarrow \forall y = \forall(\forall x \rightarrow y) = \forall 1 = 1$. From Lemma 3.1 (5), $\exists x \rightarrow \exists y = \forall(x \rightarrow \exists y) = \forall 1 = 1$. $\square$

Theorem 3.1. Let $(A, \forall, \exists)$ be a monadic MTL-algebra. Then the following conditions hold:

- 135 (i) $\neg(\exists x) = \forall(\neg x)$,
- (ii) $\exists x \leq \neg\neg(\exists x) = \neg(\forall(\neg x))$,

$$(iii) \forall x \leq \neg(\exists(\neg x)) = \forall(\neg\neg x),$$

$$(iv) \neg\neg(\forall(\neg\neg x)) = \forall(\neg\neg x) \geq \neg\neg(\forall x),$$

$$(v) \neg\neg(\exists(\neg\neg x)) = \exists(\neg\neg x) \geq \neg\neg(\exists x).$$

Proof. (i) From Lemma 3.1 (4), (5), we have:

$$\begin{aligned} \neg(\exists x) &= \exists x \rightarrow 0 \\ &= \exists x \rightarrow \exists 0 \\ &= \forall(x \rightarrow \exists 0) \\ &= \forall(x \rightarrow 0) \\ &= \forall(\neg x). \end{aligned}$$

140 (ii) By Lemma 2.1, $\exists x \leq \neg\neg(\exists x)$, from (i) we have $\neg\neg(\exists x) = \neg(\forall(\neg x))$.

(iii) From Lemma 3.1 (4), (5), we get $\neg(\exists(\neg x)) = \exists(x \rightarrow 0) \rightarrow 0 = \exists(x \rightarrow 0) \rightarrow \exists 0 = \forall((x \rightarrow 0) \rightarrow 0) = \forall(\neg\neg x)$. By Remark 3.1, $\forall x \leq \neg(\exists(\neg x)) = \forall(\neg\neg x)$.

(iv) From (ii), $\exists(\neg x) \leq \neg(\forall(\neg\neg x))$ so $\neg(\exists(\neg x)) \geq \neg\neg(\forall(\neg\neg x))$ and by (iii), $\neg(\exists(\neg x)) = \forall(\neg\neg x)$. Thus $\forall(\neg\neg x) \geq \neg\neg(\forall(\neg\neg x))$ and so $\neg\neg(\forall(\neg\neg x)) = \forall(\neg\neg x) \geq \neg\neg(\forall x)$.

145 (v) Using (i), (ii), (iii) and Lemma 3.1 (10), we obtain $\neg\neg(\exists(\neg\neg x)) = \neg(\forall(\neg\neg\neg x)) = \neg(\forall(\neg x)) = \neg\neg(\exists x)$. Also, $\exists(\neg\neg x) = \exists((x \rightarrow 0) \rightarrow 0) = \exists((x \rightarrow 0) \rightarrow \exists 0) \leq \forall(x \rightarrow 0) \rightarrow \exists 0 = \neg(\forall(\neg x)) = \neg\neg(\exists x)$. Therefore $\neg\neg(\exists(\neg\neg x)) = \exists(\neg\neg x) \geq \neg\neg(\exists x)$. $\square$

For monadic MTL-algebra $(A, \exists, \forall)$, denote:

$$A_{\forall} = \{x \in A \mid x = \forall x\},$$

$$A_{\exists} = \{x \in A \mid x = \exists x\}.$$

Put, $A_{\forall\exists} = A_{\forall} \cap A_{\exists}$.

Example 3.2. In Example 3.1, we have $A_{\forall_2} = \{0, c, 1\}$, $A_{\exists_1} = \{0, 1\}$. Thus $A_{\forall_2\exists_1} = \{0, 1\}$.

As a consequence of Lemma 3.1, we find:

Proposition 3.1. Let $(A, \exists, \forall)$ be a monadic MTL-algebra. Then $A_{\forall} = \forall(A) = \exists(A) = \exists(A_{\forall}) =$

155 $\forall(A_{\forall})$, where $\forall(A) = \{\forall x \mid x \in A\}$ and $\exists(A) = \{\exists x \mid x \in A\}$.

Proposition 3.2. *Let $(A, \exists, \forall)$ be a monadic MTL-algebra. Then $A_{\forall\exists}$ is a subalgebra of MTL-algebra A .*

Proof. Clearly, $A_{\forall\exists} \neq \emptyset$. For all $x, y \in A_{\forall\exists}$, by Lemma 3.1 (11), (12), (13) we have $\exists(x * y) = \exists(\exists x * y) = \exists x * \exists y = x * y$ and $\exists(x \vee y) = \exists x \vee \exists y = x \vee y, \exists(x \wedge y) = x \wedge y$.

160 Also, from Lemma 3.1(8), $\forall(\exists x \rightarrow \exists y) = \exists x \rightarrow \exists y$ thus by Lemma 3.1 (10), $\exists(\exists x \rightarrow \exists y) = \exists x \rightarrow \exists y$ and so $\exists(x \rightarrow y) = x \rightarrow y$. Moreover, $\exists 0 = 0, \exists 1 = 1$. $\square$

Proposition 3.3. *Let $(A, \exists, \forall)$ be a monadic MTL-algebra. Then the following identities hold, for every $x, y \in A$:*

$$(i) \exists(x \rightarrow y) \leq \forall x \rightarrow \exists y.$$

165 $(ii) \forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y.$

Proof. (i) Let x and y be in A . Because $\forall x$ and $\exists y$ are in $A_{\forall}$, also $\forall x \rightarrow \exists y \in A_{\forall}$. So $\forall x \rightarrow \exists y = \exists(\forall x \rightarrow \exists y)$ (C). Using (M_1) and Lemma 3.1(2), $\forall x \leq x, y \leq \exists y$, we find $x \rightarrow y \leq \forall x \rightarrow \exists y$ (M). Thus, $\exists(x \rightarrow y) \leq \forall x \rightarrow \exists y$ (M).

(ii) By (M_1) and (M) we have $\forall x \rightarrow \forall y \leq \forall x \rightarrow y$, so $\forall x \rightarrow \forall y = \forall(\forall x \rightarrow \forall y) \leq \forall(\forall x \rightarrow y)$ by (C) and (M). On the other hand, by (M_1) , $\forall(\forall x \rightarrow y) \leq \forall x \rightarrow y$. By the residuation principle, $\forall x * \forall(\forall x \rightarrow y) \leq y$. Moreover, $\forall x * \forall(\forall x \rightarrow y) = \forall(\forall x * \forall(\forall x \rightarrow y)) \leq \forall y$, by (C) and (M). Applying the residuation principle again, we obtain $\forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y$. $\square$

Definition 3.2. *Let $(A, \exists, \forall)$ be a monadic MTL-algebra. A monadic filter F of A is a non-empty subset F of A satisfying:*

$$(F.1) \text{ if } x \in F, y \in A \text{ and } x \leq y, \text{ then } y \in F,$$

$$(F.2) \text{ if } x, y \in F, \text{ then } x * y \in F,$$

$$(F.3) \text{ if } x \in F, \text{ then } \forall x \in F.$$

Example 3.3. *Consider Example 3.1, $F = \{b, 1\}$ is a monadic filter of $(A, \forall_3, \exists_2)$.*

180 **Definition 3.3.** *For a nonempty subset $S \subseteq A$, the smallest monadic filter of A which contains S , i.e. $\cap\{F : S \subseteq F\}$, is said to be the monadic filter of A generated by S and will be denoted by $[S]$. If $S = \{a\}$, with $a \in A$, we denoted by $[a]$ the monadic filter generated by $\{a\}$ ($[a]$ is called principal).*

We will denote the set of all monadic filters of $(A, \forall, \exists)$ by $MF(A)$.

185 **Proposition 3.4.** *Let $S \subseteq A$, a nonempty subset of A , $a \in A$. Then*

$$[S] = \{x \in A : \forall s_1 * \dots * \forall s_n \leq x, \text{ for some } n \geq 1 \text{ and } s_1, \dots, s_n \in S\}.$$

In particular, $[a] = \{x \in A : (\forall a)^n \leq x, \text{ for some } n \geq 1\}$.

Proof. Let $M = \{x \in A : \forall s_1 * \dots * \forall s_n \leq x, \text{ for some } n \geq 1 \text{ and } s_1, \dots, s_n \in S\}$. Then M is a monadic filter which contain the set S , hence $[S] \subseteq M$. Let F be a monadic filter such that $S \subseteq F$ and $x \in M$. Then there exist $s_1, s_2, \dots, s_n \in S$ such that $\forall s_1 * \dots * \forall s_n \leq x$. Since $s_1, s_2, \dots, s_n \in F$, $\forall s_1 * \dots * \forall s_n \in F$, $x \in F$. Hence $M \subseteq F$, therefore $M \subseteq F = [S]$, that is, $[S] = M$. $\square$

If F is a monadic filter and $x \notin F$, put $[F, x] := [F \cup \{x\}]$.

Theorem 3.2. *Let F, F_1, F_2 be monadic filters of A and $a \notin F$. Then*

$$(i) [F \cup a] = \{x \in A \mid x \geq f * (\forall a)^n, f \in F\} = F \vee [\forall a],$$

$$195 (ii) [F_1 \cup F_2] = \{x \in A \mid x \geq f_1 * f_2, f_1 \in F_1, f_2 \in F_2\},$$

$$(iii) \text{ if } a \leq b, \text{ then } [b] \subseteq [a],$$

$$(iv) [\forall a] = [a],$$

$$(v) [a] \vee [b] = [a \wedge b] = [a * b],$$

$$(vi) [a] \cap [b] = [\forall a \vee \forall b].$$

200 *Proof.* The proofs of (i)- (iv) are obvious.

(v) Since $a * b \leq a \wedge b \leq a, b$, By (iii), we get $[a], [b] \subseteq [a \wedge b] \subseteq [a * b]$. It follows from that $[a] \vee [b] \subseteq [a \wedge b] \subseteq [a * b]$.

Conversely, let $a \in [a * b]$. Then $a \geq (\forall(a * b))^n \geq (\forall a * \forall b)^n = (\forall a)^n * (\forall b)^n$, for some $n \in \mathbb{N}$. So $a \in [a] \vee [b]$ and $[a * b] \subseteq [a] \vee [b]$. Therefore $[a] \vee [b] = [a \wedge b] = [a * b]$.

205 (vi) Since $\forall a \leq \forall a \vee \forall b$, we have $[\forall a \vee \forall b] \subseteq [\forall a] = [a]$. Similarly, $[\forall a \vee \forall b] \subseteq [\forall b] = [b]$. It follows that $[\forall a \vee \forall b] \subseteq [a] \cap [b]$. Moreover, let $t \in [a] \cap [b]$. So, for some $n \in \mathbb{N}$ and $m \geq 1$, we deduce that $t \geq (\forall a)^m$ and $t \geq (\forall b)^n$. Hence $t \geq (\forall a)^m \vee (\forall b)^n \geq (\forall a \vee \forall b)^{nm} = (\forall(a \vee b))^{nm}$ and $a \in [\forall a \vee \forall b]$, that is $[a] \cap [b] \subseteq [\forall a \vee \forall b]$. Therefore $[a] \cap [b] = [\forall a \vee \forall b]$. $\square$

Theorem 3.3. *Let F be a monadic filter of $(A, \forall, \exists)$. Then the following conditions are equivalent:*

- 210 (i) F is a monadic filter of $(A, \forall, \exists)$,
(ii) $F = [F \cap A_\forall]$.

Proposition 3.5. *Let $(A, \forall, \exists)$ be a monadic MTL-algebra, $x \in A_\forall$ and F be a monadic filter of $(A, \forall, \exists)$. Then $[F \cup \{x\}]$ is also a monadic filter of $(A, \forall, \exists)$.*

Definition 3.4. *Let $(A, \forall, \exists)$ be a monadic MTL-algebra and θ be a congruence on A . Then θ is*
215 *called a monadic congruence on $(A, \forall, \exists)$ if $(x, y) \in \theta$ implies $(\forall x, \forall y) \in \theta$, for all $x, y \in A$. Where:*

$$(x, y) \in \theta \text{ iff } (x \rightarrow y) \wedge (y \rightarrow x) \in F.$$

Theorem 3.4. *For all monadic MTL-algebra there exists a one to one correspondence between its monadic filters and its monadic congruences.*

Let $(A, \forall, \exists)$ be a monadic MTL-algebra and F be a monadic filter. We define a mapping
220 $\forall_F : L/F \rightarrow L/F$ such that $\forall_F([x]) = [\forall x]$ and $\exists_F : L/F \rightarrow L/F$ such that $\exists_F([x]) = [\exists x]$, for all $x \in A$. It is clear that $\forall_F$ and $\exists_F$ is well defined.

Clearly, the following proposition hold:

Proposition 3.6. *Let F be a monadic filter of $(A, \forall, \exists)$. Then $(A/F, \forall_F, \exists_F)$ is a monadic MTL-algebra.*

225 **Definition 3.5.** *A proper monadic filter F of $(A, \forall, \exists)$ is called a prime monadic filter of $(A, \forall, \exists)$, if $F_1 \cap F_2 \subseteq F$, implies $F_1 \subseteq F$ or $F_2 \subseteq F$, for all $F_1, F_2 \in MF(A)$.*

Example 3.4. *In Example 3.1, $F = \{c, a, b, 1\}$ is a prime monadic filter of monadic MTL-algebra $(A, \forall_2, \exists_2)$.*

Theorem 3.5. *Let F be a monadic filter of $(A, \forall, \exists)$. Then the following are equivalent:*

- 230 (i) F is a prime monadic filter of $(A, \forall, \exists)$,
(ii) if $\forall x \vee \forall y \in F$, then $\forall x \in F$ or $\forall y \in F$, for some $x, y \in A$,
(iii) $\forall x \rightarrow \forall y \in F$ or $\forall y \rightarrow \forall x \in F$, for any $x, y \in A$,
(iv) $(A/F, \forall_F, \exists_F)$ is a linearly ordered monadic MTL-algebra.

Proof. $(i \Rightarrow ii)$ Let $\forall x \vee \forall y \in F$, for some $x, y \in A$. Then $[x] \cap [y] = [\forall x \vee \forall y] \in F$. Since F is a
235 prime monadic filter of $(A, \forall, \exists)$, then $[x] \subseteq F$ or $[y] \subseteq F$. So $\forall x \in F$ or $\forall y \in F$.

(ii $\Rightarrow$ iii) Since $(\forall x \rightarrow \forall y) \vee (\forall y \rightarrow \forall x) = 1 \in F$, we have $\forall x \rightarrow \forall y \in F$ or $\forall y \rightarrow \forall x \in F$.

(iii $\Rightarrow$ iv) It follows from Proposition 3.6, that $(A/F, \forall_F, \exists_F)$ is a monadic MTL-algebra. Now, we prove that is linearly ordered. If $[x], [y] \in A/F$, from (iii), then $[x] \leq [y]$ or $[y] \leq [x]$. Thus $(A/F, \forall_F, \exists_F)$ is a linearly ordered monadic MTL-algebra.

240 (iv $\Rightarrow$ i) It is clear.

□

Let $(A, \forall, \exists)$ be a monadic MTL-algebra. Denote:

$$Ker(\forall) = \{x \in A \mid \forall x = 1\}.$$

$$Ker(\exists) = \{x \in A \mid \exists x = 1\}.$$

245 **Definition 3.6.** A proper monadic filter F of $(A, \forall, \exists)$ is called a maximal monadic filter if it not strictly contained in any proper monadic filter of $(A, \forall, \exists)$.

Example 3.5. In Example 3.1, $F = \{c, a, b, 1\}$ is a maximal monadic filter of $(A, \forall_2, \exists_1)$. Clearly, we have $Ker(\forall_1) = \{1\}$.

Theorem 3.6. Let F be a proper monadic filter of $(A, \forall, \exists)$. Then the following conditions are
250 equivalent:

(i) F is a maximal monadic filter of $(A, \forall, \exists)$.

(ii) $\forall x \in F$ or $\neg \forall x \in F$, for all $x \in A$,

(iii) $\exists x \in F$ or $\neg \exists x \in F$, for all $x \in A$.

Proof. (i $\Rightarrow$ ii) Let F be a maximal monadic filter and there exists $x \in A$ such that $\forall x \notin F$ and
255 $\neg \forall x \notin F$. As $\neg \forall x \notin F$, we have $\forall x \cup F$ is proper. By Proposition 3.5, we obtain $[F \cup \{x\}]$ is also a monadic filter of $(A, \forall, \exists)$. According to our assumption, F is strictly contained in $\forall x \cup F$ which leads to a contradiction.

(ii $\Rightarrow$ i) Let (ii) and in contrary to (i), F is not a maximal monadic filter. Then there exists a proper monadic filter M strongly containing F . But there exists $x \in M$ such that $x \notin F$. So
260 $\forall x \notin F$. By our assumption, $\neg \forall x \in F$. Hence, $\neg \forall x \in M$. Therefore, as $x \in M$ and M is a monadic filter, $\forall x \in M$. But this contradicts the fact that M is a proper filter.

(ii $\Leftrightarrow$ iii) It is clear.

□

Theorem 3.7. *Let F be a monadic filter of monadic MTL-algebra $(A, \forall, \exists)$ and $a \notin F$. Then there exists a prime monadic filter P of $(A, \forall, \exists)$ such that $F \subseteq P$ and $a \notin P$.*

265 *Proof.* Assume

$$F_a = \{F' \mid F' \text{ is a proper monadic filter of } (A, \forall, \exists) \text{ such that } F \subseteq F', a \notin F'\}.$$

Then $F_a \neq \emptyset$, since F is a monadic filter not containing a and F_a is a partially set under inclusion relation. Let $\{F_i \mid i \in I\}$ be a chain in F_a . Then $\cup\{F_i \mid i \in I\}$ is a monadic filter of $(A, \forall, \exists)$ and it is the upper bounded of this chain. By Zorn's Lemma, there exists a maximal element P in
270 F_a . We show that P is the desire prime monadic filter of ours. Since $p \in F_a$, then P is a proper monadic filter and $a \notin P$.

On the other hand, let $x, y \in A$ and $x, y \notin P$, for some $x, y \in A$. Since P is strictly contain in $[P, x), [P, y)$ and by the maximality of P , we have $[P, x) \notin F_a$ and $[P, y) \notin F_a$. Then $a \in [P, x) = P \vee [\forall x)$ and $a \in [P, y) = P \vee [\forall y)$. Thus $a \in (P \vee [\forall x)) \wedge (P \vee [\forall y)) = P \vee ([\forall x) \wedge [\forall y)) =$
275 $P \vee [\forall (x \vee y)) \in P$, so $a \in P$, which is a contradiction. Therefore P is a prime monadic filter such that $F \subseteq P$ and $a \notin P$. $\square$

Definition 3.7. *A monadic MTL-algebra $(A, \forall, \exists)$ is said to be simple if it has exactly two monadic filters $\{1\}$ and A .*

Example 3.6. *Consider Example 3.1, clearly $(A, \forall_1, \exists_1)$ is a simple monadic MTL-algebra.*

280 **Theorem 3.8.** *Let $(A, \forall, \exists)$ be a monadic MTL-algebra. Then the following conditions are equivalent:*

(i) $(A, \forall, \exists)$ is simple,

(ii) $\forall(A)$ is simple,

(iii) $A_\forall = \{0, 1\}$,

285 (iv) $[1)$ is the only proper monadic filter in $(A, \forall, \exists)$.

Proof. $(i \Rightarrow ii)$ Let F be a filter of $A_\forall$ and $F \neq \{1\}$. Consider

$$F_f = \{t \in A \mid t \geq f \text{ for a certain } f \in F\}.$$

Then there exists $f_1, f_2 \in F$ such that $x \geq f_1, y \geq f_2$, for $x, y \in F_f$. So $x * y \geq f_1 * f_2 \in F$, hence $x * y \in F_f$. If $x \in F_f, x \leq y$, then $y \in F_f$, and if $x \in F_f$, then $x \geq f, f \in F$. Since $f \in A_\forall$, thus

290 $\forall x \geq \forall f = f$ and so $\forall x \in F_f$. Therefore F_f is a monadic filter of $A_\forall$. Since $(A, \forall, \exists)$ is simple and $F_f \neq \{1\}$, it follows that $F_f = A$. Hence $0 \in F_f$ and $0 \in F$, that is $F = \forall A$.

(ii $\Rightarrow$ i) Let F be a monadic filter of $(A, \forall, \exists)$. Then $F \cap A_\forall$ is a filter of $A_\forall$, so $F \cap A_\forall = \{1\}$ or $F \cap A_\forall = A_\forall$. If $F \cap A_\forall = A_\forall$, then $A_\forall \subseteq F$. Since $0 \in A_\forall$, then $F = A$. Since $\text{Ker}(\forall) = \{1\}$, if $F \cap A_\forall = \{1\}$ and $x \in F$, then $\forall x \in F \cap A_\forall$, so $\forall x = 1$, that is $x = 1$ so $F = \{1\}$. Thus $(A, \forall, \exists)$ is simple.
295

(ii $\Leftrightarrow$ iii) It is clear.

(i $\Leftrightarrow$ iv) It follow from Definition of simple monadic MTL-algebras. $\square$

Definition 3.8. A monadic MTL-algebra $(A, \forall, \exists)$ is said to be subdirectly irreducible if it has the least nontrivial monadic congruence.

300 Let $(A, \forall, \exists)$ be subdirectly irreducible monadic MTL-algebra. Then by Theorem 3.4, there exists a monadic filter F of $(A, \forall, \exists)$ such that $\theta_F = F$. Thus, we can conclude that a monadic MTL-algebra $(A, \forall, \exists)$ is said to be subdirectly irreducible if among the nontrivial monadic filters of $(A, \forall, \exists)$ there exists the least one, i.e., $\cap\{F \in MF(A) \mid F \neq \{1\}\} \neq \{1\}$.

Proposition 3.7. Let $(A, \forall, \exists)$ be a subdirectly irreducible monadic MTL-algebra and $F_1, F_2 \in MF(A)$. If $F_1 \cap F_2 = \{1\}$, then $F_1 = \{1\}$ or $F_2 = \{1\}$.
305

Proof. Let $F_1 \neq \{1\}$ and $F_2 \neq \{1\}$, i.e. $F_1, F_2 \in \cap\{F \in MF(A) \mid F \neq \{1\}\} \neq \{1\}$. Then $\cap\{F \in MF(A) \mid F \neq \{1\}\} \neq \{1\} \subseteq F_1 \cap F_2$. By $F_1 \cap F_2 = \{1\}$, we have $\cap\{F \in MF(A) \mid F \neq \{1\}\} = \{1\}$, which contradictions to the fact that $(A, \forall, \exists)$ is a subdirectly irreducible monadic MTL-algebra. So $F_1 = \{1\}$ or $F_2 = \{1\}$. $\square$

310 **Theorem 3.9.** The following conditions are equivalent:

(i) $(A, \forall, \exists)$ is a subdirectly irreducible monadic MTL-algebra,

(ii) there exists an element $a \in A, a < 1$, such that for all $x \in A, x < 1$ and $a \in [x]$.

Proof. (i $\Rightarrow$ ii) If $(A, \forall, \exists)$ is a subdirectly irreducible monadic MTL-algebra, i.e. , $\cap\{F \in MF(A) \mid F \neq \{1\}\} \neq \{1\}$, then $\cap\{[x] \mid x < 1\} \neq \{1\}$. Suppose that $a \in \cap\{[x] \mid x < 1\}$ satisfying $a \neq 1$.
315 Then for all $x \in A, x \neq 1$ and $a \in [x]$. By Theorem 3.2, there exists $m \in \mathbb{N}$, such that $a \geq (\forall x)^m$. Clearly, a is the element that we need.

(ii $\Rightarrow$ i) We show that for any $F \in MF(A)$, if $F \neq \{1\}$, then $a \in F$. Since $F \neq \{1\}$, there exists $x \in F, x < 1$, then by the know condition, $a \in [x]$ and $a \in F$. Hence $a \in \{F \in MF(A) \mid F \neq \{1\}\}$. Thus $\cap\{F \in MF(A) \mid F \neq \{1\}\} \neq \{1\}$, i.e. $(A, \forall, \exists)$ is a subdirectly irreducible monadic MTL-algebra. $\square$

Proposition 3.8. *Let $(A, \forall, \exists)$ be a subdirectly irreducible monadic MTL-algebras. If $x \vee y = 1$, then $x = 1$ or $y = 1$, for all $x, y \in A$.*

Proof. If $x \vee y \in A$, then by Theorem 3.2, we have $[x] \cap [y] = [\forall x \vee \forall y] = [\forall(x \vee y)] = [1] = \{1\}$. By Proposition 3.7, we have $[x] = \{1\}$ or $[y] = \{1\}$. So $x = 1$ or $y = 1$. $\square$

Proposition 3.9. *Let $(A, \exists, \forall)$ be a subdirectly irreducible monadic MTL-algebra. Then $(A, \exists, \forall)$ is a chain.*

Proof. Suppose that $(A, \exists, \forall)$ is a subdirectly irreducible monadic MTL-algebra. Since $(x \rightarrow y) \vee (y \rightarrow x) = 1$, by Proposition 3.8, we have $x \rightarrow y = 1$ or $y \rightarrow x = 1$, i.e. $x \leq y$ or $y \leq x$. So $(A, \exists, \forall)$ is a chain. $\square$

Proposition 3.10. *If $(A, \exists, \forall)$ is a subdirectly irreducible monadic MTL-algebra, then $A_{\forall}$ is linearly ordered.*

Proof. Let F be the smallest non-trivial monadic filter of $(A, \exists, \forall)$ and $x \in F \setminus \{1\}$. If $A_{\forall}$ is not linearly ordered, and $\forall a, \forall b \in A_{\forall}$ such that $\forall a \not\leq \forall b$ and $\forall b \not\leq \forall a$. Then the monadic filters $[\forall a \rightarrow \forall b]$ and $[\forall b \rightarrow \forall a]$ generated by $\forall a \rightarrow \forall b$ and $\forall b \rightarrow \forall a$ respectively are non-trivial. Thus both contain F , in particular $x \in [\forall a \rightarrow \forall b]$ and $x \in [\forall b \rightarrow \forall a]$. Since $\forall a \rightarrow \forall b \in A_{\forall}$, by Theorem 3.2, there is an n such that $x \geq (\forall(\forall a \rightarrow \forall b))^n \geq (\forall a \rightarrow \forall b)^n$ and $x \geq (\forall(\forall b \rightarrow \forall a))^n \geq (\forall b \rightarrow \forall a)^n$. By Proposition 2.1, $x \geq (\forall a \rightarrow \forall b)^n \vee (\forall b \rightarrow \forall a)^n$. Hence $x = 1$ which is a contradiction. $\square$

Theorem 3.10. *$(A, \forall, \exists)$ is a subdirect irreducible monadic MTL-algebra iff $A_{\forall}$ is a subdirectly irreducible subalgebra of A .*

Proof. Let $(A, \forall, \exists)$ be a subdirect irreducible monadic MTL-algebra. Then $MF(A) \setminus \{1\}$ has a minimal element F . Clearly, $A_{\forall}$ is a subalgebra of A . We show that $F \cap A_{\forall}$ is the minimal monadic filter of $A_{\forall}$ such that $F \cap A_{\forall} \neq \{1\}$. If $F \cap A_{\forall} = \{1\}$, since $\forall F \subseteq F \cap A_{\forall}$ then $\forall x = 1$, for all $x \in F$. Hence $F \subseteq \text{Ker}(\forall) = \{1\}$ and $F = \{1\}$, which is a contradiction. That means $F \cap A_{\forall} \neq \{1\}$.

345 Now, we show that $F \cap A_{\forall}$ is the minimal monadic filter of $(A, \forall, \exists)$. Suppose that G is a filter of $A_{\forall}$ and hence $[G]$ is the monadic filter of $(A, \forall, \exists)$ generated by G . Clearly, $[G] \cap A_{\forall} = G$. By minimality of F , $F \subseteq [G]$ and so $F \cap A_{\forall} \subseteq [G] \cap A_{\forall}$. Thus $F \cap A_{\forall}$ is the minimal filter of $A_{\forall}$ such that $F \cap A_{\forall} \neq \{1\}$. Then $A_{\forall}$ is a subdirectly irreducible subalgebra of A .

Conversely, $A_{\forall}$ is a subdirectly irreducible subalgebra of A . Then there is a minimal filter F of $A_{\forall}$ such that $F \neq \{1\}$. From Theorem 3.3, $[F \cap A_{\forall}]$ is a monadic filter of $(A, \forall, \exists)$. We show that $[F \cap A_{\forall}]$ is a minimal monadic filter of $(A, \forall, \exists)$. In fact, if G is another non-trivial monadic filter of $(A, \forall, \exists)$, then $G \cap A_{\forall} \supseteq F \cap A_{\forall}$. Then G contains the monadic filter generated by F , that is, $[F \cap A_{\forall}] \subseteq [G \cap A_{\forall}]$, i.e., $[F \cap A_{\forall}]$ is minimal. Thus, $(A, \forall, \exists)$ is a subdirectly irreducible monadic MTL-algebra. $\square$

355 **Proposition 3.11.** *Let F be a monadic filter of $(A, \forall, \exists)$. Then $F = [F \cup \{x \rightarrow y\}] \cap [F \cup \{y \rightarrow x\}]$, for all $x, y, z \in A$.*

Proof. Let z be an element of both $[F \cup \{x \rightarrow y\}]$ and $[F \cup \{y \rightarrow x\}]$. Then, there are $f_1, f_2 \in F, n_1, n_2 \in \mathbb{N}$ such that $f_1 * (\forall(x \rightarrow y))^{n_1} \leq z$ and $f_2 * (\forall(y \rightarrow x))^{n_2} \leq z$. If $f = f_1 * f_2$ and $n = \max\{n_1, n_2\}$, so $f * (\forall(x \rightarrow y))^n \leq z$ and $f * (\forall(y \rightarrow x))^n \leq z$. Thus $(\forall(x \rightarrow y))^n \leq f \rightarrow z$ and $(\forall(y \rightarrow x))^n \leq f \rightarrow z$ and so $(\forall(x \rightarrow y))^n \vee (\forall(y \rightarrow x))^n \leq f \rightarrow z$. But $(\forall(x \rightarrow y))^n \vee (\forall(y \rightarrow x))^n = (\forall((x \rightarrow y) \vee (y \rightarrow x)))^n = \forall 1 = 1$, so $f \leq z$ and $z \in F$. The forward inclusion is straightforward. $\square$

Proposition 3.12. *There exists a prime monadic filter P of $(A, \forall, \exists)$, such that $a \vee \forall r \notin P$, for all $a, r \in A, a, r \neq 1$.*

365 *Proof.* Let $C = \{a \vee \forall r \mid r \neq 1\}$. Note that $1 \notin C$. Since $a \vee \forall r = 1$ then we have $1 = \forall(a \vee \forall r) = \forall a \vee \forall r$, and so $a = 1$ or $r = 1$. Let $\mathcal{F}$ be the family of monadic filters F in $(A, \forall, \exists)$ such that $F \cap C = \emptyset$. Clearly, $1 \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$. It is straightforward to verify that any chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. By Lemma Zorn's, there exists a maximal filter P in $\mathcal{F}$. We show that P is prime monadic filter. We claim that P is prime monadic filter. Note that $P = [P \cup \{x \rightarrow y\}] \cap [P \cup \{y \rightarrow x\}]$. If we assume that neither $P = [P \cup \{x \rightarrow y\}]$ nor $P = [P \cup \{y \rightarrow x\}]$ belongs to $\mathcal{F}$, then there are $r_1, r_2 \neq 1$ such that $a \vee \forall r_1 \in P = [P \cup \{x \rightarrow y\}]$ and $a \vee \forall r_2 \in P = [P \cup \{y \rightarrow x\}]$. Since $\forall r_1$ and $\forall r_2$ are comparable, it follows that one of them belongs to both filters. Hence one of them belongs to P , which is a contradiction. This shows that either $[P \cup \{x \rightarrow y\}] \in \mathcal{F}$ or $[P \cup \{y \rightarrow x\}] \in \mathcal{F}$. By the maximality of P , we have $P = [P \cup \{x \rightarrow y\}]$ and so $x \rightarrow y \in P$. Similarly, If $P = [P \cup \{y \rightarrow x\}]$, then $y \rightarrow x \in P$. $\square$

Theorem 3.11. *Let $(A, \forall, \exists)$ be a monadic MTL-algebra. Then there exists a subdirect representation of the monadic MTL-algebra $A \leq \prod_{i \in I} A_i$, where $A_\forall$ is embedded in A_i via the corresponding projection map.*

Proof. Let P_a be one of the prime monadic filters provided by the Proposition 3.12, for all $a \in A, a \neq 1$. Clearly, $\cap_{a \neq 1} P_a = \{1\}$ and we have a natural embedding $A \rightarrow \prod_{a \neq 1} A/P_a$. We show that the natural map $A \rightarrow A/P_a$ is injective on $A_\forall$. Clearly, if there are $t_1, r_2 \in A$ such that $\forall r_1 \leq \forall r_2$ and $\forall r_1/P_a = \forall r_2/P_a$, then $\forall r_2 \rightarrow \forall r_1 = \forall(\forall r_2 \rightarrow \forall r_1)$ and so, $\forall r_2 \rightarrow \forall r_1 \neq 1$. Thus $a \vee (\forall r_2 \rightarrow \forall r_1) \notin P_a$, which is a contradiction. $\square$

3.1. Functional monadic MTL-algebras

Let A be an MTL-algebra and X be a non-empty set. Then A^X forms, with respect to the pointwise operations, an MTL-algebra (a direct power of the MTL-algebra A) containing a subalgebra isomorphic to A . For any $p \in A^X$, put $M(p) := \{p(x) \mid x \in X\}$.

Theorem 3.12. *Let A be an MTL-algebra, X a non-empty set and A a subalgebra of A^X that satisfies the following conditions:*

- (i) $\sup_A M(p) = \bigvee R(p)$, $\inf_A M(p) = \bigwedge M(p)$, for all $p \in A$.
- (ii) The constant function $\exists p$ and $\forall p$ defined such that

$$\exists p(x) := \bigvee M(p), \quad \forall p(x) := \bigwedge M(p).$$

Then $(A, \exists, \forall)$ is a monadic MTL-algebra.

Proof. We will prove that $(A, \forall, \exists)$ satisfies all axioms (M_1) - (M_5) in the definition of a monadic MTL-algebra.

For axiom (M_1) , it is obvious.

(M_2) We have $\forall(p \rightarrow \forall q) = \bigwedge_{b \in R(p)} (b \rightarrow \bigwedge R(q))$ and $\exists p \rightarrow \forall q = \bigvee R(p) \rightarrow \bigwedge R(q)$. If $b \in R(p)$, then $b \leq \bigvee R(p)$. By Lemma 2.1, $\bigvee R(p) \rightarrow \bigwedge R(q) \leq b \rightarrow \bigwedge R(q)$ and $\bigvee R(p) \rightarrow \bigwedge R(q) \leq \bigwedge_{b \in R(p)} (b \rightarrow \bigwedge R(q))$, so $\exists p \rightarrow \forall q \leq \forall(p \rightarrow \forall q)$.

On the other hand, for all $b \in R(p)$, we have $c \leq b \rightarrow \bigwedge R(q)$. Hence $c * b \leq \bigwedge R(q)$ and $c * \bigvee R(p) \leq \bigwedge R(q)$, that means $\bigwedge_{b \in R(p)} (b \rightarrow \bigwedge R(q)) \leq \bigvee R(p) \rightarrow \bigwedge R(q)$. Therefore $\forall(p \rightarrow \forall q) \leq \exists p \rightarrow \forall q$ and $\forall(p \rightarrow \forall q) = \exists p \rightarrow \forall q$. So (M_2) hold.

(M_3) By assumption, we have $\forall(\forall p \rightarrow q) = \bigwedge_{b \in R(q)} (\bigwedge R(p) \rightarrow b)$ and $\forall p \rightarrow \forall q = \bigwedge R(p) \rightarrow \bigwedge R(q)$. For all $b \in R(q)$, $\bigwedge R(q) \leq b$. By Lemma 2.1, $\bigwedge R(p) \rightarrow \bigwedge R(q) \leq \bigwedge R(p) \rightarrow b$. So
405 $\bigwedge R(p) \rightarrow \bigwedge R(q) \leq \bigwedge_{b \in R(q)} (\bigwedge R(p) \rightarrow b)$ and $\forall p \rightarrow \forall q \leq \forall(\forall p \rightarrow q)$.

Conversely, for all $b \in R(q)$, we have $c \leq \bigwedge R(p) \rightarrow b$. Thus $c * \bigwedge R(p) \leq b$ and $c * \bigwedge R(p) \leq \bigwedge R(q)$. So $c \leq \bigwedge R(p) \rightarrow \bigwedge R(q)$ and $\bigwedge_{b \in R(q)} (\bigwedge R(p) \rightarrow b) \leq \bigwedge R(p) \rightarrow \bigwedge R(q)$. Therefore $\forall(\forall p \rightarrow \forall q) \leq \forall p \rightarrow \forall q$ and $\forall(\forall p \rightarrow \forall q) = \forall p \rightarrow \forall q$. So (M_3) hold.

(M_4) We have $\forall(p \vee \exists q) = \bigwedge R(p \vee \bigvee R(q)) = \bigwedge_{a \in R(p)} (a \vee \bigvee_{b \in R(q)} b)$. On the other hand,
410 $\forall p \vee \exists q = \bigwedge R(p) \vee \bigvee R(q) = \bigwedge_{a \in R(p)} (a \vee \bigvee R(q)) = \bigwedge_{a \in R(p)} (a \vee \bigvee_{b \in R(q)} b)$. So (M_4) hold.

(M_5) $\exists p * \exists p = \bigvee R(p) * \bigvee R(p) = \bigvee_{a \in R(p)} a * \bigvee_{b \in R(p)} b = \bigvee_{a \in R(p)} \bigvee_{b \in R(p)} (a * b) = \bigvee_{a, b \in R(p)} (a * b) = \exists(p * p)$. $\square$

4. Monadic Monoidal t-norm based logic

In this section, we develop a propositional calculus, monadic monoidal t-norm based logic,
415 denoted $\mathcal{MMTL}$, which has monadic MTL-algebra as models.

The language of $\mathcal{MMTL}$ consists of:

- countably many proposition variables $(p_1, p_2, \dots)$, the constants $\bar{0}$ and $\bar{u}$,
  - the unary operators $\Box, \Diamond$, the binary operators $\sqcup, \sqcap, \&, \rightarrow$,
  - a binary logical connective $\Leftrightarrow$,
  - the auxiliary symbols (and).
- 420

The formulas of $\mathcal{MMTL}$ are defined inductively as follows:

f_1 . proposition variables, $\bar{0}$ is formula,

f_2 . if φ and ψ are formulas, then so are $(\varphi \sqcap \psi)$, $(\varphi \sqcup \psi)$, $(\varphi \& \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \Leftrightarrow \psi)$, $(\Box \psi)$ and $(\Diamond \psi)$,

425 f_3 . a string of symbols is a formula of $\mathcal{MMTL}$ iff it can be show to be a formula by a finite number of applications of f_1, f_2 .

In order to avoid unnecessary brackets, we agree on the following priority rules:

- unary operators always take precedence over binary ones, while,
- among the binary operators, $\&$ has the highest priority; furthermore $\sqcap$ and $\sqcup$ take precedence

430 over $\rightarrow$,

- the outermost brackets are not written. We also introduce some useful shorthand notations:
 $\bar{1}$ for $\bar{0} \rightarrow \bar{0}$, $\neg\varphi$ for $\varphi \rightarrow \bar{0}$, for formulas φ and ψ .

The axioms of $\mathcal{MMTL}$ are defined as follows:

I. any axioms of $(\mathcal{MMTL})$ is an axiom of $\mathcal{MTL}$, i.e. formula which has one of the following
 435 forms is an axiom (where φ, ψ and χ are arbitrary formulas):

$$(A_1) (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)),$$

$$(A_2) (\varphi \& \psi) \rightarrow \varphi,$$

$$(A_3) (\varphi \& \psi) \rightarrow (\psi \& \varphi),$$

$$(A_4) (\varphi \sqcap \psi) \rightarrow \varphi,$$

$$440 (A_5) (\varphi \sqcap \psi) \rightarrow (\psi \sqcap \varphi),$$

$$(A_6) (\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\varphi \sqcap \psi),$$

$$(A_7) (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi),$$

$$(A_8) ((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)),$$

$$(A_9) ((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$$

$$445 (A_{10}) \bar{0} \rightarrow \varphi.$$

II. a formula which has one of the following forms is an axiom (where φ and ψ are arbitrary
 formulas):

$$(\Box 1) \Box \varphi \rightarrow \varphi,$$

$$(\Diamond 1) \varphi \rightarrow \Diamond \varphi,$$

$$450 (\Box 2) \Box(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \Box \varphi),$$

$$(\Diamond 2) \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \psi),$$

$$(\Box 3) \Box(\psi \sqcup \varphi) \rightarrow (\psi \sqcup \Box \varphi),$$

$$(\Diamond 3) \Diamond(\varphi \& \varphi) \leftrightarrow \Diamond \varphi \& \Diamond \varphi.$$

The deduction rules are modus ponens (MP, from φ and $\varphi \rightarrow \psi$ infer ψ), generalization (G,
 455 from φ infer $\Box \varphi$).

If $\Gamma \subseteq \text{Form}_{\mathcal{MMTL}}$, then a formula φ is a Γ -theorem if there exists a natural number $n \geq 1$ and a sequence of formulas $\varphi_1, \dots, \varphi_n = \varphi$ such that, for any $1 \leq i \leq n$ one of the following conditions holds:

1- φ_i is an axiom,

460 2- $\varphi_i \in \Gamma$,

3- there are $j, k < i$ such that φ_j is $\varphi_k \rightarrow \varphi_i$,

The sequence $\varphi_1, \dots, \varphi_n = \varphi$ is a Γ -proof for φ . The set of Γ -theorem will be denoted by $\text{Theor}_{\mathcal{MMTL}}(\Gamma)$. The fact that a formula φ is a Γ -theorem will be simply denoted by $\Gamma \vdash_{\mathcal{MMTL}} \varphi$. A formula φ will be called a theorem if it is provable from the emptyset. This will be denoted by
 465 $\vdash_{\mathcal{MMTL}} \varphi$. In this case, a proof for φ will be a sequence of formulas $\varphi_1, \dots, \varphi_n = \varphi$ such that for any $i \in \{1, \dots, n\}$, one of the above conditions 1, 3 or 4 is satisfied. The set of all theorems of will be denoted by $\text{Theor}_{\mathcal{MMTL}}$.

Remark 4.1. *The propositional calculus $\mathcal{MMTL}$ is an extension of $\mathcal{MTL}$. It is straightforward that $\text{Form}_{\mathcal{MTL}} \subseteq \text{Form}_{\mathcal{MMTL}}$. If $\Gamma \subseteq \text{Form}_{\mathcal{MTL}}$ and $\varphi \in \text{Form}_{\mathcal{MTL}}$ such that $\Gamma \vdash_{\mathcal{MTL}} \varphi$,
 470 then $\Gamma \vdash_{\mathcal{MMTL}} \varphi$. Also, every theorem of $\mathcal{MTL}$ is a theorem of $\mathcal{MMTL}$.*

Definition 4.1. *Let $(A, \forall, \exists)$ be a monadic MTL-algebra and Γ be a theory. An e -evaluation is a mapping e from the set of formulas of $\mathcal{MMTL}$ to A that satisfies, for all two formulas φ and ψ :*

$$e(\varphi \Rightarrow \psi) = e(\varphi) \rightarrow e(\psi),$$

$$e(\varphi \sqcup \psi) = e(\varphi) \vee e(\psi),$$

475 $e(\varphi \sqcap \psi) = e(\varphi) \wedge e(\psi),$

$$e(\varphi \& \psi) = e(\varphi) * e(\psi),$$

$$e(\Box \varphi) = \forall e(\varphi),$$

$$e(\Diamond \varphi) = \exists e(\varphi),$$

$$e(\bar{0}) = 0.$$

480 *If an e -evaluation e satisfies $e(\chi) = 1$, for any χ in Γ , it is called an $\mathcal{A}$ -model of Γ .*

4.1. Lindenbaum- Tarski algebra of $\mathcal{MMTL}$

In the following $\Gamma \subseteq \text{Form}_{\mathcal{MMTL}}$ is a fixed set of formulas. Let φ and ψ be formulas, we define

$$\varphi \equiv_{\Gamma} \psi \text{ iff } \Gamma \vdash_{\mathcal{STL}} \varphi \rightarrow \psi \text{ and } \Gamma \vdash_{\mathcal{STL}} \psi \rightarrow \varphi.$$

It is straightforward that the relation $\equiv_{\Gamma}$ is an equivalence relation on $Form_{\mathcal{MMTL}}$. For any formula $\varphi \in Form_{\mathcal{MMTL}}$ we will denote by $[\varphi]_{\Gamma}$ the equivalence class of φ with respect to $\equiv_{\Gamma}$. The set $Form_{\mathcal{MMTL}} / \equiv_{\Gamma} = \{[\varphi]_{\Gamma} \mid \varphi \in Form_{\mathcal{MMTL}}\}$ is the quotient of $Form_{\mathcal{STL}}$ with respect to $\equiv_{\Gamma}$. On $Form_{\mathcal{STL}} / \equiv_{\Gamma}$ we define the following operations:

$$[\varphi]_{\Gamma} \rightarrow [\psi]_{\Gamma} := [\varphi \Rightarrow \psi]_{\Gamma},$$

$$[\varphi]_{\Gamma} \vee [\psi]_{\Gamma} := [\varphi \sqcup \psi]_{\Gamma},$$

$$[\varphi]_{\Gamma} \wedge [\psi]_{\Gamma} := [\varphi \sqcap \psi]_{\Gamma},$$

$$[\varphi]_{\Gamma} * [\psi]_{\Gamma} := [\varphi \& \psi]_{\Gamma},$$

$$\nu[\varphi]_{\Gamma} := [\Box \varphi]_{\Gamma},$$

$$\mu[\varphi]_{\Gamma} := [\Diamond \varphi]_{\Gamma},$$

$$1_{\Gamma} := Theor_{\mathcal{STL}}(\Gamma),$$

$$0_{\Gamma} := \neg 1_{\Gamma},$$

This algebras is denoted by $MMTL(\Gamma) = (Form_{\mathcal{MMTL}} / \equiv_{\Gamma}, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1)$.

Proposition 4.1. *The structure $MMTL(\Gamma) = (Form_{\mathcal{MMTL}} / \equiv_{\Gamma}, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1)$ is a monadic MTL-algebra.*

Proof. The proof is similar to that of Proposition 3.6. □

Theorem 4.1. *$\mathcal{MMTL}$ is complete if and only if the monadic NM-algebra $(Form_{\mathcal{MMTL}} / \equiv_{\Gamma}, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1)$ is linearly ordered.*

Proof. The proof is similar to that of Theorem 3.5. □

It is easily verified that $\mathcal{MMTL}$ is sound with respect to the variety of monadic MTL-algebras, i.e. if a formula φ can be deduced from a theory Γ , then for any A -model e of $\mathcal{MMTL}$, $e(\varphi) = 1$. Indeed, we need to verify the soundness of the new axioms and deductions of $\mathcal{MMTL}$. For the axioms, it is easy, as they are straightforward generalizations of axioms for monadic MTL-algebras. We will now verify the soundness of the new deduction rules.

Proposition 4.2. *The deduction rules of $\mathcal{MMTL}$ are sound in the following sense, for any formula φ, ψ and A -model e of Γ ,*

- 510 (i) *If $e(\varphi) = 1$, then $e(\forall\varphi) = 1$.*
(ii) *If $e(\varphi) = 1$ and $e(\varphi \Rightarrow \psi) = 1$, then $e(\psi) = 1$.*

Proof. (i) If $e(\varphi) = 1$, then $e(\forall\varphi) = \forall e(\varphi) = \forall 1 = 1$ by Lemma 3.1 (3) and Definition 4.1.

(ii) If $e(\varphi) = 1$ and $e(\varphi \Rightarrow \psi) = 1$, then $1 = e(\varphi \Rightarrow \psi) = e(\psi)$ by Lemma 2.1 and Definition 4.1. $\square$

515 The following theorem shows a completeness theorem of $\mathcal{MMTL}$.

Theorem 4.2. *Let Γ be a theory over $\mathcal{MMTL}$ and φ be a formula. Then the following statements are equivalent:*

- (i) $\Gamma \vdash \varphi$,
(ii) *for each monadic MTL-algebra A and each A -model e of Γ , $e(\varphi) = 1$.*

520 *Proof.* (i) $\Rightarrow$ (ii) Soundness is proved as usual.

(ii) $\Rightarrow$ (i) To this end recall Proposition 4.1, among other things, that the algebra $\mathcal{MMTL}_\Gamma$ of classes of equivalent formulas of $\mathcal{MMTL}$ is a monadic MTL-algebra, thus, a φ satisfying (ii) is an A -model. In particular, let $e(p_i) = [p_i]_\Gamma$ and $e(\forall p_i) = [\forall p_i]_\Gamma$ for all propositional variables p_i . Then $e(\varphi) = [\varphi]_\Gamma = [1]_\Gamma$, thus $\Gamma \vdash \varphi \equiv 1$, hence $\Gamma \vdash \varphi$. $\square$

525 5. Conclusion and future work

In this work, we have considered the expansion of MTL-algebras with two unary operators $\forall$ and $\exists$. The notion of monadic filters has been introduced and considered them in detail. These types of filters play a basic role and thus the extended filters behave differently. Based on these facts, we give a classification for MTL-algebras. The relationship between monadic MTL-algebras and some related structures is studied. The notion of functional monadic MTL-algebras had
530 been introduced and we show that these structures are particular cases of monadic MTL-algebras. Finally, we introduced monadic monoidal t-norm based logic and proved its completeness based on monadic MTL-algebras.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal participants This article does not contain any studies with human participants or animals performed by any of the authors.

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