A search for a stochastic archetype of quantum probability

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Research Article

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Posted Date: October 11th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-954460/v3

A search for a stochastic archetype of quantum probability

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The author of this article does not have any conflicts of interest in connection with it.

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Abstract

Superposed wavefunctions in quantum mechanics lead to a squared amplitude that introduces interference into a probability density, which has long been a puzzle because interference between probability densities exists nowhere else in probability theory. In recent years, Man'ko and coauthors have successfully reconciled quantum and classical probability using a symplectic tomographic model. Nevertheless, there remains an unexplained coincidence in quantum mechanics, namely, that mathematically, the interference term in the squared amplitude of superposed wavefunctions has the form of a variance of a sum of correlated random variables, and we examine whether there could be an archetypical variable behind quantum probability that provides a mathematical foundation that observes both quantum and classical probability directly. The properties that would need to be satisfied for this to be the case are identified, and a generic variable that satisfies them is found that would be present everywhere, transforming into a process-specific variable wherever a quantum process is active. This hidden generic variable appears to be such an archetype.

1 Introduction

Using the Schödinger picture and the causal interpretation as accounted by Holland in his quantum theory of motion, in particular [1],[2],[10],[11], and confining ourselves to a *single spatial dimension* to keep a focus on concepts, the probability distribution of the location of a moving particle in the spatial dimension x at time t is derived from the wavefunction $\psi(\boldsymbol{x}, t)$, which is defined and continuous everywhere. The squared amplitude of the wavefunction is $|\psi(x,t)|^2 = \psi(x,t)\psi^*(x,t)$, where $\psi^*$is the complex conjugate of ψ and the probability density $f(x, t)$ associated with the location of the particle being between x and $x + dx$ at time t is given by

$$f(x,t) = \frac{\psi(x,t)\psi^*(x,t)}{C(t)}$$

where $C(\mathrm{t})$ is a normalization constant $\int_{-\infty}^{\infty} \psi\psi^* dx$ [1], which varies with *t*. Provided that $\int_{-\infty}^{\infty} \psi\psi^* dx$ is finite, *C*(*t*) rescales the squared amplitude to a probability density by satisfying $\int_{0}^{\infty} f(x,t)\, dx = 1$ and reflects the fact that the particle exists somewhere at time *t*.

The one-dimensional Schödinger equation is:

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\right)\psi \tag{1}$$

where $V = V(x,t)$ is the potential energy due to an external classical potential field, $\hbar = \frac{h}{2\pi}$, and h is Planck's constant [1].

Consider now two distinct solutions of this equation, expressed in polar form:

$$\psi_1 = R_1\, e^{\frac{iS_1}{\hbar}}, \text{and} \psi_2 = R_2\, e^{\frac{iS_2}{\hbar}}$$

where R_1, R_2 are the respective wave amplitudes and are real functions of space and time, $\psi_1\psi_1^* = R_1^2, \psi_2\psi_2^* = R_2^2$ are the respective squared amplitudes, and S_1, S_2 are the respective wave phases, also real functions of space and time.

Forming a wavefunction solution ψ_{sum} that is the sum of these wavefunctions, namely,

$$\psi_{sum} = \psi_1 + \psi_2$$

and calculating the squared amplitude using $e^{i\vartheta} = cos\theta + i\, sin\theta,\ e^{-i\vartheta} = cos\theta - i\, sin\theta$ gives

$$R^2 = \psi_{sum}\psi_{sum}^* = \left(R_1\, e^{\frac{iS_1}{\hbar}} + R_2\, e^{\frac{iS_2}{\hbar}}\right)\left(R_1\, e^{-\frac{iS_1}{\hbar}} + R_2\, e^{-\frac{iS_2}{\hbar}}\right)$$

$$= R_1^2 + R_2^2 + 2R_1\, R_2 \cos\left[\frac{2\pi(S_1 - S_2)}{h}\right] \tag{2}$$

The third term on the right describes the interference between the superposed waves[2], and its sign is variable depending on the phase difference [2]. Using the standard measure of h and expressing $\cos\left[\frac{2\pi(S_1-S_2)}{h}\right]$ in the form of $2\pi\mathrm{n} + \theta$, where n is an integer and $0 \leq \theta \leq 2\pi$, then $\cos\left[\frac{2\pi(S_1-S_2)}{h}\right] = \cos\ (2\pi\mathrm{n} + \theta) = \cos\ \theta$ [3] and the interference term simplifies to $2R_1\, R_2 \cos\theta$.

The remarkable feature of equation (2) is that, as is well known, the cosine is mathematically equivalent to and has all the attributes of a coefficient of correlation [3]. It follows, from the standard result in statistics for the variance of the sum of two correlated random variables [4], that the squared amplitude is mathematically equivalent to this variance, where the two variables have variance R_1^2 and R_2^2 and a correlation coefficient that equals $\cos\theta$.

There are also interference terms in the momentum field determined by ψ_{sum} [2], but as ψ_{sum} is a function of (*x, t*), we must express statistics such as the mean and variance of the

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[1] The integration is over the support of $\psi\psi^*$ if different from $-\infty \leq x \leq \infty$.
[2] Note the shift from $\hbar$ to $h/2\pi$.
[3] Using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\ \beta$.

momentum p as functions of x and t, but the uncertainty relationship $\Delta p\ \Delta x \geq \hbar/2$ means that in principle knowing x rules out knowing p, its expected value or other statistics, and we do not consider momentum further.

2 What might equivalence between $\psi_{sum}(x,t)\psi^*_{sum}(x,t)$ and a variance signify?

John von Neumann believed [5] that quantum randomness cannot be reduced to a statistical variation of properties in an ensemble of systems, in the way classical randomness is, and likewise Richard Feynman [6] believed that probability interference in quantum mechanics showed that the latter describes statistical properties in microscopic phenomena, where classical Kolmogorov probability theory is not applicable. Later, no-go theorems such as Bell's theorem confirmed that one cannot reproduce quantum probabilities from classical probability theory, but the possibility nevertheless still remains that hidden variables may exist that would bypass this limitation.

In work from 1996 to 2012, Vladimir Man'ko and others [7] were able, working only with classical Kolmogorov probabilities, to reproduce quantum probability results. They showed how to describe quantum states using a positive probability distribution called a symplectic tomogram, which is a nonnegative function $W(x, \mu, \nu)$ of the random position x measured in reference frames in phase space with rotated and scaled axes $q \rightarrow \mu q, p \rightarrow \nu p$ where $\mu = e^{\lambda} \cos\theta$, $\nu = e^{-\lambda} \sin\theta$, θ is the angle of rotation and e^{λ} is the scaling parameter.

Quantum mechanics is a physical system that is described in probabilistic terms, and despite this achievement by Man'ko and his colleagues, it is nevertheless impossible to ignore the paradox presented by the coincidence that mathematically, the squared amplitude of superposed wavefunctions has the form of a variance of the sum of correlated random variables, not the form of a probability.

The undoubted empirical success of quantum mechanics establishes that after normalization, the squared amplitude of two superposed wavefunctions gives the probability density $f(x, t)$ associated with the particle whose motion is described by the superposition being at the point (x, t)[4], but the paradox suggests that there may be a reconciliation between quantum and classical probability that is different from the tomographic model and that there may be deeper physics within quantum mechanics. This article will explore how the paradox might be resolved.

First, it is useful to picture a quantum mechanical process such as the self-interference of electrons sent one at a time through a two-slit interferometer to a detection screen. First, after 100 electrons have been dispatched, there will be no discernible pattern, but after 100,000 electrons have been dispatched, a pattern of multiple bands of dense detection separated by sparse detection will take shape across the entire detection screen. These bands of constructive and destructive interference and the varying detection intensity within the bands

[4] For economy of notation (x, t) denotes a position between x and $x + dx$ at time t.

from peak to trough will become increasingly distinct as the cumulative number of electrons sent through the interferometer increases.

Likewise, before normalization, the squared amplitude relating to a quantum mechanical process as described by its wavefunction represents the mean number of detections of a phenomenon in the interval between x and $x+dx$ at experiment time t resulting from a large number of experiments. In the case of the self-interference example, the squared amplitude is proportional to the build-up of particle detection intensity at (x, t), and when normalized by dividing it by the sum of the squared amplitudes over all x at time t is the expected relative frequency of detections there, and by definition the probability of finding a particle at (x, t) from one trial.

It is instructive to recall a little of what Von Weizsäcker covered in his 1973 paper on probability and quantum mechanics [8]. Noting that a single measurement will not suffice to convince us whether a result is to be considered a confirmation or a refutation of that we expected and we will repeat the measurement many times and apply the theory of errors, he summarised his analysis of the structure of an empirical test of a theoretical prediction in an abbreviated statement by saying "An empirical confirmation or refutation of any theoretical prediction is never possible with certainty but only with a higher or lesser degree or probability." He pointed out that in practice, every application of the theory of errors implies that we consider relative frequencies of events to be predictable quantities. He said that in this sense, probability is a measured quantity and that his abbreviated statement also applies to probability itself, leading to the conclusion that the empirical test of a theoretical probability is only possible with some degree of probability.

At the conclusion of his paper, Von Weizsäcker expressed the belief that "there is a quantum theory behind quantum theory, precisely because probabilities can only be defined with the help of probabilities".

Taking these observations into account, a starting point towards explaining why the squared amplitude of two superposed waves has the form of a variance of the sum of two correlated random variables and not the form of a probability would therefore be to suppose that before normalization, the squared amplitude is associated with a hidden variable whose mean equals the variance of another closely related and relevant hidden variable, a property we call the *mean/variance property*. Finding a pair of such variables with the properties needed to be consistent with the formulation of quantum mechanics is the purpose of this article.

3 Properties that must be displayed by the variables

The search begins with a statement of the properties that such a pair of variables must display to fit in with quantum mechanics. We call the generic form of these variables the *unit base variable* and the *unit squared amplitude variable,* respectively. Their necessary properties are that:

1. The unit base variable must have a mean of zero and variance of one, and the unit squared amplitude variable must be the square of the unit base variable in order that its mean equals 1, the variance of the unit base variable (the mean/variance property).

2. The element of a wavefunction capable of having these properties is its real part, the product of its amplitude and the cosine of its argument, which captures its essence. Accordingly, the unit base variable should be the product of two independent variables, an amplitude variable, and a cosine variable, and providing that the properties in 1 above are satisfied, the unit squared amplitude variable forms a pair with the closely related unit base variable.

3. These two generic variables transform by scaling the generic background variables to quantum process-specific variables where there is quantum activity.
    a. The transformed unit squared amplitude variable, which we call the *squared amplitude variable,* is the stochastic analogue of the deterministic squared amplitude, and its transforming scale factor must equal the deterministic squared amplitude.
    b. The transforming scale factor of the unit base variable is the square root of the scale factor of the squared amplitude variable, making the transformed base variable the stochastic analogue of the square root of the deterministic squared amplitude.

4. The mean of the squared amplitude variable must also equal the variance of the transformed base variable.

5. The squared amplitude variable must accommodate nonlocal and entangled quantum activity as well as local activity.

As a result of these properties, the Law of Large Numbers will ensure that with repeated experiments under the same conditions, the mean of the realisations of the squared amplitude variable converges to the deterministic squared amplitude as formulated by quantum mechanics.

The conjectures

The search for variables that have the properties that the task requires therefore begins with the following set of conjectures:

- There is a generic variable Z, the randomness of which is equivalent to that of the square of another generic variable W, which is the product of two independent seed variables:
    - Variable $A \sim U$ (0,3), which is a generic stochastic analogue of the amplitude of a wavefunction, and
    - Variable $C \sim U$ (-1,1), which is a generic stochastic analogue of the cosine of the argument of a wavefunction.

- The dependent unit base variable W, which is the product of the variables A and C, is a generic stochastic analogue of the real part of a wavefunction.
- The dependent unit squared amplitude variable Z, which is the square of W, is a generic stochastic analogue of the square of the real part of a wavefunction.
- When and where a quantum process is active, the generic variables W and Z instantaneously transform through scaling at every applicable point (x, t) into process-specific variables. The transforming scale factor for Z is SF and for W is $\sqrt{SF}$, where SF equals the deterministic squared amplitude of the process. As a result:
    - W transforms into the process-specific stochastic analogue of the square root of the squared amplitude of the wavefunction and is denoted X.
    - Z transforms into the process-specific squared amplitude variable, which is the stochastic analogue of the squared amplitude of the wavefunction, denoted Y.
- The generic variables W and Z are present everywhere unless there is quantum activity present to trigger their transformation into the specific variables X and Y and maintain each as a stochastic process.

Our central aim is to demonstrate that with these conjectures, the means of Z and Y equal the variances of W and X, respectively, the property we have called the mean/variance property.

4 The base variable

The unit base variable

The distribution of the product of two independent uniform variables, A and C, whose product forms the variable W, is known [9], and in this case, pdf is (see Appendix A):

$$f_W(w) = -\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\ln\left(\frac{|w|}{3}\right), \; (-3 \leq w \leq 3) \qquad (3a)$$

W is i.i.d. at all points x at time t. The mean and variance of pdf (3a) are 0 and 1, respectively (see Appendix A), as required.

The cumulative density function is also needed for simulation and goodness of fit testing (see the next subsection). By symmetry of pdf (3a), the cumulative density, in a form that is useful in stepping around the singularity at $w = 0$, is (see Appendix A):

$$F_W(|w|) = -\left(\ln\left(\frac{|w|}{3}\right) - 1\right)\frac{|w|}{3}. \; (0 < |w| \leq 3) \qquad (3b)$$

The scaled base variable

The transformation to the specific variable X when and where a quantum process is active is achieved by multiplying the unit base variable by the scale factor $\sqrt{SF}$, where

$$SF = R_1^2(x,t) + R_2^2(x,t) + 2R_1(x,t)R_2(x,t)\cos\theta(x,t)$$

and SF is the scale factor that is applied to the squared amplitude variable (see Section 5).

From (3a) and the standard method for deriving the probability density of a function of a random variable, the scaled base variable pdf is:

$$f_{W(\sqrt{SF})}(\sqrt{SF}w) = -\left(\frac{1}{6\sqrt{SF}}\right)\ln\left(\frac{|w|}{3\sqrt{SF}}\right),\ (-3\sqrt{SF} \le w \le 3\sqrt{SF}) \qquad (4a)$$

The transformed pdf (4a) has a mean of 0 and variance of *SF,* which equals the mean of the squared amplitude variable as required (see Section 5). Scaling the unit base variable cdf (3b) is needed for simulation testing to check for correctness within a margin that allows for randomness in the simulation. This is done in a manner ready for squaring when the squared amplitude variable is tested and involves multiplying the absolute value of the simulated *W* data by the scale factor $\sqrt{SF}$ to put it in the form $\sqrt{SF}\,|w|$, transforming the unit cdf (3b) to the scaled base variable cdf (4b) below, and testing it as described in Appendix C:

$$F_{W(\sqrt{SF})}(\sqrt{SF}|w|) = -\left(\ln\left(\frac{|w|}{3\sqrt{SF}}\right) - 1\right)\frac{|w|}{3\sqrt{SF}},\ (0 < |w| \le 3\sqrt{SF}\ ) \qquad (4b)$$

5 The squared amplitude variable

The unit squared amplitude variable

It is conjectured that *W* and *Z* are present everywhere at all times. *Z* is the square of W, and the distribution of the square of a random variable is well known in this case as pdf (see Appendix B):

$$f_Z(z) = -\frac{1}{6\sqrt{z}}\ln\frac{\sqrt{z}}{3},\ (0 < z \le 9) \qquad (5a)$$

The variable *Z* is i.i.d. at all points *x* at time *t.* The mean of *Z* is 1, which agrees with the variance of the unit base variable, as required. Its variance is $56/25 = 2.24$ (see Appendix B), and its coefficient of variation is $\sqrt{2.24} \cong 1.5$, or 150%, compared with, for example, 100% for a gamma distribution with a shape parameter of 1.

As with the base variable, the cumulative density function is needed for simulation and goodness of fit testing, and being the resultant variable of the chain that comprises the family of variables is important in its own right. The cumulative density of (5a) is (see Appendix B):

$$F_Z(z) = -\left(\ln\left(\frac{\sqrt{z}}{3}\right) - 1\right)\left(\frac{\sqrt{z}}{3}\right),\ (0 < z \le 9) \qquad (5b)$$

The squared amplitude variable

As with the base variable, the transformation to the squared amplitude variable Y when and where a quantum process is active is achieved by multiplying the unit variable by a scale factor, in this case *SF* (defined in the previous section), which aligns the mean of the squared amplitude variable with the deterministic squared amplitude. The scaled pdf is:

$$f_{Z(SF)}(SFz) = -\frac{1}{6\sqrt{zSF}}\ln\left(\frac{1}{3}\sqrt{\frac{z}{SF}}\right),\ (0 < z \le 9SF) \qquad (6a)$$

The scale factor $SF = R(x,t)^2 = R_1^2(x,t) + R_2^2(x,t) + 2R_1(x,t)R_2(x,t)\cos\theta(x,t)$, which provides the link to point (x, t). If there is quantum activity but no interference, SF equals one or another of $R_1^2(x,t)$ or $R_2^2(x,t)$.

The generic unit squared amplitude variable Z is present in the background everywhere all the time and is transformed to the squared amplitude variable Y across the set of stochastic processes behind all the active quantum events at time t. Z is therefore the generic archetype we set out to find.

The specifics of pdf (6a) depend on the quantum process, and the specific variable Y that relates to the j^{th}quantum process of the set of N active processes at time t will be denoted $\mathrm{Y}(j,t)$. In this conception, the unit variable Z is present everywhere with the essential characteristics of a precursor of a squared amplitude. From when and where quantum process j becomes active, the unit base variable W is instantly transformed by the scale factor $\sqrt{\sum SF(j,x,t)}$ to the variable $\mathrm{X}(j,x,t)$ and the unit squared amplitude variable Z by the scale factor $SF(j,x,t)$ to the squared amplitude variable $\mathrm{Y}(j,x,t)$.[5] Using the shortened notation, the mean of $\mathrm{Y}(j)$ equals the scale factor $SF(j)$, which equals the formulation of the squared amplitude of process j in quantum mechanics. The mean of $\mathrm{Y}(j)$ also equals the variance of $\mathrm{X}(j)$ - the mean/variance property.

In summary, Z is an enduring and universal process, and the transformed set $\mathrm{Y}(j,x,t), j \in [1,N]$ comprises the universe of active processes at time t with each member specifying the details of its process at every applicable point x at that time. In particular, the probability that a particular particle in a particular process j is at a particular point x at a particular time t is the normalised variable $\mathrm{Y}(x,j,t)$ and has a mean equal to the squared amplitude as formulated in quantum mechanics.

The squared amplitude cdf is needed for simulation testing to check for correctness. This involves squaring the simulated W data, transforming the unit cdf (5b) to the scaled cdf (6b) below and testing it as described in Appendix C:

$$F_{Z(SF)}(SFz) = -\left(\ln\left(\frac{1}{3}\sqrt{\frac{z}{SF}}\right) - 1\right)\left(\frac{1}{3}\sqrt{\frac{z}{SF}}\right), (0 < z \leq 9SF) \qquad (6b)$$

Note that the scaled cdf reaches a value of 1 when $z = 9SF$, where SF can be any nonnegative number, giving pdf (6a) and cdf (6b) a support capable of being anywhere in the range $0 < z < \infty$, depending on SF.

This means that the squared amplitude variable can be expanded with $SF > 1$ or compressed with $0 < SF < 1$ as required to match the deterministic squared amplitude. This is illustrated in Table 1, where the cumulative density in the first bin of a set of bins of equal width is shown for a range of scale factors.

[5] Where the context permits $SF(j,x,t)$, $\mathrm{X}(j,x,t)$ and $\mathrm{Y}(j,x,t)$ will be denoted $SF(j)$, $\mathrm{X}(j)$ and $\mathrm{X}(j)$.

Table 1 Percentage of the cumulative density of the squared amplitude variable Y in the first bin of a set of bins of equal width 0.25											
SF	72	36	18	9	3	1	.72	.3	.18	.09	.03
% (rounded)	9.7	12.7	16.6	21.6	32.1	46.5	51.6	66.6	76.0	88.2	99.9

6 The significance of the squared amplitude variable

The identification of a squared amplitude variable that satisfies the properties needed to be compatible with the formulation of quantum mechanics suggests the possibility that the formulation of probability in quantum mechanics could be pointing to and employing the normalised mean of a random variable, with the average of a large number of experiments converging to that mean. Quantum probability would then be a prediction of the expected relative frequencies of possible experimental outcomes, which is the very definition of a probability.

What is suggested here is that the squared amplitude variable springs from a generic stochastic variable *Z,* which is an archetype in the Platonic concept of pure form, satisfying the properties in Section 3 and embodying the essential characteristics of a precursor variable of quantum probability. Under this hypothesis, quantum mechanics provides the prediction of the probability of a quantum event occurring at a spatial point and experiment time, and wherever there is quantum activity, the generic archetype is transformed into a process-specific squared amplitude variable and the cumulative number of its realisations at each point and time as a fraction of the cumulative total of all such realisations over all possible points at that time would if known be seen to converge to the set of probabilities formulated by quantum mechanics of the particle whose motion is governed by the wavefunction being at these points. Of course, realisations of such a squared amplitude variable at a spatial point and time and their sum across all such possible points at that time would not be countable or known, but even if such realisations and their relative frequencies cannot be counted or known, they can be hypothesized to exist.

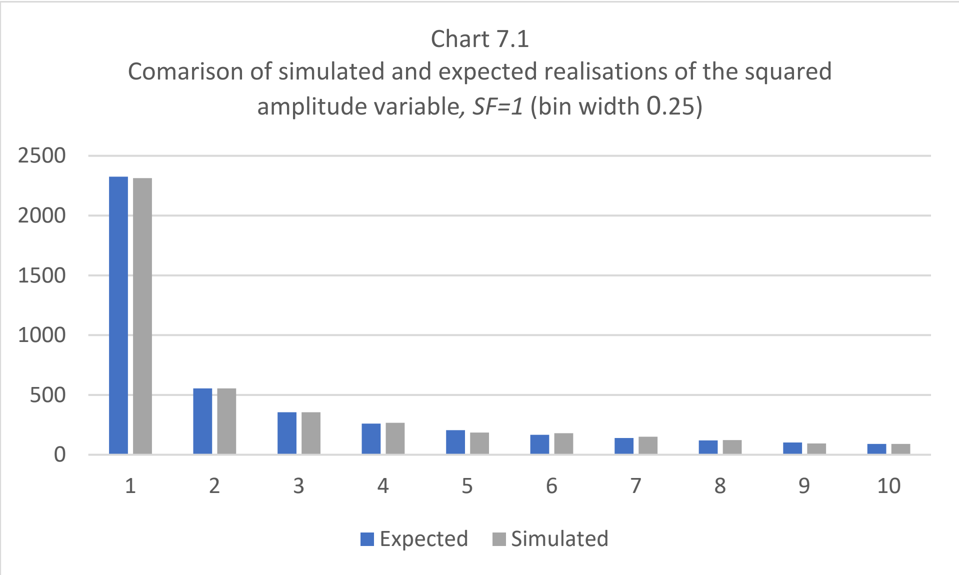
7 Simulation and goodness of fit testing of the squared amplitude variable

To test the fit between expected and simulated results, the squared amplitude variable was simulated using 5000 simultaneous random realisations of the amplitude-like and cosine seed variables A and C, with realisations of the scaled base and squared amplitude variables X and Y calculated from them: in the case of X by multiplying the A and C simulated data and then multiplying the absolute values of the result by $\sqrt{SF}$ to get simulated $\sqrt{SF}\,|W|$data; and in the case of Y by squaring the simulated $\sqrt{SF}\,|W|$ data to get simulated $SFW^2 \equiv SFZ$ data; and then proceeding to test the results against those expected as described in Appendix C.

As a consequence, the simulated values provide an independent test of the derivation of the squared amplitude variable Y. The three charts that follow compare the simulated results with those expected using the cumulative density function (6b), and Table 2 following the charts provides some key statistics.

It can be seen from the charts that there is a good fit of the results expected from the derived squared amplitude variable with the results of the simulations. Appendix C provides the data underlying the charts and describes the method used.

It can also be seen from the key statistics in Table 2 that the scaled base and squared amplitude variables have the required properties set out in Section 3, including the essential mean/variance property. The table confirms a good fit between expected and simulated means and variances, including the expected and simulated variances of the squared amplitude variable, which have been included as a matter of interest.



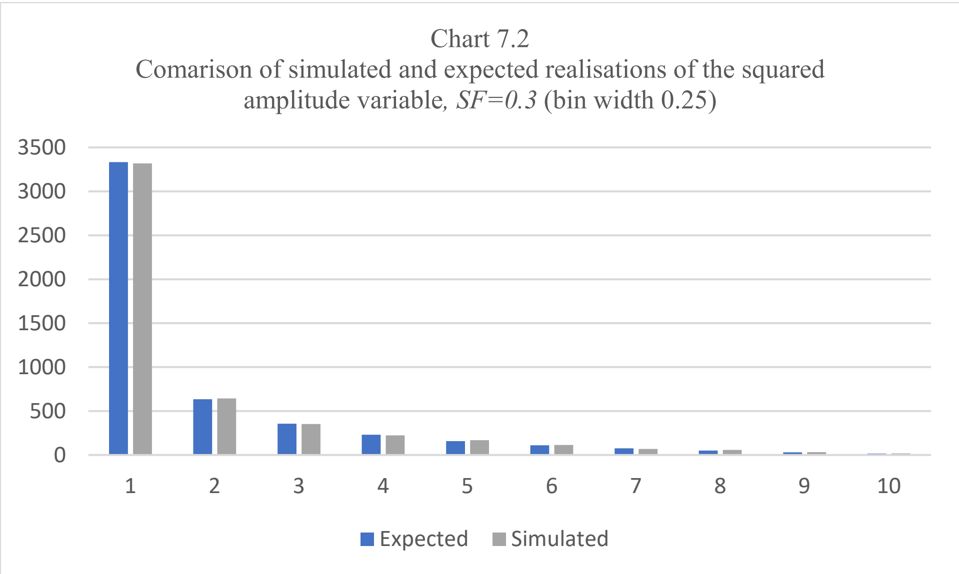


Chart 7.3
Comarison of simulated and expected realisations of the squared amplitude variable, *SF=3* (bin width 0.25)

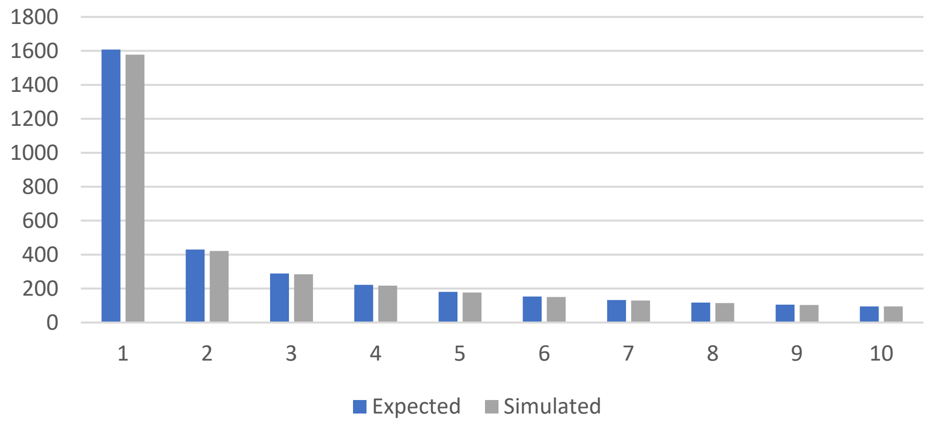


Table 2 Key statistics						
Variable that is scaled	X	Y	X	Y	X	Y
Scaling factor	3		1		.3	
Expected mean Simulated mean	0 .002	3 3.04	0 .001	1 1.01	0 .001	.3 .304
Expected variance Simulated variance	3 3.04	20.16 20.48	1 1.01	2.24 2.28	.3 .304	.2016 .2048

8 Consistency of the hypothesis with nonlocality and entanglement

The generic variable Z is hypothesized to be present everywhere all the time. As discussed above, it is transformed by local quantum activity into a specific variable Y in a localized stochastic process.

However, the universality of Z means that Y can also relate to interactions such as measurements relating to two particles that are too far apart in space and too close together in time for them to be connected even by signals moving at the speed of light. This is because such interactions can nevertheless be physically connected through Holland's quantum theory of motion, as described in his account of the de Broglie-Bohm causal interpretation of quantum mechanics and his treatment of a many-body system [10].

Holland defines an individual n-body system as comprising:

a) A wavefunction $\psi = \psi(\mathbf{x_1}, \ldots\ldots, \mathbf{x}_n, t)$ is defined in a $3n$-dimensional configuration space in which $\mathbf{x_1}, \ldots\ldots, \mathbf{x_n}$ provides a set of rectangular Cartesian coordinates.

b) A set of n point particles pursuing trajectories $\mathrm{x}_{\boldsymbol{i}}(t), i = 1, \ldots\ldots, n$ in three-dimensional Euclidean space. A single configuration space trajectory is equivalent to n particle trajectories in Euclidean space.

As Holland explains, a many-body system then means a *single* wavefunction with a set of particles, with no individual wave associated with a particle. While the particles each move in 3-space, the guiding wave is defined in $3n$-space. Since in this interpretation the wave is a physical influence on the particles, the configuration space is attributed with as much physical reality as that of Euclidean space in the one-body theory. These notions in Holland's extension of the quantum theory of motion to many-body systems, namely, that an individual physical system resides in a multidimensional configuration space and that the latter is physically real, are in Hollands words "striking features not evident in the one-body case" [10].

Holland points out that this conception, with these features, including that the configuration space wavefunction depends on a single evolution parameter *t,* implies that the state of the *n* particles is specified at a common time and that there is a nonlocal connection between them brought about by the classical and quantum potentials, with the instantaneous motion of any one particle depending on the coordinates of all the other particles at the same time. If part of the system is disturbed in a localized region of three-dimensional space, the configuration space as a whole will respond, and consequently, all the particles making up the system will be affected *instantaneously* [10].

Holland studies the conditions in a two-body system under which we can expect to find correlations in particle motions and the conditions under which they are independent [11]. The case where $\psi(\mathbf{x_1}, \mathbf{x}_2, t)$ is factorizable and is the product of two wavefunctions associated with each of the particles is one in which the two particles are physically independent. To examine a case with correlation and entanglement, Holland moves to a wavefunction expressed as the sum of two such factorizable wavefunctions [11]:

$$\psi(\mathbf{x}_1, \mathbf{x}_2, t) = \psi_A(\mathbf{x}_1)\psi_B(\mathbf{x}_2) + \psi_C(\mathbf{x}_1)\psi_D(\mathbf{x}_2) \tag{8a}$$

Noting that such solutions may be constructed when there is no classical interaction between the particles, Holland states that when the summands in (8a) overlap in configuration space, that is, when $\psi_A \cap \psi_C \neq \emptyset$ and $\psi_B \cap \psi_D \neq \emptyset$, the wavefunction is nonfactorizable and entangled [11], and he gives the joint squared amplitude of (8a) in Euclidean space as follows [11][6]:

$$R^2 = R_A^2 R_B^2 + R_C^2 R_D^2 + 2R_A R_B R_C R_D \cos\left[(S_A + S_B - S_C - S_D)/\hbar\right] \tag{8b}$$

and that the interference term in (8b) is finite, the particle motions are nonlocally correlated, and a particle is no longer associated with wave $\psi_A\psi_B$ or wave $\psi_C\psi_D$ -

[6] Which when normalised is the joint probability.

both particles are guided by one wavefunction $\psi(\mathbf{x}_1, \mathbf{x}_2, t)$ [7]. To quote Holland, “bearing in mind that the primary property of the wavefunction is the influence it exerts on the particles, we deduce that the particles are *statistically correlated because they are physically connected*”.

When the cosine term in (8b) is simplified to $\cos\ \theta$, as was done in Section 1 and the squared amplitude (8b) is used as the scale factor SF in pdf (6a), Y becomes a specific stochastic analogue of the squared amplitude (8b) and is part of its stochastic process, including its inherent nonlocality. We can take from this that the universal generic variable Z, the process-specific variable Y, and the hypothesis as a whole are consistent with nonlocality in quantum mechanics in the de Broglie-Bohm causal interpretation as embodied in Holland’s quantum theory of motion and that they encompass both local and nonlocal quantum events.

9 Conclusion

This article presents a hypothesis that behind both the squared amplitude of a superposed or individual wavefunction as formulated by quantum mechanics, there is an associated specific variable Y at each applicable point in space and time that springs from a universal generic archetypical variable Z. The variable Y has a mean at each point at time that equals and with repeated trials of an experiment converges to the squared amplitude. When the number of realisations there is normalised to a relative frequency, it converges with repeated trials to the probability predicted by quantum mechanics that the particle whose motion is described by the wavefunction is at each such point. Importantly, as shown in the previous section, despite being developed using classical probability theory, the variable Y and its stochastic process can relate to either a local or a nonlocal quantum mechanical process.

Quantum mechanics works perfectly well without a mathematical reconciliation between the formulation of quantum probability and the axioms of probability. However, the results provide insight that quantum probability itself could originate from a generic and universal stochastic archetype that triggers a specific variable when and where there is quantum activity and continues as a stochastic process while the activity continues. When normalized, this variable would be a stochastic analogue of quantum probability. This prospect not only raises intriguing questions about the nature of the underlying physics that would be described by such a process but might also inform or otherwise prove useful in quantum technology.

[7] In this article the R and S terms here are functions of position at time t of the particles 1 and 2 on their single dimension lines x_1, x_2 and guided by $\psi(x_1, x_2, t) = \psi_A(x_1)\psi_B(x_2) + \psi_C(x_1)\psi_D(x_2)$.

APPENDIX A

The unit base variable

To derive the pdf of the product of the two independent uniform random variables, $A \sim U$ (0,3) and $C \sim U$ (-1,1) begin with the probability density of a Γ(2,1)[8] distribution $f(s)ds = se^{-s}\,ds\ (0 < s < \infty)$. Let $s = -\ln w$ so that $ds = -d(\ln w) = -\,dw/w\ \ (0 < w < 1) \rightarrow\ f(s) = -\ln w\ \ (0 < w < 1)$. As Huber saw [9], this causes larger values of *s t*o lead to smaller values of *w* so that when the substitution is reversed, a minus sign must be attached to the result. Thus:

$$f(s)ds = -\left(-\ln w\, e^{-(-\ln w)}(-\,dw/w)\right) = -\ln w\, dw, (0 < w < 1)$$

Allowing for the support of *A* gives$-\ln(w/3)\ d(w/3)$= -$\frac{1}{3}$ln $(w/3)dw$ (0 <*w* <3). Replacing *w* with |*w*| and spreading the pdf across -3<*w*<*3* gives[9]:

$$f_W(w) = -\frac{1}{2}\frac{1}{3}\ln(|w|/3),\ (-3 < w < 3) \qquad \text{(A1)}$$

Because of the symmetry of (A1), we can use the following alternate form to derive a useful form of the cumulative density function, using only the positive half of the support:

$$f_W(|w|) = -\frac{1}{3}\ln(|w|/3),\ (0 < |w| \leq 3) \qquad \text{(A2)}$$

To integrate (A2), substitute $u = \frac{|w|}{3}$, giving $d|w| = 3du$ so that (A2) becomes $\int -\ln(u)\, du = -\int \ln(u) . 1\, du$, treat factor 1 as the derivative of u and integrate by parts giving $\int -\ln(u)\, du = u - u\ln u$. Reversing the substitution gives:

$$F_W(|w|) = -\left(\ln\left(\frac{|w|}{3}\right) - 1\right)\frac{|w|}{3},\ (0<|w|\leq 3) \qquad \text{(A3)}$$

F(|*w*|) = 1 when *w* = 3, confirming that (A2) is a probability density. By inspection, the mean of W is zero, and drawing on the means and variances of *A*, $\left(\frac{3}{2}, \frac{9}{12}\right)$ and *C*, $\left(0, \frac{4}{12}\right)$ and using the standard result for the variance of the product of two independent variables, the variance of *W* is:

$$(\sigma_A^2 + \mu_A^2)(\sigma_C^2 + \mu_C^2) - \mu_A\mu_C = \left(\frac{9}{12} + \frac{9}{4}\right)\left(\frac{4}{12}\right) = 1$$

[8] The negative logarithm of a *U* (0,1) variable has an exponential distribution with rate 1, so the negative log of the product of two of them has the distribution of the sum of two exponential variables. As the exponential with rate 1 is a Γ (1,1) variable, then adding the shape parameters the sum of two of them is Γ (2,1).

[9] For a deeper and more explanatory derivation see the elegant treatment by Huber [9].

APPENDIX B

The unit squared amplitude variable

The density of the square of a random variable is well known and, allowing for symmetry of the variable W that is being squared, is by inspection of pdf (A1):

$$Z \sim -\frac{1}{6\sqrt{z}}\ln\frac{\sqrt{z}}{3},\ (0 < z \leq 9) \tag{B1}$$

and

$$f_Z(z) = -\frac{1}{6\sqrt{z}}\ln\frac{\sqrt{z}}{3},\ (0 < z \leq 9) \tag{B2}$$

To integrate (B2) to provide the cumulative density function put $u = \frac{\sqrt{z}}{3}$ then $\frac{du}{dz} = \frac{1}{6\sqrt{z}}$ giving $\frac{1}{6\sqrt{z}}dz = du \rightarrow -\int \ln u\, du = u - u\ln u$, and reversing the substitution gives:

$$F_Z(z) = -\left(\ln\left(\frac{\sqrt{z}}{3}\right) - 1\right)\frac{\sqrt{z}}{3},\ (0 < z \leq 9) \tag{B3}$$

$F(z) = 1$ when $z = 9$, confirming that (B2) is a probability density.

Using (B2) to integrate for the mean we have:
$-\int \frac{z}{6\sqrt{z}}\ln\frac{\sqrt{z}}{3}dz = -\int\frac{\sqrt{z}}{6}\ln\frac{\sqrt{z}}{3}dz = -\frac{1}{12}\int\sqrt{z}\ln\frac{z}{9}dz$, put $u = \frac{z}{9}$, then $\frac{du}{dz} = \frac{1}{9}$ and $\frac{1}{9}dz = du$, also $\sqrt{z} = 3\sqrt{u}$ and simplying $\rightarrow -27\int\sqrt{u}\ln u\ du$. Integrating by parts, using $\sqrt{u} = \frac{d}{du}\frac{2u^{3/2}}{3}$ and dropping the factor -27 for the moment gives $\frac{2(\ln u)\,u^{3/2}}{3} - \int\frac{2\sqrt{u}}{3}du$, and the latter integration gives $\frac{4u^{3/2}}{9}$. Reversing the substitution and returning the factor -27 gives $-\frac{\left(3\ln\frac{z}{9}-2\right)z^{3/2}}{54}$, and we have:

$$\mu_Z = -\left[\frac{\left(3\ln\frac{z}{9}-2\right)z^{3/2}}{54}\right]_0^9 = 1. \tag{B4}$$

The approach is the same to integrate for $\mathrm{E}[z^2]$. The substitution $u = \frac{z}{9}$ is the same, and $z^{3/2} = 27u^{3/2} \rightarrow -243\int u^{3/2}\ln u\ du$. Integrating by parts using $u^{3/2} = \frac{d}{du}\frac{2u^{5/2}}{5}$ and dropping the factor -243 temporarily $\rightarrow \frac{2(\ln u)\,u^{5/2}}{5} - \int\frac{2u^{5/2}}{5}du$, the integration term gives $\frac{4u^{5/2}}{25}$. Reversing the substitution and returning -243 gives $-\frac{\left(5\ln\frac{z}{9}-2\right)z^{5/2}}{150}$ and:

$$\sigma_Z^2 = \mathrm{E}[z^2] - {\mu_Z}^2 = -\left[\frac{(5\ln\frac{z}{9}-2)z^{5/2}}{150}\right]_0^9 - 1 = \frac{81}{25} - 1 = \frac{56}{25} = 2.24 \tag{B5}$$

APPENDIX C

Data and method for Charts 7.1, 7.2 and 7.3

Table 3
Data underlying the Charts comparing simulated and expected realisations of the squared amplitude variable (bin width 0 .25)

SF	Bin	1	2	3	4	5	6	7	8	9	10
Chart 7.1 SF=1	xpected	2326	555	355	261	205	167	140	120	103	90
	Simulated	2313	555	356	267	186	180	151	122	95	90
Chart 7.2 SF=.3	Expected	3332	634	357	231	158	110	76	51	31	16
	Simulated	3319	643	353	224	168	114	68	57	33	18
Chart 7.3 SF=3	Expected	1607	430	289	221	181	153	133	117	105	95
	Simulated	1578	422	284	217	177	150	130	115	103	95

Method of testing the scaled base variable

The results of simulation testing of the base variable have not been included, but the method used to test cdf (4b) is similar to that described below for the squared amplitude variable and create Table 3 above. As with the squared amplitude variable, testing revealed a good fit. Of course, the cdf and its support, the bin width, and the data transformation $\sqrt{SF}|W|$ referred to in Section 7 are all different, and the method is adjusted accordingly. For example, the base variable data counted in a particular bin for a particular *SF* comprise all simulated base variable values $\sqrt{SF}|W|$ that fall in the range of that bin, and the total count in it is compared with the expected count using cdf (4b). Note that because testing the base variable involves simulated data in the form $\sqrt{SF}|W|$ when these data are squared during testing of the squared amplitude variable, it is in the correct form SFW^2 for that purpose.

Method of testing the squared amplitude variable

To test cdf (6b), there are 5000 simulations and ${}^{9SF}/_{bw}$ bins of equal width *bw* = .25 to count the simulated data described in Section 7 in, progressing from the first bin covering values 0 to *bw* through to the last bin covering values from $(9SF - bw)$ to $9SF$, at which point cdf (6b) reaches a value of 1, the bin width having been chosen to give an integer number of bins at this point. Considering the data counted in say the 5^{th} bin for each *SF*, which go from *SF* to 1.25SF, all simulated squared amplitude values SFW_5^2 that fall in the range of this bin for their *SF* are counted in it and the total count is compared with the expected count in this bin, which using cdf (6b) is $5000\big(F_{Z(SF)}(1.25SF) - F_{Z(SF)}(SF)\big)$.

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