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Combinatorial Optimization with Photonics-Inspired Clock Models

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Abstract

NP-hard combinatorial optimization problems are in general hard problems that their computational complexity grows faster than polynomial scaling with the size of the problem. Thus, over the years there has been a great interest in developing unconventional methods and algorithms for solving such problems. Here, inspired by the nonlinear optical process of q -photon down-conversion, we introduce a single-variable nonlinear dynamical model that allows for efficiently approximating the ground state of the classical q -state planar Potts Hamiltonian. This reduces the exhaustive search in the large discrete solution space of a large class of combinatorial problems that are represented by the Potts Hamiltonian to solving a system of coupled dynamical equations. We introduce two different annealing mechanisms, adiabatic deformation and chaotic annealing, and show that numerical simulations of the proposed dynamical model can efficiently approximate the solution of hard combinatorial problems. The proposed algorithm is applied to graph- q -partitioning as an example.

Introduction

Combinatorial optimization problems such as graph partitioning and coloring problems appear in a wide range of applications [1]. These problems involve discrete search spaces and target finding the optimal solution from a finite set of objects. In computational complexity theory, it is known that many combinatorial problems are NP-hard [2]. Thus, solving large-scale combinatorial problems often requires finding solutions with certain accuracy in a limited time. On the other hand, it is well known that many important classes of combinatorial problems can be formulated as finding the ground state of lattice spin models including the Ising and Potts models [3, 4]. In this regard, over the years, there has been a growing interest in simulating these models through

quantum and classical physical systems for solving combinatorial optimization problems. These include superconducting qubits [5], trapped ions [6] and most recently photonic oscillator networks [7–9].

Along different lines, numerical simulation of dynamical models governing physical systems has shown promise for solving combinatorial problems [10–17]. These schemes are built on direct simulation of nonlinear dynamical equations governing networks of interacting neurons with continuous degrees of freedoms that collectively evolve toward equilibrium points with binary states [12]. Despite their partial success, however, these methods suffer from two fundamental problems; (i) limitations of scaling the size of the network due to an increasing computational burden, and (ii) trapping into local minima, which could be far from the global minimum of the governing cost function. Subsequently, there has been a continuous interest in developing novel physics-inspired algorithms that allow for finding good approximations of the solutions to large-scale combinatorial problems.

In this work, inspired by the dynamical equations governing a network of coupled q -Photon Down-Conversion (q -PDC) oscillators, we introduce a new method for approximating the ground state of the planar Potts model. The proposed method is built on the dynamical evolution of a network of first-order nonuniform phase oscillators toward equilibrium states that generally represent the local minima of the Potts Hamiltonian. In order to mitigate the problem of trapping into the local minima, we utilize two different annealing mechanisms that show significant improvement in minimizing the Potts Hamiltonian. The first approach is based on adiabatically deforming the underlying cost function through a parameter that controls the convexity at discrete phase values. The second approach is based on an adiabatic passage of the dynamical system through a chaotic regime which is enabled through numerical chaos that can be controlled by time-discretization step size. Through examples, we show that the proposed minimalistic model in conjunc-

tion with a proper annealing mechanism enables efficient solution of combinatorial optimization problems.

The Planar Potts Model

The planar Potts model describes a network of interacting q -state spin degrees of freedom, where each spin can take an angle of $\theta = (2p - 1)\pi/q$ ($p = 1, \dots, q$) in the two-dimensional plane [4]. The interaction energy of a pair of spins (i, j) depends on their coupling strength J_{ij} and the cosine of the relative angles $(\theta_i - \theta_j)$ according to $\mathcal{H}_{ij} \propto -J_{ij} \cos(\theta_i - \theta_j)$. The planar Potts Hamiltonian is expressed as the sum of the pairwise interactions as [4]:

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \cos(\theta_i - \theta_j), \quad \theta_i = \frac{(2p - 1)\pi}{q}, \quad p = 1, \dots, q \quad (1)$$

which, is also referred to as the q -state clock model [18]. For $q = 2$, it reduces to the Ising Hamiltonian and for $q \rightarrow \infty$ it approaches the XY Hamiltonian. Here, it is important to note the difference between the planar and the standard Potts models. The standard Potts Hamiltonian is defined as $\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \delta(\theta_i, \theta_j)$, where, δ represents the Kronecker delta function. Clearly, the two models are identical for $q = 2$ and 3, while the planar Potts deviates from the standard model for $q \geq 4$.

The Potts Optimization Problem

The Potts problem can be stated as finding the spin configuration that minimizes the Potts Hamiltonian. It can be readily viewed as coloring the vertices of a graph using q colors (q -coloring problem), where each vertex is represented by a spin and its color is represented by the spin state, and the graph edges weights are equal to the coupling strengths [3]. A number of other combinatorial optimization problems such as Max- q -Cut can be expressed in terms of the Potts problem. In this problem, we want to divide the vertices into q disjoint groups such that the total weight of edges between these groups is maximized. For $q = 2$, the Potts problem reduces to the Ising problem which represents the Max-Cut problem. These problems belong to the NP-hard computational complexity class meaning the solution space grows exponentially with problem size and it becomes prohibitively difficult to find optimum solutions to large-scale problems. For this reason, unconventional computing systems are being developed to find approximate solutions to large-scale problems in shorter times.

It is worth mentioning that much of the research on unconventional methods and devices that target solving combinatorial optimization problems is focused on the Ising problem. These include, the D-Wave's quantum annealing machine [5, 19], the coherent Ising machine [7, 8], Hitachi's CMOS Ising solver [20], Fujitsu's digital annealer [21], and Toshiba's simulated bifurcations machine [16]. In contrast, developing methods for

solving the Potts problem offers an advantage in treating combinatorial problems that can be mapped more efficiently onto the Potts Hamiltonian.

Results

The q -PDC Oscillator

Recently, it was shown that the 3-state Potts model can be simulated with networks of 3PDC oscillators [22]. This is based on the tristable phase of the parametric spontaneous three-photon down-conversion process [23]. The idea can be extended for building a classical nonlinear oscillator model for the q -photon down-conversion (q -PDC) process. In this manner, we introduce a second-order nonlinear oscillator with complex amplitude a with the dynamics described as $\dot{a} = -\gamma_l a + g(a^*)^{q-1} - g_s |a|^{2(q-1)} a$, where $q = 2, 3, 4, \dots$, and “*” shows complex conjugation. Here, g is the small signal oscillator gain, g_s is the gain saturation coefficient, and γ_l is the rate of intrinsic oscillator losses. The dynamics of this oscillator is governed by a Lyapunov function described as $\mathcal{F}_0 = \gamma_l |a|^2 - \frac{g}{q} (a^q + (a^*)^q) + \frac{g_s}{q} |a|^{2q}$. [24]

Figures 1(a) schematically depict the q -PDC oscillator and the associated virtual energy diagram for different q values. The corresponding single-oscillator Lyapunov functions are shown in Figs. 1(b). These figures clearly indicate a phase transition as a result of increasing the gain g , which marks the onset of self-sustained oscillations with a finite nonzero amplitude. The equilibrium oscillator amplitude $|\bar{a}|$ obeys the nonlinear relation $-\gamma_l |\bar{a}| + g |\bar{a}|^{q-1} - g_s |\bar{a}|^{2q-1} = 0$, which can be solved for various gain levels as depicted in Figs. 1(c). According to these figures, the phase transition is second-order for $q = 2$ and first-order for $q = 3, 4, \dots$, where by increasing the gain a solution branch appears out of the blue sky [25]. For the purpose of optimization, a first-order phase transition causes additional complexities because it creates the undesired possibility of the death of oscillators [22]. However, as discussed later in this paper, by reducing the q -PDC oscillators to phase oscillators this problem will be effectively circumvented.

Next we consider a network of linearly coupled q -PDC oscillators for $q = 2, 3, 4, \dots$. The equations governing the complex amplitudes of the oscillator network follows:

$$\dot{a}_i = -\gamma_i a_i + g(a_i^*)^{q-1} - g_s |a_i|^{2(q-1)} a_i - \sum_j \kappa_{ij} a_j, \quad (2)$$

where, a_i represents the complex amplitude of the i 'th oscillator, κ_{ij} is the coupling coefficient between the i 'th and j 'th oscillators, and $\gamma_i = \gamma_l + \sum_j |\kappa_{ij}|$ is the damping factor of the i 'th oscillator which is composed of the intrinsic oscillator loss γ_l and external losses due to dissipative interaction with other oscillators $\sum_j |\kappa_{ij}|$ [22, 26, 27]. The relative strength of the self-oscillation g and coupling coefficients κ_{ij} plays a critical role in the dynamics and accordingly in the success of the optimization process which will be discussed later in this paper.

The dynamical system described by equations 2 admits a Lyapunov function $\mathcal{F}$, such that $\dot{a}_i = -\partial\mathcal{F}/\partial a_i^*$ and $\dot{a}_i^* = -\partial\mathcal{F}/\partial a_i$. The function Lyapunov function,

$$\mathcal{F} = \sum_i \left[\gamma_i |a_i|^2 - \frac{g}{q} (a_i^q + (a_i^*)^q) + \frac{g_s}{q} |a_i|^{2q} \right] + \frac{1}{2} \sum_{i,j} a_i^* \kappa_{ij} a_j, \quad \dot{\phi}_i = g \sin(q\phi_i) - \sum_j \kappa_{ij} \sin(\phi_j - \phi_i). \quad (3)$$

serves effectively as a cost function for the oscillator network. The Potts problem can be solved by numerically integrating the dynamical equations (2) which is equivalent to applying a gradient-descent minimization of the cost function of Eq. (3). Alternatively, one can start from the cost function $\mathcal{F}$ and apply any gradient-based technique for minimizing $\mathcal{F}$ over its continuous $2n$ -dimensional phase space [28].

The Nonuniform Phase Oscillator

The cost function of Eq. eqF embeds the planar Potts Hamiltonian, while a faithful representation of the Potts Hamiltonian requires uniform amplitudes for all oscillators [13, 26, 29, 30]. To reduce the computational burden of numerical simulations, we consider uniform amplitudes for all oscillators, i.e., $a_i \rightarrow e^{i\phi_i}$, which results in great reduction of the governing $2n$ -dimensional Lyapunov function to an n -dimensional potential, $\mathcal{F} \rightarrow \mathcal{V}$, where $\mathcal{V}$ is a function of the phase degrees of freedom $\phi_i, \dots, \phi_n$ [31]:

$$\mathcal{V} = \frac{g}{q} \sum_i \cos(q\phi_i) + \frac{1}{2} \sum_{i,j} \kappa_{ij} \cos(\phi_i - \phi_j). \quad (4)$$

This relation is in essence the Lyapunov function governing a network of nonuniform phase oscillators [25]. This new potential carries the parameter g , which remains as a control parameter to adjust the strength of the self-oscillation of each oscillator compared to its interaction rate with the network. It should be noted that for $q = 2$ this relation reduces to the cost function introduced in Ref. [32] and has been implemented with circuits for solving the Ising model [33, 34].

In the energy potential of relation (4), the first operates as a penalty term that tends to enforce the evolution of each oscillator toward a q -ary state. The second term in isolation is equivalent to the XY Hamiltonian without the external field term. Therefore, assuming that the first term can successfully drive the oscillators to q -ary states, $\phi_i = (2p-1)\pi/q, p = 1, \dots, q$, the potential reduces to the planar Potts Hamiltonian. However, this critically depends on the parameter g , which controls the strength of the penalty term.

Dynamical Model and Optimization

Considering the energy potential of Eq. (4), we can now write a dynamical model that guarantees descent motion toward the equilibrium points, ideally the global minimum. In doing so, we have the flexibility to deviate from the physics of the system and devise a dynamical model

that results in faster and better numerical convergence to equilibrium points. Here, the straightforward approach is a gradient descent according to $\dot{\phi}_i = -\partial\mathcal{V}/\partial\phi_i$, which results in:

$$\dot{\phi}_i = g \sin(q\phi_i) - \sum_j \kappa_{ij} \sin(\phi_j - \phi_i). \quad (5)$$

Alternatively, we can consider a pair of phase and angular velocity (ϕ_i, ω_i) with dynamical equations $\dot{\phi}_i = \omega_i$ and $\dot{\omega}_i = -\partial\mathcal{V}/\partial\phi_i - \gamma\omega_i$, where γ is an effective friction coefficient and η provides an additional control parameter for the strength of the gradient. This choice is associated with the following second-order dynamics for the phase oscillators:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i = g \sin(q\phi_i) - \sum_j \kappa_{ij} \sin(\phi_j - \phi_i) \quad (6)$$

To numerically find the equilibrium points, one can apply finite difference to either of Eqs. (5) or Eqs. (6) to iteratively update the phases at discrete time steps until reaching stable solutions. By applying different finite-difference formulas, different algorithms are obtained, while in general, momentum methods obtained from a discretization of Eqs. (6) show faster convergence (please see Methods).

Annealing Mechanisms

In general, restricting the dynamics to a descent motion on the potential landscape of Eq. (4) could result in trapping into local minima. The problem of trapping into the local minima is a long-standing challenge in optimization of non-convex functions. An interesting approach for avoiding this is to adiabatically increase the gain parameter [16, 22, 29]. This approach can be compared with quantum adiabatic optimization [35, 36] difference is a lack of a classical counterpart for the quantum adiabatic theorem which ensures a gradual evolution of the state of the system to the ground state of the final Hamiltonian. In here, the classical potential adiabatically deforms toward the desired combinatorial Hamiltonian while the shallow local minima are less likely to arise during this process [22]. In the case of the phase oscillator networks discussed here, by gradually increasing the gain from zero, one can slowly guide the system into a valid (discrete) solution while staying inside the deeper regions of the coupling potential, possibly leading to a good approximation of the ground state solution. Clearly, it is important to increase the gain at an optimal speed that gives the network enough time to search for better solutions, while the caveat is that, slow tuning of the gain generally slows down the stabilization time. Figures 2(a-c) show the landscape potential for an all-to-all connected network of $n = 3$ oscillators for three different gain levels. According to these figures, the potential deforms drastically by increasing the gain.

An alternative approach for bypassing the local minima is a controlled use of chaos. Chaos is by nature different from random fluctuations. However, the possibility of harnessing chaos for optimization has been

suggested in previous works and surprisingly it is suggested that chaotic fluctuations can significantly outperform random noise [37, 38]. A controlled transition of the oscillator network from chaos to fixed-point attractors allows for utilizing the exponentially fast searching capability of the network in its chaotic regime to largely avoid trapping into the local minima when reaching an equilibrium. It should be noted that an individual oscillator in equation 5, described by the dynamical equation $\dot{\phi} = g \sin(\phi)$, is a first-order dynamical system and lacks the minimum dimensionality required for bifurcation and chaos. However, in principle, the dynamical model will be solved numerically, and as it is well known, numerical simulation of ordinary differential equations can induce chaos [39]. In this case, by using Euler’s method, the phase of the oscillator at a time step $t^{(k+1)}$ can be obtained in terms of the previous step $t^{(k)}$ according to $\phi_i^{(k+1)} = \phi_i^{(k)} + \Delta g \sin(q\phi^{(k)})$, where $\Delta = t^{(k+1)} - t^{(k)}$ is the time step. It is straightforward to show that this simple iterative relation creates a chaotic map of the equilibrium phase ϕ versus the time-step Δ . The chaotic map is depicted as an inset in Fig. 2 for a single oscillator with $q = 2$. As a result, by time-discretization, computational chaos can be incorporated in the system. Fig. 2(b) depicts the dynamics of the three coupled oscillators for different values of the step size which clearly shows the chaotic dynamics for large step sizes. Tuning the parameter Δ allows a transition between equilibrium and chaos. Here, by starting with a large time-step, the randomly initialized system evolves into a chaotic regime, and then by reducing the time-step, the system makes an adiabatic passage from chaos to equilibrium.

Optimization Results

We consider the Ising problem ($q = 2$) on an instance from the Biq Mac Library [40], which involves a collection of quadratic binary programming problems. We chose the example of the “g05_100.6” graph, which is an unweighted graph with $n = 100$ nodes and an edge probability of 0.5 between any two nodes, containing a total of 2475 edges. Any other graph from this dataset essentially provides similar insights. Figure 3 shows the results of numerical simulation of the dynamical model of equation (5) with constant gain (first row), adiabatic gain (second row), and chaotic annealing (third row).

The first column depicts the time-evolution of the Ising Hamiltonian (Eq. (1)) for 1000 random initial conditions. The distribution of the Ising energy associated with the steady-state solutions is shown in the second column. The last column exemplifies the dynamics of the phases for one simulation instant. For adiabatic deformation (d-f), the gain is linearly increased from zero. It is observed that at over 45% of the initial conditions lead to the best-known solution (ground state), compared to 5% for the constant gain scenario. By comparing the dynamics of the phases in panels (c) and (f), it is apparent that for the case of constant gain, the phases quickly

settle into their final values. This is expected given the presence of many local minima that could involve the initial phase state in their basin of attraction. In this case, the dynamical system rapidly stabilizes in a local minimum. In the adiabatic gain regime, on the other hand, excursions continue to happen for longer simulation times, which indicates the evolving nature of the energy landscape.

For the case of the chaotic annealing (Figs. 3(g-i)), the gain is set to a constant value, similar to the first row, but the step size is reduced linearly, from $\Delta t_i = 10\Delta_0$ to $\Delta t_f = \Delta_0$. According to Fig. 3(h), 40% of initial conditions lead to the ground state. Here, the distribution of the steady-state energies turns out to be broader than the adiabatic gain case, and spanning over higher energy levels that are associated with local minima. Here, the rapidly evolving chaotic motion of the phases as depicted in Fig. 3(i) allows the system to explore distant parts of the phase space that otherwise would not be accessible.

Next, we apply the adiabatic deformation method to solve the Ising problem on 100-node instances from all classes in the Biq-Mac Library [40]. This library involves 9 classes of 100-node graphs, each containing 10 instances. The results are shown in Fig. 4. In this figure, each panel shows the distribution of the steady-state energies for 1000 random initial conditions. The horizontal axis is the total number of steps, ranging from 10 steps up to 700. According to this figure, in almost all cases the top 20% of the ensemble reach the ground state with under 200 time-steps.

In contrast with the Ising problem, to the best of our knowledge, the Potts problem (considering the case of $q = 3$) lacks a collection of benchmark problems with known solutions. Here, we generated sets of randomly connected 3-colorable graphs with different sizes and connection densities, with each set having 100 random instances. In general, average graph density or average degree of the graph is known to be a measure of the hardness for coloring problems, acting as an order parameter, with low-density graphs being generally harder to color [41]. Therefore, we focused on relatively low density graphs with 500 and 1000 nodes and average degree of 9 (Fig. 5 (a) and (c)) and 18 (Fig. 5 (b) and (d)) (see Methods for details). In these simulations, the time-step size was set to a constant 0.01, while the gain is adiabatically tuned from 0 to 2.5. Each panel in Fig. 5 shows the distribution of the Potts energy associated with the steady-state phase of the oscillator networks versus the total number of time steps. The phases are initiated with uniform random values in the range $-\pi \leq \phi_i(0) \leq \pi$. Here, again, the top 20% of the solutions find the ground state in less than 200 steps.

Conclusion

In summary, inspired by the q -photon down-conversion parametric oscillator, we introduced a nonuniform phase oscillator that emulates a q -state spin in the two-

dimensional plane. This provided an efficient route for searching for the ground state of the planar Potts Hamiltonian by numerically integrating the dynamical equations governing a network of phase oscillators that collectively evolve toward an equilibrium point. The dynamics of the oscillator network is restricted in the phase space to a Lyapunov function which is, in general, a non-convex function. Therefore, it is important to devise schemes that can prevent trapping into the local minima of the Lyapunov function. Here, we considered two different annealing mechanisms based on adiabatic deformation of the cost function and by passage through chaos. Our numerical results confirm that both schemes greatly improve the success of solving the optimization problem.

The proposed method is ideally suited for parallel computing and application to large-scale optimization problems that can be formulated as the planar Potts model. This method is based on two parallel matrix-vector multiplications in conjunction with non-linear function evaluations. In terms of the computational burden, the matrix-vector multiplication is the bottleneck of the proposed method. Therefore, speed-up can be achieved by parallel computing methods and systems.

It is worth stressing that for $q \geq 4$, the clock Hamiltonian deviates from the standard Potts Hamiltonian. In this case, terms associated with unequal spin angles are penalized unevenly and depending on the relative angle of the two interacting spins. In this case, the clock model can still be used to approximate the ground state of the standard Potts model, while the approximation clearly deteriorates for larger q values.

Methods

For numerical calculations, a simple approach is to discretize the first order dynamical model of Eq. (5) by using first-order forward difference relation for the derivative term. This results in the Euler's method for solving ordinary differential equations which is based on a single-step updating of the dynamical variables in subsequent time steps k according to:

$$\Phi^{(k+1)} = \Phi^{(k)} - \Delta_k \nabla \mathcal{V}^{(k)} \quad (7)$$

where, $\Delta_k = t^{(k+1)} - t^{(k)}$ is the time step size, $\Phi = (\phi_1, \dots, \phi_n)^t$, and $\nabla \mathcal{V}$ is the potential gradient,

$$\nabla \mathcal{V} = -g \sin(q\Phi) + \cos(\Phi) \odot K \sin(\Phi) - \sin(\Phi) \odot K \cos(\Phi), \quad (8)$$

where, K is the connectivity matrix with elements κ_{ij} and ' $\odot$ ' shows entryways vector product.

In addition, we considered discretization of the second-order model of Eq. (6) by utilizing the central difference formula for the second derivative and the backward difference formula for the first derivative. This results in a two-step updating rule (known as momentum method in

numerical analysis):

$$\Phi^{(k+1)} = (1 + \mu_k) \Phi^{(k)} - \mu_k \Phi^{(k-1)} - \eta_k \nabla \mathcal{V}^{(k)} \quad (9)$$

where, $\mu_k = 1 - \gamma \Delta_k$ and $\eta_k = \Delta_k^2$. The convergence rate of the difference relations 9 can be controlled with the two parameters μ_k and η_k . In numerical simulations, for the adiabatic deformation approach we used the two-step relation (9) by taking $\mu_k = (k-1)/(k+2)$ and with a constant η_k . For the chaotic annealing approach, we used the single-step relation (7), with a linearly decreasing η_k .

The random graphs used for the 3-coloring problem were generated as described below. For a given number of nodes n , three partitions with n_1, n_2 , and n_3 nodes were considered such that $n_1 + n_2 + n_3 = n$. Then, a full 3-partite graph was generated by connecting each node from one partition to all nodes of the other two partitions, while no edges exist in each partition. Next, the existing edges were removed at random with a probability of $0 < p_0 < 1$, to create a random 3-colorable graph with the desired edge density.

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Figure 1: (a.1-a.4) A schematic of the q -photon down-conversion oscillator for $q = 1, 2, 3$, and 4 . (b.1-b.4) The associated energy landscapes above oscillation thresholds. (c.1-c.4) The phase transition associated with the onset of oscillations. (d.1-d.4) The one-dimensional azimuthal potential governing the dimension-reduced phase oscillators. (e.1-e.4) The associated multi-state spin degrees of freedom.

Figure 2: An illustration of the potential landscape for different gain levels (top row) and the dynamics for different discretization time-steps (bottom row) for a 3-oscillator network (inset). These two parameters form the basis of the adiabatic deformation and chaotic annealing mechanisms that assist the optimization search process. (a-c) The local minima are shallower for lower than higher gains, which facilitates escaping the local minima when the gain is adiabatically increased. (d-h) The route to chaos as a result of increasing discretization time-step, which creates the possibility of harnessing chaos for dynamical complexity in the optimization process. The inset depicts the chaotic map of a single phase oscillator for $q = 2$.

Figure 3: The optimization results for the Ising problem on the “g05 100.6” graph ($n = 100$) with constant gain (a-c), adiabatic gain (d-f), and chaotic annealing (g-i). The first column depicts the time-evolution of the Ising Hamiltonian for 1000 random initial conditions. The second column shows the distribution of the Ising energy associated with the steady-state solutions. The third column exemplifies the dynamics of the phases for one simulation instant. In (a-c), the gain is set to $g = 2.5$ and the step size is set to $\Delta = 0.01$. In (d-f) the gain is linearly increased from $g = 0$ to $g = 2.5$ and the step size is set to $\Delta = 0.01$. In (g-i), the gain is set to $g = 2.5$ and the step size is linearly decreased from $\Delta = 10 \times 0.01$ to $\Delta = 0.01$. In the first column, the transient energy is calculated based on the XY Hamiltonian which transforms to the Ising Hamiltonian as the phases become binary after several iterations. These figures show the the minimum-to-maximum range of the energy distribution for all random initializations (light shaded area), the first standard deviation of the associated distribution (dark shaded area), and the average energy (red line) as a function of the iteration number.

Figure 4: Solution to the Ising ($q = 2$) problem on 100-node graphs from all 9 classes in the Biq-Mac Library with known ground states. (a-i) The distribution of the equilibrium state energies versus the number of iterations for 1000 random initial conditions. The horizontal axis shows the total number of steps (from 10 to 700). In all cases, the gain is linearly increased from $g = 0$ to $g = 2.5$, while the time step-size is $\Delta = 0.01$.

Figure 5: Solution to the Potts-3 ($q = 3$) problem on randomly generated ensembles of 3-colorable graphs of $n = 500$ (top) and $n = 1000$ (bottom) nodes and with average edge density of $p_0 = 0.03$ (left) and $p_0 = 0.06$ (right). For all panels 100 random graphs are generated and each case is simulated with 100 random initial conditions to create an energy distribution based on 10000 data points for a given number of iterations. In all cases, the gain is linearly increased from $g = 0$ to $g = 2.5$, while the time step-size is $\Delta = 0.01$.