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Elastic wave dispersion equation considering material and geometric nonlinearities

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Abstract In this letter, we describe the propagation of longitudinal waves in one-dimensional nonlinear elastic thin rods with material nonlinearity and geometric nonlinearity. Mathematical analysis is used to derive the analytical dispersion relationship of longitudinal waves; subsequently, the numerical Fourier spectrum method is used to solve the nonlinear wave equation directly and the results are used to verify the correctness of the derived nonlinear dispersion relation.

Keywords *Material nonlinearity, geometry nonlinearity, nonlinear dispersion relationship, Fourier spectrum method*

1 Introduction

Wave motion is a ubiquitous phenomenon in nature as evidenced by sound waves, water waves, electromagnetic waves, waves in solids, and even chemical reactions, life process, population migration, and traffic flows. Understanding the governing principles and nature of wave propagation is of great significance for controlling wave propagation and benefiting human society¹. Because of this, research on wave motion has attracted researchers from across a variety of fields. The theory of linear dispersion waves is almost complete, but the nonlinear behavior has not yet been fully characterized.

In the study of waves, the dispersion relationship can provide valuable information about the dynamics of the propagation medium². Most of the early studies on elastic wave dispersion were based on linear governing equations but now, nonlinear effects are deemed more important because they can describe the potential motion of waves more accurately. The first nonlinear wave theory was studied by Stokes (1847). However, because the study of nonlinear waves

involves very complicated mathematical models, the progress of research is very slow. In this regard, G. B. Whitham systematically introduced different wave phenomena in physics and mechanics in his work, especially the problem of wave propagation in solids and fluids². P. L. Bhatnager (1979) introduced the interaction between shock waves, solitary waves in a simple one-dimensional dispersion system in a monograph³. Nariboli, Clarkson P. A, Dreiden, etc. have studied the propagation characteristics of physical nonlinear waves in the elastic rod while considering the nonlinear wave propagation characteristics including the dispersion effect⁴⁵⁶.

In 2013, M.H. Abedinnasab and M.I. Hussein proposed a theory to derive the precise nonlinear dispersion relationship (NDR) while studying finite strain elastic waves in rods and beams⁷; this theory derives an accurate NDR for nonlinear elastic thin rods. Without limiting the amplitude of the elastic waves, numerical simulation can be used to predict the instantaneous dispersion of large-amplitude waves. In 2019, R.K proposed an NDR theory to predict the propagation of longitudinal waves⁸ and used it to predict the generation of harmonics while considering the geometric nonlinearity. In China, Hu and others have studied the numerical simulation of the propagation of longitudinal waves in nonlinear elastic rods. In recent years, scholars have mainly studied nonlinear elastic waves in phononic crystals or artificial metamaterial^{[10][11][12][13][14]}.

However, research that considers both geometric nonlinearity and material nonlinearity in the nonlinear dispersion relationship is not available; in actual engineering, there are indeed two nonlinearities in a material, such as in the case of rubbers or in the case of explosions and high-speed impact, where two nonlinear factors must be considered simultaneously. In this study, we consider thin rods (ignoring lateral inertial effects) with two simultaneous nonlinearities: geometric nonlinearity and material nonlinearity and study the influence of these nonlinear factors on the dispersion relationship. Initially, the analytical method is used to derive the nonlinear dispersion relationship by considering the linear factors (lateral inertial effects) and nonlinear factors (geometric nonlinearities). Subsequently, we ignore the linear factor and concentrate on the influence of the nonlinear factors on the wave displacement gradient solution. From three different instants, we can see that the nonlinear factor makes the wavefront steeper. Using sinusoidal excitation as the initial condition, the nonlinear dispersion relation with

a specific excitation wavenumber is derived; subsequently, the nonlinear wave equation with initial excitation is solved directly, and finally, the correctness of the deduced dispersion relation is verified by wavenumber-frequency spectrum analysis.

2 Dispersion relationship including material nonlinearity and geometric nonlinearity

Consider an infinitely long one-dimensional rod with a radius r , where unidirectional axial stress $\sigma(x, t)$ and longitudinal displacement $u(x, t)$ are functions of position x and time t , respectively. The governing equation can be obtained from Hamilton's principle, namely $\delta \int_{t_0}^t L dt = 0$. Without considering the external non-conservative forces and moments, the elastic Lagrangian density function is $L = T - U$, where $T = \frac{1}{2} \rho \left(\frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} \rho v^2 r^2 \left(\frac{\partial^2 u}{\partial x \partial t} \right)^2$ is the system kinetic energy density and $U = \frac{1}{2} \sigma \varepsilon$ is the elastic potential energy density of the system. Here ρ and ν are the mass density and Poisson's ratio, respectively, and the uniaxial force is a nonlinear relationship^[9] $\sigma = E\varepsilon + E\varepsilon^2$, where E is the Young's modulus. The finite strain here is the Green-Lagrange strain(GLS):

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \quad (1)$$

and Hencky strain (HS):

$$\varepsilon = \ln \left(1 + \frac{\partial u}{\partial x} \right) \quad (2)$$

The Lagrangian density function is substituted into the following Euler-Lagrangian equation, namely:

$$\frac{\partial}{\partial x} \frac{\partial L}{\partial \left(\frac{\partial u}{\partial x} \right)} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \left(\frac{\partial u}{\partial t} \right)} - \frac{\partial^2}{\partial x \partial t} \frac{\partial L}{\partial \left(\frac{\partial^2 u}{\partial x \partial t} \right)} = 0 \quad (3)$$

Take the derivative with respect to x on both sides of the above equation and set

$\bar{u} = \frac{\partial u}{\partial x}$ to obtain:

$$\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial^2 \left(\alpha \bar{u} + \beta N(\bar{u}) + \gamma \frac{\partial^2 \bar{u}}{\partial t^2} \right)}{\partial x^2} = 0 \quad (4)$$

For GLS, $\alpha = \beta = c^2$, $N(u) = 3\bar{u}^2 + \frac{7}{2}\bar{u}^3 + \frac{15}{8}\bar{u}^4 + \frac{3}{8}\bar{u}^5$; For HS, $\alpha = 0$, $\beta = c^2$,

$$N(u) = \frac{\ln(1+\bar{u})}{1+\bar{u}} + \frac{3}{2} \frac{\ln(1+\bar{u})^2}{1+\bar{u}}. \text{ For both cases, } \gamma = r^2 v^2 \text{ and } c = \sqrt{E/\rho}.$$

To facilitate the description of the dispersion, the transformation $\xi = kx - \omega t$ is introduced, where k is the wave number and ω is the frequency of the traveling wave. Equation (4) can be written as:

$$\omega^2 \frac{\partial^2 \bar{u}}{\partial \xi^2} - k^2 \frac{\partial^2 \left(\alpha \bar{u} + \beta N(\bar{u}) + \omega^2 \gamma \frac{\partial^2 \bar{u}}{\partial \xi^2} \right)}{\partial \xi^2} = 0 \quad (5)$$

We integrate the two ends of equation (5) twice and set the integral constant term to 0 to ensure that the traveling wave solution of the initial bounded displacement field is bounded at infinity before solving the equation. After replacing the above variables, the initial displacement field^[15] is a sinusoidal displacement $u(\xi) = B \sin(\xi)$, it satisfies $\bar{u}(0) = Bk$ and $\frac{\partial^2 \bar{u}(0)}{\partial \xi^2} = -Bk$ at $\xi = 0$, and an accurate dispersion relationship can be obtained. The dispersion relationship corresponding to the GLS and HS strain is as follows:

$$w = kc \sqrt{\frac{3B^4 k^4 + 15B^3 k^3 + 28B^2 k^2 + 24Bk + 8}{8(1 + k^2 r^2 v^2)}} \quad (6)$$

And

$$w = kc \sqrt{\frac{\ln(1+Bk)(2+3\ln(1+Bk))}{2Bk(1+Bk+k^2 r^2 v^2 + Bk^3 r^2 v^2)}} \quad (7)$$

If the amplitude B of the wave and the initial excitation of the wave number k_e are given, the initial conditions become:

$$\bar{u}(0) = Bk_e \quad (8)$$

And

$$\frac{\partial^2 \bar{u}}{\partial \xi^2} = -Bk_e \quad (9)$$

Then, we get the nonlinear dispersion with a single harmonic excitation:

$$w_{GLSe} = kc \sqrt{\frac{3B^4 k_e^4 + 15B^3 k_e^3 + 28B^2 k_e^2 + 24Bk_e + 8}{8(1 + k^2 r^2 v^2)}} \quad (10)$$

And

$$w_{HSe} = kc \sqrt{\frac{\ln(1 + Bk_e)(2 + 3\ln(1 + Bk_e))}{2Bk_e(1 + Bk_e + k^2 r^2 v^2 + Bk_e^2 k_e^2 r^2 v^2)}} \quad (11)$$

3 Numerical verification

The Fourier spectrum method is used to solve equation (4) and the precise nonlinear dispersion relationship (10) and (11) may thus be verified.

Given an initial single harmonic excitation, $\bar{u}(x, t) = Bk_e(1 + \cos(k_e(x - ct)))/2$, the amplitude is $B=0.1$, and the wave number k_e is 4.5. Since it is impossible to simulate an infinite rod, the simulated rod is assumed to have a finite length. The space-time domain is defined by $-x^* < x \leq x^*$ (long enough to avoid any reflections from boundaries) with a grid spacing of $h=1$ mm, and $0 < t < t^*$ (set just before the breaking point of the wave profile) with a constant time stepping of $\Delta t = 1\mu s$. The material considered is aluminum which has property values of $\rho=2700kg/m^3$, $\nu=0.33$, and $E=70GPa$. We display the results of the WFS analysis using the logarithmic value of the spectrum $S(k, \omega) = \sum_{k_e} \ln |s|$, scaled in the unit interval spanning the normalized axes of Bk and w/c .

Thin-Material nonlinearity + GLS

Thin-Material nonlinearity + HS

Figure 1(color). Profile snap shots for three different times and their corresponding harmonics.

(a) Profiles captured at three different time steps. (b) Normalized distribution of the harmonics generated over time. (c) Wavenumber-frequency spectrum at t_2 represented by the logarithmic spectrum S^2 for the considered value of $k_e = 4.5$.

We prescribe the $\bar{u}(x, t) = Bk_e(1 + \cos(k_e(x - ct)))/2$ field along the entire computational domain at time $t=0$. Beyond the initial time, the wave is allowed to propagate freely in the simulation with no further prescription of displacement. Three different snapshots of the simulated motion are shown in Fig. 1(a). Performing Fourier analysis in space on the wave function at the time of excitation (t_0) and two timesteps later (t_1 and t_2) shows the evolution of the energy spectrum, K . At time t_0 , only a single harmonic exists, which is the cosine excitation signal. As the wave evolves, the nonlinear effects increasingly cause distortion and generate higher harmonics, as shown in Fig.1(b). The WFS analysis results corresponding to this excitation is also presented in Fig.1(c). The frequency wavenumber spectrum analysis is performed, and the linear dispersion relationship, nonlinear dispersion relationship, and dispersion relationship with harmonic excitation are superimposed on the spectrum. A careful observation of the red dotted line in Figure(c) shows that this curve passes accurately through the bright spots in the figure, and these bright spots represent the generation of harmonics. Therefore, the deduced nonlinear dispersion relationship can predict

the harmonics accurately. The occurrence of the wave further proves the correctness of the deduced nonlinear dispersion relation containing the harmonics Equations (10) and (11).

4 Conclusion

In this letter, we describe how we considered material and geometric nonlinearities in a long, thin rod simultaneously. Based on the given material nonlinearity, we deduced the nonlinear longitudinal wave equations under two different geometric nonlinearities, GLS and HS, respectively, and used analytical mathematical methods to derive the nonlinear dispersion relationship. Finally, the obtained dispersion relationship was verified using a numerical simulation.

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Data availability

The data will be made available upon reasonable request for academic use and within the limitations of the provided informed consent by the corresponding author upon acceptance.

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