Influence of a Turbulent Jet Engine Exhaust on Laguerre-Gaussian Correlated Shell-Model Beams

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Influence of a turbulent jet engine exhaust on Laguerre-Gaussian correlated Shell-model beams

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Abstract

In this paper, we investigated the influence of a turbulence jet engine exhaust on Laguerre-Gaussian correlated shell-model beams (LGSMs). The analytical formulae of the cross-spectral density function as well as the beam width are derived based on the Huygens-Fresnel diffraction principle and the second-order moments of the Wigner distribution function, respectively. From our main results, the spectral density, the degree of coherence and beam width of a LGSM are analyzed numerically. It is found that for high source coherence width, the spectral density changes gradually its profiles from circular to elliptical shape at short propagation distance, then the beam transforms into a well like Gaussian at long propagation distance. Although, at very short propagation distance, the beam becomes an elliptical dark hollow if the source coherence is very lower. Also, the numerical results show that the LGSM spreads more rapidly than the GSM in the same conditions.

Keywords: *Laguerre-Gaussian correlated Schell-model; Gaussian Schell-model; Jet engine exhaust; Huygens-Fresnel diffraction; Second-order moments; Spectral density; Degree of coherence.*

1. Introduction

In recent years, a great deal of research has been focused to study the properties of laser beams passing through turbulent atmosphere due to their interesting applications such as imaging system, LIDAR and optical communications [1-6]. In addition, after the invention of the laser, the increased attention has been paid to generate and develop a partially coherent beam propagating through a turbulence atmosphere and have been extensively studied because they have various important characteristics and are useful in many applications, such as free space optical communication [7], optical scattering [8], and remote detection [9]. Moreover, in the study triggered by Jian [10], it was demonstrated that the partially coherent

beams are less affected by atmospheric turbulence compared to fully coherence beams. Dogariu and Amande [11] have shown experimentally, that the propagation of partially coherent beams are key mitigation tools of optical turbulence. Recently, Mei and Korotkova [12] introduced two kinds of partially coherent beams with nonconventional correlation function such as LGSMBs and Bessel-Gaussian correlated Schell-model beams (BGSMBs). In the source plane, the LGSMB profile can be a Gaussian one, while in far-field the LGSMB exhibits dark hollow. Wang et al. [13] proposed an experimental setup for generating partially coherent beams, and reported the generation of LGSMBs. Up to now, various kinds of partially coherent beams propagating through atmospheric turbulence have been examined, such as: cosine-Gaussian correlated Schell-model beams [14], LGSMBs and BGSMBs [15,16], Generalized Flattened Hermite Cosh-Gaussian beams [17] and so on.

Since 1970, experimental results found by Consortini et al. [18] have shown that the optical turbulence can exhibit anisotropic characteristics in some regions of the Earth's atmosphere. Recently, several campaigns have experimentally measured the presence of anisotropy turbulence near the ground [19, 20]. Many works on the propagation properties of partially coherent beams in anisotropic non-Kolmogorov turbulent atmospheric have been extensively investigated [21-24].

On the other hand, airborne laser systems have attracted great attention due to their broad application notably in the protection and security of military and civilian airlines. The main issue related to performance of these systems arise from turbulence in close vicinity of jet engine plume. The efflux, from the jet engine, contains the hot exhaust gases mixing with surrounding ambient air occur severe turbulence effect on propagation of laser beams. Thereafter, the jet engine plume produces a non-homogeneous and anisotropy zone. Consequently, this strong zone can be described by the power spectrum anisotropy which has been proposed by Sirazetdinov [25]. In the last few years, the effect of jet engine plumes on propagation of laser beam has experimentally been reported by Hogge and Visinsky [26], and reviewed by Sjöqvist [27]. Recently, Ding et al. [28] have studied theoretically the propagation of partially coherent GSMB through a jet engine exhaust. To our knowledge, the influence of a turbulence jet engine exhaust on the evolution behavior of partially coherent LGSMBs has not been reported.

In this work, the propagation of partially coherent LGSMB through a jet engine exhaust plum region is investigated. In Section 2, we derive the analytical expressions of the cross-spectral density, spectral density and degree of coherence of a LGSMB propagating in jet

engine exhaust. In Section 3, we deduce the beam width of a LGSMB propagating through a jet engine exhaust. Some numerical simulations are presented and discussed in Section 4. Finally, we give the conclusion of this paper.

2. Analytical model of the propagation of a LGSMB in jet engine exhaust

We assume a scalar LGSMB source propagates along the z direction in a jet engine exhaust (see Fig. 1). The second order statistical properties of LGSMB situated at plane $z=0$ are characterized by the cross-spectral density function which can be written as [12]

$$W^{(0)}(\mathbf{r}_1, \mathbf{r}_2, 0) = \exp\left(-\frac{\mathbf{r}_1^2 + \mathbf{r}_2^2}{4\sigma_0^2} - \frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_g^2}\right) L_n\left(\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_g^2}\right), \quad (1)$$

where $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ are two arbitrary points on the transverse plane ($z=0$), σ_0 and σ_g denote the initial beam width and source coherent width, respectively, and L_n is the Laguerre polynomial of mode order n . If $n=0$, Eq. (1) reduces to the cross spectral density function of a GSMB beam in the source plane.

Figure 1: Illustration of the geometry of beam propagation through a jet engine exhaust.

The spatial anisotropic power spectrum within an exhaust region from a jet engine can be expressed as [28]

$$\Phi_n(\boldsymbol{\kappa}) = 0,033 C_n^2 \cdot \left\{ \left(\frac{L_{0x} L_{0y}}{1 + (\kappa_x L_{0x})^2 + (\kappa_y L_{0y})^2} \right)^{11/6} \exp\left(-\frac{\kappa_x^2}{\kappa_{mx}^2} - \frac{\kappa_y^2}{\kappa_{my}^2}\right) + B \left[\left(\frac{2\pi}{L_s} \right)^2 + \kappa^2 \right]^{\frac{11}{6}} \exp\left(-\frac{\kappa^2}{\kappa_m^2}\right) \right\}, \quad (2)$$

where L_{0i} ($i = x, y$) denotes the outer scales in the x and y directions, $\kappa_m = c(\alpha)/l_0$ with $l_0 = \sqrt{l_{0x}^2 + l_{0y}^2}$ being the inner scale, $\kappa_{mi} = c(\alpha)/l_{0i}$ with l_{0i} ($i = x, y$) is the inner scale in the x

or y directions, B is a constant and α is the generalized power slope. The parameter $c(\alpha)$ is defined as

$$c(\alpha) = \left(\frac{2\pi \Gamma(5-\alpha/2) A(\alpha)}{3} \right)^{1/(\alpha-5)}, \quad (3)$$

where $\Gamma(\cdot)$ is the Gamma function, and $A(\alpha) = \Gamma(\alpha-1) \cos(\alpha\pi/2) / 4\pi^2$ with $3 < \alpha < 4$. For our calculation, the parameter power slope takes $\alpha = 11/6$. On the other hand, Eq. (2) can be rewritten as

$$\Phi_n(\boldsymbol{\kappa}) = \Phi_{n_1}(\boldsymbol{\kappa}) + \Phi_{n_2}(\boldsymbol{\kappa}), \quad (4)$$

where

$$\Phi_{n_1}(\boldsymbol{\kappa}) = 0,033C_n^2 \cdot \left(\frac{L_{0x}L_{0y}}{1 + (\kappa_x L_{0x})^2 + (\kappa_y L_{0y})^2} \right)^{11/6} \exp\left(-\frac{\kappa_x^2}{\kappa_{mx}^2} - \frac{\kappa_y^2}{\kappa_{my}^2} \right), \quad (5)$$

and

$$\Phi_{n_2}(\boldsymbol{\kappa}) = 0,033C_n^2 \cdot B \left(\left(\frac{2\pi}{L_s} \right)^2 + \kappa^2 \right)^{-11/6} \exp\left(-\frac{\kappa^2}{\kappa_m^2} \right). \quad (6)$$

Based on the extended Huygens-Fresnel principle and paraxial approximation, the cross-spectral density function of a partially coherent beam at z plane through an atmospheric turbulence can be expressed as [1]

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{1}{(\lambda z)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W^{(0)}(\mathbf{r}_1, \mathbf{r}_2, 0) \exp\left(-\frac{ik}{2z}(\boldsymbol{\rho}_1 - \mathbf{r}_1)^2 + \frac{ik}{2z}(\boldsymbol{\rho}_2 - \mathbf{r}_2)^2 \right) \\ \times \left\langle \exp\left[\psi^*(\mathbf{r}_1, \boldsymbol{\rho}_1, z) + \psi(\mathbf{r}_2, \boldsymbol{\rho}_2, z) \right] \right\rangle d^2\mathbf{r}_1 d^2\mathbf{r}_2, \quad (7)$$

where $\boldsymbol{\rho}_i = (\xi_i, \eta_i)$ ($i = 1, 2$) is the position vector of two points at the plane z, k is the wave number related to the wavelength λ by $k = 2\pi/\lambda$, $\psi(\mathbf{r}, \boldsymbol{\rho}, z)$ denotes the complex phase perturbation of a spherical wave through the random medium from the source plane to the output plane, the asterisk $*$ specifies the complex conjugate, and the angular brackets $\langle \cdot \rangle$ represents the ensemble average over the medium fluctuations. The last term in Eq. (7) of the general anisotropic power spectra of turbulence is given by [29]

$$\left\langle \exp\left[\psi^*(\mathbf{r}_1, \boldsymbol{\rho}_1, z) + \psi(\mathbf{r}_2, \boldsymbol{\rho}_2, z) \right] \right\rangle \\ = \exp\left\{ -2\pi k^2 z \int_0^1 dt \int d^2\boldsymbol{\kappa} \Phi_n(\boldsymbol{\kappa}) \left[1 - \exp\left\{ -i(t\boldsymbol{\rho}_d \cdot \boldsymbol{\kappa} + (1-t)\mathbf{r}_d \cdot \boldsymbol{\kappa}) \right\} \right] \right\}, \quad (8)$$

In this last equation we have $\boldsymbol{\rho}_d = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$, $\mathbf{r}_d = \mathbf{r}_1 - \mathbf{r}_2$, $\boldsymbol{\kappa} = (\kappa_x, \kappa_y)$ is the two-dimensional spatial frequency and $\Phi_n(\boldsymbol{\kappa})$ is the spatial power spectrum of the refractive index fluctuations of the turbulent indicated by Eq. (2).

According to Eqs. (5) and (6), Eq. (8) can be rewritten as a sum of two terms as [28]

$$\langle \exp[\psi^*(\mathbf{r}_1, \boldsymbol{\rho}_1, z) + \psi(\mathbf{r}_2, \boldsymbol{\rho}_2, z)] \rangle = \psi_A + \psi_B, \quad (9)$$

where

$$\psi_A = \exp \left\{ -2\pi k^2 z \int_0^1 dt \int d^2 \boldsymbol{\kappa} \Phi_{n_1}(\boldsymbol{\kappa}) \left[1 - \exp \left\{ -i(t\boldsymbol{\rho}_d \cdot \boldsymbol{\kappa} + (1-t)\mathbf{r}_d \cdot \boldsymbol{\kappa}) \right\} \right] \right\}, \quad (10)$$

and

$$\psi_B = \exp \left\{ -2\pi k^2 z \int_0^1 dt \int d^2 \boldsymbol{\kappa} \Phi_{n_2}(\boldsymbol{\kappa}) \left[1 - \exp \left\{ -i(t\boldsymbol{\rho}_d \cdot \boldsymbol{\kappa} + (1-t)\mathbf{r}_d \cdot \boldsymbol{\kappa}) \right\} \right] \right\}. \quad (11)$$

To evaluate the term ψ_A , we introduce coordinate changes $\kappa_i = \kappa'_i / \mu_i$ ($i = x, y$) where μ_i represents the anisotropic factors in two mutually directions x and y. The ratio of the outer and the inner scales in the x and the y direction can be constant and expressed as $L_{0x} = \mu_x L_0$, $L_{0y} = \mu_y L_0$, $l_{0x} = \mu_x l_0$ and $l_{0y} = \mu_y l_0$ with considering the turbulent eddies transport the energy from the outer scale to the inner scale in the inertial sub-range.

The term ψ_A can be written as

$$\psi_A = \exp \left\{ -\frac{2\pi k^2 z}{\mu_x \mu_y} \int_0^1 dt \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\kappa'_x d\kappa'_y \Phi_{n_1}'(\boldsymbol{\kappa}') \left[1 - \exp \left\{ -i(t\boldsymbol{\rho}'_d \cdot \boldsymbol{\kappa}' + (1-t)\mathbf{r}'_d \cdot \boldsymbol{\kappa}') \right\} \right] \right\}, \quad (12)$$

where $\boldsymbol{\rho}'_d = (\xi_d / \mu_x, \eta_d / \mu_y)$ and $\mathbf{r}'_d = (x_d / \mu_x, y_d / \mu_y)$.

From Eq. (5), the power spectrum Φ_{n_1} turns out to be

$$\Phi_{n_1}'(\boldsymbol{\kappa}') = 0.033 C_n^2 \cdot \left(\frac{\mu_{0x} \mu_{0y} L_0^2}{1 + (\kappa' L_0)^2} \right)^{11/6} \exp \left(-\frac{l_0^2}{c(\alpha)^2} \kappa'^2 \right). \quad (13)$$

Using $d\kappa'_x d\kappa'_y = \kappa' d\kappa' d\theta$ and $d\kappa_x d\kappa_y = \kappa d\kappa d\theta$, the polar coordinates with recalling the integral formulas [30]

$$\int_0^{2\pi} \exp(-i x \cos(\theta)) d\theta = 2\pi J_0(x), \quad (14)$$

after integrating over θ , then Eqs. (11) and (12) become

$$\psi_A = \exp \left\{ -\frac{4\pi^2 k^2 z}{\mu_x \mu_y} \int_0^1 dt \int_0^{+\infty} \kappa' d\kappa' \Phi_{n_1}'(\kappa') \left[1 - J_0(\kappa' |t\boldsymbol{\rho}'_d + (1-t)\mathbf{r}'_d|) \right] \right\}, \quad (15)$$

and

$$\psi_B = \exp \left\{ -4\pi^2 k^2 z \int_0^1 dt \int_0^{+\infty} \kappa d\kappa \Phi_{n_2}(\kappa) \left[1 - J_0(\kappa |t\boldsymbol{\rho}_d + (1-t)\mathbf{r}_d|) \right] \right\}. \quad (16)$$

Now by using the Taylor expansion the Bessel function of the first kind and zero order $J_0(x) \approx 1 - x^2/4$, and after some algebraic calculations, Eqs. (15) and (16) are reduced to

$$\begin{aligned} \psi_A &= \exp \left\{ -\frac{\pi^2 k^2 z}{\mu_x \mu_y} \left(\frac{\xi_d^2 + \xi_d x_d + x_d^2}{3\mu_x^2} + \frac{\eta_d^2 + \eta_d y_d + y_d^2}{3\mu_y^2} \right) \int_0^{+\infty} \kappa^3 \Phi_{n_1}'(\kappa') d\kappa' \right\} \\ &= \exp \left\{ -\frac{\pi^2 k^2 z T_A (\xi_d^2 + \xi_d x_d + x_d^2)}{3\mu_x^2} \right\} \exp \left\{ -\frac{\pi^2 k^2 z T_A (\eta_d^2 + \eta_d y_d + y_d^2)}{3\mu_y^2} \right\}, \end{aligned} \quad (17)$$

and

$$\begin{aligned} \psi_B &= \exp \left\{ -\pi^2 k^2 z \left(\frac{\xi_d^2 + \xi_d x_d + x_d^2}{3} + \frac{\eta_d^2 + \eta_d y_d + y_d^2}{3} \right) \int_0^{+\infty} \kappa^3 \Phi_{n_2}(\kappa) d\kappa \right\} \\ &= \exp \left\{ -\frac{1}{3} \pi^2 k^2 z T_B (\xi_d^2 + \xi_d x_d + x_d^2) \right\} \exp \left\{ -\frac{1}{3} \pi^2 k^2 z T_B (\eta_d^2 + \eta_d y_d + y_d^2) \right\}, \end{aligned} \quad (18)$$

where T_A and T_B denote the strength of turbulence expressed as [28]

$$T_A = \frac{1}{\mu_x \mu_y} \int_0^{+\infty} \kappa^3 \Phi_{n_1}'(\kappa') d\kappa' = 0,033 C_n^2 (\mu_x \mu_y)^{\frac{5}{6}} \left[\frac{\left(5 + \frac{6}{\kappa_m^2 L_0^2} \right) \exp\left(\frac{1}{\kappa_m^2 L_0^2}\right) \Gamma\left(\frac{1}{6}, \frac{1}{\kappa_m^2 L_0^2}\right)}{10 \left(\frac{1}{\kappa_m}\right)^{\frac{1}{3}}} - \frac{3}{5} \left(\frac{1}{L_0}\right)^{\frac{1}{3}} \right], \quad (19)$$

and

$$T_B = \int_0^{+\infty} \kappa^3 \Phi_{n_2}(\kappa) d\kappa = 0.033 C_n^2 B \left[\frac{\left(5 + \frac{24\pi^2}{k_m^2 L_s^2} \right) \exp\left(\frac{4\pi^2}{k_m^2 L_s^2}\right) \Gamma\left(\frac{1}{6}, \frac{4\pi^2}{k_m^2 L_s^2}\right)}{10 \left(\frac{1}{k_m}\right)^{\frac{1}{3}}} - \frac{3}{5} \left(\frac{2\pi}{L_s}\right)^{\frac{1}{3}} \right]. \quad (20)$$

For the convenience of integration, we introduce the “sum” and “difference” coordinates

$$\boldsymbol{\rho}_s = (\xi_s, \eta_s) = \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2} = \left(\frac{\xi_1 + \xi_2}{2}, \frac{\eta_1 + \eta_2}{2} \right), \quad (21)$$

$$\boldsymbol{\rho}_d = (\xi_d, \eta_d) = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 = (\xi_1 - \xi_2, \eta_1 - \eta_2), \quad (22)$$

$$\mathbf{r}_s = (x_s, y_s) = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right), \quad (23)$$

and

$$\mathbf{r}_d = (x_d, y_d) = \mathbf{r}_1 - \mathbf{r}_2 = (x_1 - x_2, y_1 - y_2). \quad (24)$$

Then, Eq. (7) can also be written in the form

$$\begin{aligned} W(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) &= \frac{1}{(\lambda z)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W^{(0)}(\mathbf{r}_s, \mathbf{r}_d, 0) \exp\left(-\frac{ik}{2z}[2x_s x_d + 2\xi_s \xi_d - 2x_s \xi_d - 2x_d \xi_s]\right) \\ &\times \exp\left(-\frac{ik}{2z}[2y_s y_d + 2\eta_s \eta_d - 2y_s \eta_d - 2y_d \eta_s]\right) \left\langle \exp[\psi^*(\mathbf{r}_d, \boldsymbol{\rho}_d, z) + \psi(\mathbf{r}_d, \boldsymbol{\rho}_d, z)] \right\rangle d^2 \mathbf{r}_s d^2 \mathbf{r}_d, \end{aligned} \quad (25)$$

where

$$W^{(0)}(\mathbf{r}_s, \mathbf{r}_d, 0) = \exp\left[-\frac{\mathbf{r}_s^2}{2\sigma_0^2} - \left(\frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2}\right) \mathbf{r}_d^2\right] L_n\left(\frac{\mathbf{r}_d^2}{2\sigma_g^2}\right). \quad (26)$$

The cross spectral density of the LGSMB propagation through a jet engine exhaust can be expressed as

$$W(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) = W_A(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) + W_B(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z), \quad (27)$$

where $W_A(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z)$ and $W_B(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z)$ are the cross spectral density function of the LGSMB, they are found from terms ψ_A and ψ_B , respectively.

Now, we will evaluate $W_A(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z)$ and $W_B(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z)$. On substituting Eq. (17) into Eq. (25) and after carrying on tedious calculations, we obtain the cross spectral density function of the LGSMB from term ψ_A in the plane $z > 0$ as

$$\begin{aligned} W_A(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) &= \frac{k^2 \sigma_0^2}{2z^2} \exp\left(-\frac{\pi^2 k^2 z T_A}{3} \left(\frac{\xi_d^2}{\mu_x^2} + \frac{\eta_d^2}{\mu_y^2}\right)\right) \exp\left(-\frac{ik}{z}(\xi_s \xi_d + \eta_s \eta_d)\right) \\ &\times \exp\left(\frac{1}{4M_x} \left(\frac{ik}{z} \xi_s + \left(\frac{k^2 \sigma_0^2}{z^2} - \frac{\pi^2 k^2 z T_A}{3\mu_x^2}\right) \xi_d\right)^2\right) \exp\left(\frac{1}{4M_y} \left(\frac{ik}{z} \eta_s + \left(\frac{k^2 \sigma_0^2}{z^2} - \frac{\pi^2 k^2 z T_A}{3\mu_y^2}\right) \eta_d\right)^2\right) \\ &\times \exp\left(-\frac{k^2 \sigma_0^2}{2z^2}(\xi_d^2 + \eta_d^2)\right) \sum_{p=0}^n \sum_{m=0}^p \binom{n}{p} \binom{p}{m} \frac{1}{p! 2^{3p} \sigma_g^{2p} M_x^{p-m+1/2} M_y^{m+1/2}} \\ &\times H_{2p-2m}\left(\frac{-k}{2z\sqrt{M_x}} \left(\xi_s - i\left(\frac{\sigma_0^2 k}{z} - \frac{\pi^2 k z^2 T_A}{3\mu_x^2}\right) \xi_d\right)\right) H_{2m}\left(\frac{-k}{2z\sqrt{M_x}} \left(\eta_s - i\left(\frac{\sigma_0^2 k}{z} - \frac{\pi^2 k z^2 T_A}{3\mu_y^2}\right) \eta_d\right)\right), \end{aligned} \quad (28)$$

where $H_n(\cdot)$ denotes the Hermite polynomial of mode order n , and M_x and M_y are given by

$$M_x = \frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2} + \frac{\sigma_0^2 k^2}{2z^2} + \frac{\pi^2 k^2 z T_A}{3\mu_x^2}, \quad (29)$$

and

$$M_y = \frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2} + \frac{\sigma_0^2 k^2}{2z^2} + \frac{\pi^2 k^2 z T_A}{3\mu_y^2}. \quad (30)$$

On substituting from Eq. (18) into Eq. (25), and after carrying on tedious operations, we obtain the cross spectral density function of a LGSMB from term ψ_B in the plane $z>0$ as

$$\begin{aligned} W_B(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) = & \frac{k^2 \sigma_0^2}{2z^2} \exp\left(-\frac{\pi^2 k^2 z T_A}{3}(\xi_d^2 + \eta_d^2)\right) \exp\left(-\frac{ik}{z}(\xi_s \xi_d + \eta_s \eta_d)\right) \\ & \times \exp\left(\frac{1}{4M}\left(\frac{ik}{z}\xi_s + \left(\frac{k^2 \sigma_0^2}{z^2} - \frac{\pi^2 k^2 z T_B}{3}\right)\xi_d\right)^2\right) \exp\left(\frac{1}{4M}\left(\frac{ik}{z}\eta_s + \left(\frac{k^2 \sigma_0^2}{z^2} - \frac{\pi^2 k^2 z T_B}{3}\right)\eta_d\right)^2\right) \\ & \times \exp\left(-\frac{k^2 \sigma_0^2}{2z^2}(\xi_d^2 + \eta_d^2)\right) \sum_{p=0}^n \sum_{m=0}^p \binom{n}{p} \binom{p}{m} \frac{1}{p! 2^{3p} \sigma_g^{2p} M^{p+1}} \\ & \times H_{2p-2m}\left(\frac{-k}{2z\sqrt{M_x}}\left(\xi_s - i\left(\frac{\sigma_0^2 k}{z} - \frac{\pi^2 k z^2 T_B}{3}\right)\xi_d\right)\right) H_{2m}\left(\frac{-k}{2z\sqrt{M_x}}\left(\eta_s - i\left(\frac{\sigma_0^2 k}{z} - \frac{\pi^2 k z^2 T_B}{3}\right)\eta_d\right)\right), \end{aligned} \quad (31)$$

where M is given by expression

$$M = \frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2} + \frac{\sigma_0^2 k^2}{2z^2} + \frac{\pi^2 k^2 z T_B}{3}. \quad (32)$$

In the above calculations, we have applied the following expansion and integral formulas [30-32]

$$L_n(x) = \sum_{p=0}^n \binom{n}{p} \frac{(-1)^p}{p!} x^p, \quad (33)$$

$$(x+y)^p = \sum_{m=0}^p \binom{p}{m} x^{p-m} y^m, \quad (34)$$

$$\int_{-\infty}^{+\infty} x^n e^{-(x-\beta)^2} dx = (2i)^{-n} \sqrt{\pi} H_n(i\beta), \quad (35)$$

and

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}ax^2 + bx\right) dx = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right). \quad (36)$$

On placing $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = (\xi, \eta)$ into Eq. (27), the spectral density at the z plane shown by

$$\begin{aligned}
S(\boldsymbol{\rho}, z) &= W(\boldsymbol{\rho}, \boldsymbol{\rho}, z) = W_A(\boldsymbol{\rho}, \boldsymbol{\rho}, z) + W_B(\boldsymbol{\rho}, \boldsymbol{\rho}, z) \\
&= \frac{k^2 \sigma_0^2}{2z^2} \exp\left(-\frac{k^2}{4z^2 M_x} \xi_s^2 - \frac{k^2}{4z^2 M_y} \eta_s^2\right) \\
&\times \sum_{p=0}^n \sum_{m=0}^p \binom{n}{p} \binom{p}{m} \frac{1}{p! 2^{3p} \sigma_g^{2p} M_x^{p-m+1/2} M_y^{m+1/2}} H_{2p-2m}\left(-\frac{k}{2z\sqrt{M_x}} \xi_s\right) H_{2m}\left(-\frac{k}{2z\sqrt{M_y}} \eta_s\right) \\
&+ \frac{k^2 \sigma_0^2}{2z^2} \exp\left(-\frac{k^2}{4z^2 M} (\xi_s^2 + \eta_s^2)\right) \\
&\times \sum_{p=0}^n \sum_{m=0}^p \binom{n}{p} \binom{p}{m} \frac{1}{p! 2^{3p} \sigma_g^{2p} M^{p+1}} H_{2p-2m}\left(-\frac{k}{2z\sqrt{M}} \xi_s\right) H_{2m}\left(-\frac{k}{2z\sqrt{M}} \eta_s\right).
\end{aligned} \tag{37}$$

The degree of coherence of the beam between two arbitrary points $\boldsymbol{\rho}_1 = (\xi_1, \eta_1)$ and $\boldsymbol{\rho}_2 = (\xi_2, \eta_2)$ is defined as

$$\begin{aligned}
\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) &= \frac{W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)}{\sqrt{S(\boldsymbol{\rho}_1, z) S(\boldsymbol{\rho}_2, z)}} \\
&= \frac{W_A(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) + W_B(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)}{\sqrt{W_A(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, z) + W_B(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, z)} \sqrt{W_A(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2, z) + W_B(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2, z)}}.
\end{aligned} \tag{38}$$

Eqs. (37) and (38) are the main mathematical analytical formulae to study the evolution properties of the spectral density and the spectral degree of coherence of a LGSMB propagating through jet engine exhaust. If we set $n = 0$ in Eqs. (28) and (31), those equations can be simplified to Eqs. (18) and (25) of the cross spectral density of GSMB through jet engine exhaust, which is consistent with Ref. [28].

3. Beam width of a LGSMB propagating through jet engine exhaust

The spectral density of the LGSMB is a sum of two cross spectral density W_A and W_B . We have noticed that W_A depends on the anisotropic term than W_B . To understand and examine the spreading properties of this beam, we derived the root-mean-square (r.m.s) beam width by using the cross spectral density W_A .

At the propagation distance z , the Wigner distribution function is defined by the Fourier transformation of the cross spectral function as [33]

$$h(\boldsymbol{\rho}_s, \boldsymbol{\theta}, z) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) \exp(-ik\boldsymbol{\theta} \cdot \boldsymbol{\rho}_d) d^2 \boldsymbol{\rho}_d, \tag{39}$$

where $\theta \equiv (\theta_x, \theta_y)$ denotes the angle which the vector of interest makes with the propagation axis z , and $(k\theta_x, k\theta_y)$ are the components of the wave vector along the x and y axis.

Eq. (2) can be written as

$$W(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) = \left(\frac{k}{2\pi z} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(\mathbf{r}_s, \mathbf{r}_d, 0) \exp \left\{ \frac{ik}{z} [(\boldsymbol{\rho}_s - \mathbf{r}_s)(\boldsymbol{\rho}_d - \mathbf{r}_d)] - H(\boldsymbol{\rho}_d, \mathbf{r}_d, z) \right\} d^2 \mathbf{r}_d d^2 \mathbf{r}_s, \quad (40)$$

where $H(\boldsymbol{\rho}_d, \mathbf{r}_d, z)$ denotes the effect of the turbulent.

Based on the inverse Fourier transform of Dirac delta function, we write the cross spectral density function in the source plane as

$$W(\mathbf{r}_s, \mathbf{r}_d, 0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(\mathbf{r}'_s, \mathbf{r}_d, 0) \exp(i\mathbf{k}_d \cdot (\mathbf{r}'_s - \mathbf{r}_d)) d^2 \mathbf{r}'_s d^2 \mathbf{k}_d. \quad (41)$$

By substituting Eq. (41) into Eq. (40), one obtains

$$W(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W\left(\mathbf{r}'_s, \boldsymbol{\rho}_d + \frac{z}{k} \mathbf{k}_d, 0\right) \exp(-i\boldsymbol{\rho}_s \cdot \mathbf{k}_d) \exp(i\mathbf{k}_d \cdot \mathbf{r}'_s) \\ \times \exp \left\{ -H\left(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k} \mathbf{k}_d, z\right) \right\} d^2 \mathbf{r}'_s d^2 \mathbf{k}_d, \quad (42)$$

where

$$W\left(\mathbf{r}'_s, \boldsymbol{\rho}_d + \frac{z}{k} \mathbf{k}_d, 0\right) = \exp \left[-\frac{\mathbf{r}'_s{}^2}{2\sigma_0^2} - \left(\frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2} \right) \left(\boldsymbol{\rho}_d + \frac{z}{k} \mathbf{k}_d \right)^2 \right] L_n \left[\frac{1}{2\sigma_g^2} \left(\boldsymbol{\rho}_d + \frac{z}{k} \mathbf{k}_d \right)^2 \right]. \quad (43)$$

The effect turbulence can be determined from the term ψ_A as

$$H\left(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k} \mathbf{k}_d, z, z\right) = \frac{\pi^2 k^2 z T_A}{3} \left(3\boldsymbol{\rho}_d'^2 + 3\frac{z}{k} \mathbf{k}_d' \cdot \boldsymbol{\rho}_d' + \left(\frac{z}{k} \right)^2 \mathbf{k}_d'^2 \right) \\ = \frac{\pi^2 k^2 z T_A}{3} \left(3 \left(\frac{\xi_d^2}{\mu_x^2} + \frac{\eta_d^2}{\mu_y^2} \right) + 3\frac{z}{k} \left(\frac{\xi_d k_{dx}}{\mu_x^2} + \frac{\eta_d k_{dy}}{\mu_y^2} \right) + \left(\frac{z}{k} \right)^2 \left(\frac{k_{dx}^2}{\mu_x^2} + \frac{k_{dy}^2}{\mu_y^2} \right) \right). \quad (44)$$

By using Eqs. (42) and (44), and after integrating, we obtain the Wigner distribution function given by Eq. (39)

$$h(\boldsymbol{\rho}_s, \theta, z) = \frac{2\pi\sigma_0^2 k^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_{dx} dk_{dy} d\xi_d d\eta_d \exp \left[\exp(-a_x k_{dx}^2 - b_x \xi_d^2 - c_x k_{dx} \xi_d - i\xi_s k_{dx} - ik\theta_x \xi_d) \right] \\ \times \exp \left[(-a_y k_{dy}^2 - b_y \eta_d^2 - c_y k_{dy} \eta_d - i\eta_s k_{dy} - ik\theta_y \eta_d) \right] \\ \times L_n \left[\frac{1}{2\sigma_g^2} \left(\left(\xi_d + \frac{z}{k} k_{dx} \right)^2 + \left(\eta_d + \frac{z}{k} k_{dy} \right)^2 \right) \right], \quad (45)$$

where

$$a_x = \frac{\sigma_0^2}{2} + \left(\frac{z}{k}\right)^2 \left(\frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2}\right) + \left(\frac{z}{k}\right)^2 \frac{\pi^2 k^2 z T_A}{3\mu_x^2}, \quad (46)$$

$$b_x = \frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2} + \frac{\pi^2 k^2 z T_A}{\mu_x^2}, \quad (47)$$

$$c_x = \frac{z}{k} \left(\frac{1}{4\sigma_0^2} + \frac{1}{\sigma_g^2}\right) + \frac{z}{k} \frac{\pi^2 k^2 z T_A}{\mu_x^2}, \quad (48)$$

$$a_y = \frac{\sigma_0^2}{2} + \left(\frac{z}{k}\right)^2 \left(\frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2}\right) + \left(\frac{z}{k}\right)^2 \frac{\pi^2 k^2 z T_A}{3\mu_y^2}, \quad (49)$$

$$b_y = \frac{1}{8\sigma_0^2} + \frac{1}{2\sigma_g^2} + \frac{\pi^2 k^2 z T_A}{\mu_y^2}, \quad (50)$$

and

$$c_y = \frac{z}{k} \left(\frac{1}{4\sigma_0^2} + \frac{1}{\sigma_g^2}\right) + \frac{z}{k} \frac{\pi^2 k^2 z T_A}{\mu_y^2}. \quad (51)$$

The moments of order $n_1 + n_2 + m_1 + m_2 = 0$ of the Wigner distribution function for the beam is defined by

$$\langle \xi_s^{n_1} \eta_s^{n_2} \theta_x^{m_1} \theta_y^{m_2} \rangle = \frac{1}{P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \xi_s^{n_1} \eta_s^{n_2} \theta_x^{m_1} \theta_y^{m_2} h(\rho_s, \theta, z) d\xi_s d\eta_s d\theta_x d\theta_y, \quad (52)$$

where P is the total energy carried by the beam and given by

$$P = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\rho_s, \theta, z) d\xi d\eta d\theta_x d\theta_y = 2\pi\sigma_0^2. \quad (53)$$

Substituting Eqs. (45) and (53) into Eq. (52) and after tedious calculations, one obtains the following expressions for the r.m.s beam width and the angular beam width of a LGSMB after propagation

$$\sigma_i(z) = \left[\sigma_0^2 + \frac{z^2}{k^2} \left(\frac{1}{4\sigma_0^2} + \frac{(1+n)}{\sigma_g^2} \right) + \left(\frac{z}{k} \right)^2 \frac{2\pi^2 k^2 z T_A}{3\mu_i^2} \right]^{\frac{1}{2}}, \quad i = (x, y). \quad (54)$$

$$\langle \theta_i^2 \rangle = \frac{1}{k^2} \left(\frac{1}{4\sigma_0^2} + \frac{(1+n)}{\sigma_g^2} \right) + \frac{2\pi^2 z T_A}{\mu_i^2}, \quad i = (x, y). \quad (55)$$

$$\langle \xi_s \theta_x \rangle = \frac{z}{k^2} \left(\frac{1}{4\sigma_0^2} + \frac{1+n}{\sigma_g^2} \right) + \frac{\pi^2 z^2 T_A}{\mu_x^2}, \quad (56)$$

and

$$\langle \eta_s \theta_y \rangle = \frac{z}{k^2} \left(\frac{1}{4\sigma_0^2} + \frac{1+n}{\sigma_g^2} + \right) + \frac{\pi^2 z^2 T_A}{\mu_y^2}. \quad (57)$$

In the above calculations, we have been using the following expansion and integral formulae [30- 32]

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}ax^2 + bx\right) dx = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right), \quad (58)$$

$$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0), \quad (59)$$

$$\int_{-\infty}^{+\infty} f(x) \delta^{(n)}(x) dx = (-1)^n f^{(n)}(0), \quad (60)$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} (-ix)^n \exp(-isx) dx = \delta^{(n)}(s), \quad (61)$$

and

$$\delta(r'_s - r_s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(\pm ik_d(r'_s - r_s)) d^2 k_d. \quad (62)$$

Eqs. (54-57) are the principal analytical results of this Section. Eq. (54) gives the propagating beam width of the LGSMB in jet engine exhaust. Note that, if $n = 0$, this equation reduces to the expression of the r.m.s beam width of GSMB through a jet engine exhaust (see Eq. (30) of Ref. [28]).

4. Numerical results

In this Section, we will study the evolution properties of the spectral density, degree of coherence and beam width of the LGSMB passing through a jet engine exhaust, by using the analytical expressions established in the above Sections. The parameters used in the present simulations are taken to be: $\sigma_0 = 5mm$, $C_n^2 = 1.6 \times 10^{-9} m^{-2/3}$, $L_0 = 0.5m$, $L_s = 1mm$, $\mu_x = 0.7$, $\mu_y = 1.4$, $\lambda = 632.8nm$, $B=6$, $l_0 \approx 0.67mm$ and $n = 2$. Our analysis focuses on the small propagation distances of the order of meter compared to natural anisotropic turbulence where the beam is analyzed on the propagation of the order of kilometer rangers and farther. To illustrate the effect of turbulent which increases with increasing distances, we plot the results at some propagation distances from the source until 20m. Fig. 2 shows the

density plots of the spectral density of the LGSMB with different values of source coherence width at several propagation distances.

Figure 2: Density plots of the spectral density of the LGSMB at several propagation distances z ,

With: (a)-(c) $\sigma_g = 0.3mm$, (d)-(f) $\sigma_g = 0.02mm$ and (g)-(i) $\sigma_g = 5\mu m$.

We can see from this figure in the case of the source coherence width $\sigma_g = 0.3mm$, the beam profile will gradually changes to elliptical shape at short ranges, then the central beak submerges and the beam has a well like at large distance the propagation. However, when the source coherence width is very smaller the beam profile takes a form elliptical dark hollow beam at only a short propagation distance $z=5m$ and conserves its elliptical dark hollow on further propagation. Fig. 3 shows the density plots of the modulus of the degree of coherence of the LGSMB at two points $\rho_1 = (\xi, \eta)$ and $\rho_2 = (0, 0)$ at several propagation distances z , with different values of the source coherence width.

Figure 3: Density plots for the modulus of the degree of coherence of the LGSMB at several propagation distances z , with: (a)-(c) $\sigma_g = 0.3\text{mm}$, (d)-(f) $\sigma_g = 0.01\text{mm}$ and (g)-(i) $\sigma_g = 5\mu\text{m}$.

This figure shows that when the source coherence width decreasing for example $\sigma_g = 5\mu\text{m}$, the side lobe disappearance and the beam profile takes a Gaussian profile at short propagation distances than to elliptical shape for larger distance $z=20\text{m}$ (see Fig. 3(i)).

We notice that the major axis of the degree of coherent profile is along y direction (vertical), whereas for the spectral density profile the major axis is along x direction (horizontal).

We present in Fig. 4 the cross line ($y=0$) of the normalized spectral density distribution of the LGSMB propagating in jet engine exhaust at various values of propagation distance for two values of source coherence width and by varying mode order n .

Figure 4: Cross-line ($y=0$) of the normalized spectral density of the LGSMB at several propagation distances z **(a-e)**: $z=1\text{m}$, **(b-f)**: $z=5\text{m}$, **(c-g)**: $z=10\text{m}$, **(d-h)**: $z=20\text{m}$ for three values of n , with: **(a)-(d)** $\sigma_g = 0.3\text{mm}$ and **(e)-(h)** $\sigma_g = 5\mu\text{m}$.

It is found that, in the case of the mode order n equal zero, the spectral density profile reduces to Gaussian shape at any values of the propagation distances. When the source coherence width $\sigma_g = 0.3mm$, at large propagation distance as the mode order n increases, the central spot of the spectral density profile submerges, which becomes a well like shape (see Fig. 4(d)). Whereas for very small source coherence width $\sigma_g = 0.5\mu m$, at short distance propagation, the beam has a hollow beam shape during its propagation (see Fig. 4(f)-(h)). It is can also observe the central dark spot of this profile is widens with the increasing of mode order n .

Similarly, Fig. 5 illustrates the evolution properties of the modulus of the degree of coherence of the LGSMB at two position distances z (2m and 10m) with two values of source coherence width and by varying mode order n .

Figure 5: Cross-line ($y=0$) of the Modulus of the degree of coherence of the LGSMB at two propagation distances z : **(a-c)**: $z=2m$, **(b-d)**: $z=10m$, for three values of n with: **(a)-(b)** $\sigma_g = 0.2mm$ and **(c)-(d)** $\sigma_g = 5\mu m$.

This figure shows, if $n=0$ the modulus of degree of coherence profile takes a Gaussian distribution upon propagation. Also, one deduces that with the increasing of the mode order n the profile of the degree of coherence has a number of side lobes around the main lobe and start to disappear at larger distance. It can be also observed that the width of the main lobes becomes small when the value of n is larger.

Now, let us turn to study the evolution properties of the r.m.s beam width of the LGSMB propagating through a jet engine exhaust.

In Fig. 6, we plot the r.m.s beam width of the LGSMB propagating through a jet engine exhaust along two directions x and y as a function of propagation distance z with different mode order n . From these plots, the LGSMB is more spreading then GSMB on the same conditions. When comparing Fig. 6(a) with Fig. 6(b), it is needed to mention that the LGSMB is more spreading along x direction then the y direction.

Figure 6: The r.m.s beam width of the LGSMB along the x direction (a) and the y direction (b)

versus the propagation distance z for three values of mode order n , with $\sigma_g = 0.5mm$.

Fig. 7 presents the beam width of the LGSMB as function of propagation distance z in jet engine exhaust region for different values of the source coherence width. As shown from this figure, the beam width of LGSMB depends on the source coherence width. For a given propagation distance, the smaller source coherence width corresponds a larger beam width spreading. This result is similar to the phenomenon discussed previously (see Fig. 7 in Ref [34]).

Figure 7: The r.m.s beam width of a LGSMB along the x and y directions versus the propagation distance for three values of coherence width σ_g .

We illustrate in Fig. 8 the influence of the structure constant on the beam width of the LGSMB along the x direction versus propagation distances in jet engine exhaust for two values of source coherence width ($\sigma_g = 5mm$ and $\sigma_g = 0.3mm$).

Figure 8: The r.m.s beam width of the LGSMB along the x direction versus the propagation distance for different values of structure constant C_n^2 with (a) $\sigma_g = 5mm$ and (b) $\sigma_g = 0.3mm$.

It can be inferred that from these plots, the beam width increases with the increasing of C_n^2 , which implies that the beam spreads more largely in the stronger turbulence. As seen from both curves in this figure, as source coherence width decreases, the beam width rises rapidly.

This result means that the LGSMB will have a stronger capability of resisting the destructive effects of turbulence with lower source coherence.

5. Conclusion

In this work, we have studied the propagation of the LGSMB through jet engine exhaust. Based on the generalized Huygens-Fresnel integral formula, and the second order moments of the Wigner distribution function, we have derived some analytical expressions of the spectral density, the degree of coherence and the r.m.s beam width of the LGSMB propagating through the turbulent medium. It is shown that from numerical calculations, the profile of the considered beam changes to elliptical shape then the beam evolves into a well like for large propagation distance. Also, we found that the beam evolves into an elliptical dark hollow beam for lower coherence width during the propagation. We have also investigated and studied numerically the effects of beam parameters such as mode order, structure constant and the source coherence width on the evolution of r.m.s beam width. It is found that for comparison, the LGSMB spreads more rapidly than the GSMB. Additionally, from the present study, the LGSMB is less affected by turbulence induced from the jet engine exhaust when coherence width is very low. We note that we found published results of the propagation of the GSMB through the medium and our results can be regarded as a generalized case of some previous works. Our main results investigated here can be beneficial in some potential applications such as optical imaging, ranging, tracking and communication systems which operate within the jet engine exhaust region.

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