Application To Lipschitzian and Integral Systems Via Quadruple Coincidence Point in Fuzzy Metric Spaces

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Application to Lipschitzian and integral systems via quadruple coincidence point in fuzzy metric spaces

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Abstract

Without a partially ordered set, in this manuscript, we investigate quadruple coincidence point (QCP) results for commuting mapping in the setting of fuzzy metric spaces (FMSs). Furthermore, some relevant findings are presented to generalize some of the previous results in this direction. Ultimately, non-trivial examples and applications to find a unique solution for Lipschitzian and integral quadruple systems are provided to support and strengthen our theoretical results.

Keywords: Quadruple coincidence point, commuting mapping, Lipschitzian mappings, an integral equation, fuzzy metric spaces

Mathematics Subject Classification (2010): 47H10, 54H25.

1 Introduction

There is no doubt that the study of fuzzy sets is extremely important for its multiple applications, such as control ill-defined, complex and non-linear systems. It is more common to find solutions to control problems that are difficult to solve with the classical control theory. Fuzzy set theory is becoming an increasingly important tool, especially in the rapidly evolving disciplines of artificial intelligence: expert systems and neural networks. It creates completely new opportunities for the application of fuzzy sets in chemical engineering [1, 2, 3, 4].

The concept of fuzzy sets initiated by Zadeh [5] in 1965. Many mathematicians considered these sets to introduce interesting concepts into the field of mathematics like fuzzy logic, fuzzy differential equations, and fuzzy metric spaces. It is known that the FMS is an important generalization of the ordinary metric space where the extended topological definitions and possible applications in

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several areas. Many mathematicians have considered this problem in many ways. For example, the authors in [6] modified the concept of a FMS that initiated by Kramosil and Michalek [7] and defined the Hausdorff topology of a FMS. For more details about this idea we advise the reader to view [8, 9, 10, 11, 12].

In 2011, the coupled fixed point (FP) [13] result extended to a tripled FP in partially ordered metric spaces by Berinde and Borcut [14]. Under the mentioned spaces they introduced exciting results about tripled FP theorems. For more details, see [15, 16, 17, 18, 19, 20].

In the setting of FMSs, coupled FP results are presented and some important theorems are given by Zhu and Xiao [21] and Hu [22]. Elagan *et al.* [23] studied the existence of a FP in locally convex topology generated by fuzzy n -normed spaces.

Motivated by the results of coupled and tripled FP notions in partially ordered metric spaces, Karapinar [24] suggested the concept of quadruple FP and proved some related FP consequences in the same spaces. Also, see [25, 26, 27].

Based on the last two paragraphs above, in this manuscript, a QCP is considered and some new related FP results are presented in FMSs. The strength of our paper depends on two directions. Firstly, we can customize it to complete metric spaces (CMSs) such as obtaining the results of Karapinar [24] (in non fuzzy sets). So our paper covered and unify a lot of results in the same direction, also covers coupled FP results. Secondly, we can use the theoretical results to find a unique solution for Lipschitzian and integral quadruple systems. Ultimately, non-trivial examples are stated and discussed.

2 Preliminaries

Hereinafter, we will refer to ζ with a non-empty set, $\zeta^4 = \zeta \times \zeta \times \zeta \times \zeta$, $\Omega(\rho, \sigma, \tau, v)$ with $\Omega_{\rho\sigma\tau v}$, $\Psi(\rho, \sigma, \kappa)$ with $\Psi_{\rho\sigma}(\kappa)$ and $\omega(\rho, \sigma)$ with $\omega_{\rho\sigma}$.

The usual MS is a non-empty set ζ defined on a distance $\omega : \zeta \times \zeta \rightarrow \mathbb{R}$ verifying, for all $\rho, \sigma, \tau \in \zeta$,

- $\omega_{\rho\sigma} \geq 0$,
- $\omega_{\rho\sigma} = 0$ iff $\rho = \sigma$,
- $\omega_{\rho\sigma} \leq \omega_{\rho\tau} + \omega_{\tau\sigma}$.

The pair (ζ, ω) is called a MS.

A mapping $\Upsilon : \zeta \rightarrow \zeta$ on a MS (ζ, ω) is called Lipschitzian if there is $\varpi \geq 0$ so that

$$\omega_{\Upsilon\rho\Upsilon\sigma} \leq \varpi\omega_{\rho\sigma}, \quad \forall \rho, \sigma \in \zeta.$$

The smallest constant ϖ denoted by ϖ_{Υ} satisfy the above inequality is called Lipschitz constant for Υ . It is clear that a Lipschitzian mapping (LM) is a contraction with $\varpi_{\Upsilon} < 1$.

Theorem 2.1. A contraction mapping defined on a MS (ζ, ω) have a unique FP if the pair (ζ, ω) verify the completeness property.

For examples on LMs, let $\zeta = \mathbb{R}$ and $\mathcal{T}_i : \zeta \rightarrow \zeta$ defined by $\mathcal{T}_1(\rho) = \Lambda$, $\mathcal{T}_2(\rho) = \mu\rho$, $\mathcal{T}_3(\rho) = \cos \rho$, $\mathcal{T}_4(\rho) = \frac{1}{1+\rho}$, $\mathcal{T}_5(\rho) = \frac{1}{(1+\rho)^2}$, $\mathcal{T}_6(\rho) = \arcsin \rho$.

Definition 2.2. A mapping $\star : [0, 1]^2 \rightarrow [0, 1]$ is called a κ -norm if it is nondecreasing in both arguments, associative, commutative and has 1 as identity. For all $\ell \in [0, 1]$, the sequence $\{\star^m \ell\}_{m=1}^\infty$ is defined inductively by $\star^1 \ell = \ell$, $\star^m \ell = (\star^{m-1} \ell) \star \ell$. A triangular norm $\star$ is called Υ -type [29] if $\{\star^m \ell\}_{m=1}^\infty$ is equicontinuous at $\ell = 1$, that is, for each $\epsilon \in (0, 1)$, there is $\varkappa \in (0, 1)$ so that if $\ell \in (1 - \varkappa, 1]$, then $\star^m \ell > 1 - \epsilon$, for each $m \in \mathbb{N}$.

The most famous continuous κ -norm of Υ -type is $\star = \min$, that satisfy $\min(\ell_1, \ell_2) \geq \ell_1 \ell_2$ for all $\ell_1, \ell_2 \in [0, 1]$.

The result below discuss a wide range of κ -norms of Υ -type.

Lemma 2.3. Assume that $\star$ is a κ -norm and $\varrho \in (0, 1]$ is a real number. Define $\star_\varrho$ by $\rho \star_\varrho \sigma = \rho \star \sigma$, if $\max\{\rho, \sigma\} \leq 1 - \varrho$, and $\rho \star_\varrho \sigma = \min\{\rho, \sigma\}$ if $\max\{\rho, \sigma\} > 1 - \varrho$. Then $\star_\varrho$ is a κ -norm of Υ -type.

Definition 2.4. [7] Let $\zeta \neq \emptyset$ be an arbitrary set, $\star$ be a continuous κ -norm and $\Psi : \zeta \times \zeta \times [0, \infty) \rightarrow [0, 1]$ be a fuzzy set. We say that $(\zeta, \Psi, \star)$ is a FMS if the function Ψ satisfy the hypotheses below, for each $\rho, \sigma, \tau \in \zeta$, and $\kappa, \mu > 0$:

- (fms 1) $\Psi_{\rho\sigma}(0) = 0$;
- (fms 2) $\Psi_{\rho\sigma}(\kappa) = 1 \Leftrightarrow \rho = \sigma$;
- (fms 3) $\Psi_{\rho\sigma}(\kappa) = \Psi_{\sigma\rho}(\kappa)$;
- (fms 4) $\Psi_{\rho\sigma}(\cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (fms 5) $\Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\mu) \leq \Psi_{\rho\tau}(\kappa + \mu)$.

Here, we also called (ζ, Ψ) is a FMS under $\star$ and we will only consider FMS verifying:

$$(D) \lim_{\kappa \rightarrow \infty} \Psi_{\rho\sigma}(\kappa) = 1, \forall \rho, \sigma \in \zeta.$$

Lemma 2.5. On the infinite set $[0, \infty)$, $\Omega_{\rho\sigma}(\cdot)$ is a non-decreasing function.

Definition 2.6. Assume that (ζ, Ψ) is a FMS under some κ -norm, a sequence $\{\rho_m\} \subset \zeta$ is called:

- Convergent to $\rho \in \zeta$, and we write $\lim_{\kappa \rightarrow \infty} \rho_m = \rho$ if, for every $\epsilon > 0$, $\kappa > 0$, there is $m_0 \in \mathbb{N}$ so that $\Psi_{\rho_m \rho} > 1 - \epsilon$ for all $m \geq m_0$.
- A Cauchy sequence if for every $\epsilon > 0$, $\kappa > 0$, there is $m_0 \in \mathbb{N}$ so that $\Psi_{\rho_m \rho_j} > 1 - \epsilon$ for all $m, j \geq m_0$.

If a Cauchy sequence is convergent, then a FMS is called complete.

Definition 2.7. We say that a function $\top : \zeta \rightarrow \zeta$ defined on a FMS is continuous at $\rho_0 \in \zeta$ if, $\lim_{m \rightarrow \infty} \top \rho_m = \top \rho_0$, for any $\{\rho_m\} \in \zeta$ so that $\lim_{m \rightarrow \infty} \rho_m = \rho_0$. As familiar, we will denote for $\rho_0 \in \zeta$, $\top^{-1}(\rho_0) = \{\rho \in \zeta : \top \rho = \rho_0\}$.

Remark 2.8. If $\ell_1 \leq \ell_2$, then $\rho^{\ell_1} \geq \rho^{\ell_2}$ provided that $\rho \in [0, 1]$ and $\ell_1, \ell_2 \in (0, \infty)$. This fact will be expressed here as follows: $0 < \ell_1 \leq \ell_2 \leq 1$ implies that $\Psi_{\rho\sigma}(\kappa)^{\ell_1} \geq \Psi_{\rho\sigma}(\kappa)^{\ell_2} \geq \Psi_{\rho\sigma}(\kappa)$.

For any κ -norm $\star$, it is obvious that $\star \leq \min$. So, if (ζ, Ψ) is a FMS via $\min$, then (ζ, Ψ) is a FMS under any κ -norm.

In examples below, we only define $\Psi_{\rho\sigma}(\kappa)$ for $\kappa > 0$ and $\rho \neq \sigma$.

Example 2.9. For $\kappa > 0$, and $\rho \neq \sigma$ define a FMS in different ways from a MS (ζ, ω) as follows:

$$\bullet \Psi_{\rho\sigma}^{\omega}(\kappa) = \frac{\kappa}{\kappa + \omega_{\rho\sigma}} \quad \bullet \Psi_{\rho\sigma}^e(\kappa) = e^{-\frac{\omega_{\rho\sigma}}{\kappa}} \quad \bullet \Psi_{\rho\sigma}^o(\kappa) = \begin{cases} 0, & \text{if } \kappa \leq \omega_{\rho\sigma}, \\ 1, & \text{if } \kappa > \omega_{\rho\sigma}. \end{cases}$$

It is obvious that under the product $\star = \cdot$, (ζ, Ψ^{ω}) is a FMS, called the standard FMS on (ζ, ω) . Also, (ζ, Ψ^{ω}) , (ζ, Ψ^e) and (ζ, Ψ^o) are FMSs under $\min$, it is a standard method to see the MS (ζ, ω) as a FMS though it is lesser-known.

Moreover, (ζ, ω) is a CMS iff (ζ, Ψ^{ω}) or (ζ, Ψ^e) or (ζ, Ψ^o) is a complete FMS.

3 Main results

We begin this section with the simple definition below

Definition 3.1. Assume that $\Omega : \zeta^4 \rightarrow \zeta$ and $\top : \zeta \rightarrow \zeta$ are two mappings.

- We say that Ω and $\top$ is commuting if $\top \Omega_{\rho\sigma\tau v} = \Omega_{\top\rho\top\sigma\top\tau\top v}$, $\forall \rho, \sigma, \tau, v \in \zeta$.
- We say that $(\rho, \sigma, \tau, v) \in \zeta^4$ is a QCP of Ω and $\top$ if

$$\Omega_{\rho\sigma\tau v} = \top_{\rho}, \Omega_{\sigma\tau v\rho} = \top_{\sigma}, \Omega_{\tau v\rho\sigma} = \top_{\tau} \text{ and } \Omega_{v\rho\sigma\tau} = \top_v.$$

Theorem 3.2. Assume that $\star$ is a κ -norm of Υ -type so that $\mu \star \kappa \geq \mu\kappa$, for all $\mu, \kappa \in [0, 1]$. Suppose that $(\zeta, \Psi, \star)$ is a complete FMS and $\Omega : \zeta^4 \rightarrow \zeta$, $\top : \zeta \rightarrow \zeta$ are two mappings so that

- (a) $\Omega(\zeta^4) \subseteq \top(\zeta)$,
- (b) $\top$ is continuous,
- (c) $\top$ is commuting with Ω ,
- (d) for all $\rho, \sigma, \tau, v, \hat{\rho}, \hat{\sigma}, \hat{\tau}, \hat{v} \in \zeta$,

$$\Psi_{\Omega_{\rho\sigma\tau v} \Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{v}}}(\kappa\varpi) \geq \Psi_{\top\rho\top\hat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\top\sigma\top\hat{\sigma}}(\kappa)^{\ell_2} \star \Psi_{\top\tau\top\hat{\tau}}(\kappa)^{\ell_3} \star \Psi_{\top v\top\hat{v}}(\kappa)^{\ell_4}, \quad (3.1)$$

where $\varpi \in (0, 1)$ and $\ell_1, \ell_2, \ell_3, \ell_4$ are real numbers in $[0, 1]$ so that $\ell_1 + \ell_2 + \ell_3 + \ell_4 \leq 1$. Then the conclusions below hold.

- (1) There is a unique $\rho \in \zeta$, so that $\rho = \top_{\rho} = \Omega_{\rho\rho\rho\rho}$. In particular,
- (2) There is at least QCP for the mappings $\top$ and Ω , moreover, in the case of $\Omega = \rho_0$ is a constant on ζ^4 , this holds only if we let $\top^{-1}(\rho_0) = \{\rho_0\}$, we have
- (3) (ρ, ρ, ρ, ρ) is a unique QCP of $\top$ and Ω .

Note, to avoid the unidentified quantity 0^0 , we consider here $\Psi_{\top_{\rho}\top_{\hat{\rho}}}(\kappa)^0 = 1$ for all $\kappa > 0$ and all $\rho, \hat{\rho} \in \zeta$.

Proof. We divide the proof into two cases:

Case 1. When $\Omega \in \zeta$ is constant, that is there is $\rho_0 \in \zeta$ so that, for all $\rho\sigma\tau v \in \zeta$, $\Omega_{\rho\sigma\tau v} = \rho_0$. Since Ω and $\top$ are commuting, then one can write $\top\rho_0 = \top\Omega_{\rho_0\rho_0\rho_0\rho_0} = \Omega_{\top\rho_0\top\sigma_0\top\tau_0\top v_0} = \rho_0$. Therefore, $\rho_0 = \top\rho_0 = \Omega_{\rho_0\rho_0\rho_0\rho_0}$ and $(\rho_0, \rho_0, \rho_0, \rho_0)$ is a QCP of Ω and $\top$. On the other hand, assume that $\top^{-1}(\rho_0) = \{\rho_0\}$ and $(\rho, \sigma, \tau, v) \in \zeta^4$ is another QCP of Ω and $\top$. Then $\top\rho = \Omega_{\rho\sigma\tau v} = \rho_0$, so $\rho \in \top^{-1}(\rho_0) = \{\rho_0\}$. With the same manner we can write $\rho = \sigma = \tau = v = \rho_0$, hence $(\rho_0, \rho_0, \rho_0, \rho_0)$ is a unique QCP of Ω and $\top$.

Case 2. Assume that $\Omega \in \zeta$ is not constant, for this let $(\ell_1, \ell_2, \ell_3, \ell_4) \neq (0, 0, 0, 0)$. In this case, we consider j and m are non-negative integers and $\kappa \in [0, \infty)$. This case is divided into five steps:

St₁. Deriving four sequences $\{\rho_m\}$, $\{\sigma_m\}$, $\{\tau_m\}$ and $\{v_m\}$. Suppose that $\rho_0, \sigma_0, \tau_0, v_0$ are arbitrary points in ζ . As $\Omega(\zeta^4) \subseteq \top(\zeta)$, we can select $\rho_1, \sigma_1, \tau_1, v_1 \in \zeta$ so that $\top\rho_1 = \Omega_{\rho_0\sigma_0\tau_0v_0}$, $\top\sigma_1 = \Omega_{\sigma_0\tau_0v_0\rho_0}$, $\top\tau_1 = \Omega_{\tau_0v_0\rho_0\sigma_0}$ and $\top v_1 = \Omega_{v_0\rho_0\sigma_0\tau_0}$. Again by $\Omega(\zeta^4) \subseteq \top(\zeta)$, we can select $\rho_2, \sigma_2, \tau_2, v_2 \in \zeta$ so that $\top\rho_2 = \Omega_{\rho_1\sigma_1\tau_1v_1}$, $\top\sigma_2 = \Omega_{\sigma_1\tau_1v_1\rho_1}$, $\top\tau_2 = \Omega_{\tau_1v_1\rho_1\sigma_1}$ and $\top v_2 = \Omega_{v_1\rho_1\sigma_1\tau_1}$. By continuing with the same scenario, we can construct $\{\rho_m\}$, $\{\sigma_m\}$, $\{\tau_m\}$ and $\{v_m\}$ so that for $m \geq 0$, $\top\rho_{m+1} = \Omega_{\rho_m\sigma_m\tau_mv_m}$, $\top\sigma_{m+1} = \Omega_{\sigma_m\tau_mv_m\rho_m}$, $\top\tau_{m+1} = \Omega_{\tau_mv_m\rho_m\sigma_m}$ and $\top v_{m+1} = \Omega_{v_m\rho_m\sigma_m\tau_m}$.

St₂. $\{\rho_m\}$, $\{\sigma_m\}$, $\{\tau_m\}$ and $\{v_m\}$ are Cauchy sequences. For $m \geq 0$ and all $\kappa \geq 0$, define

$$\Xi_m(\kappa) = \Psi_{\top\rho_m\top\rho_{m+1}}(\kappa) \star \Psi_{\top\sigma_m\top\sigma_{m+1}} \star \Psi_{\top\tau_m\top\tau_{m+1}} \star \Psi_{\top v_m\top v_{m+1}}.$$

As Ξ_m is a non-decreasing function and $\kappa - \kappa\varpi \leq \kappa \leq \frac{\kappa}{\varpi}$, so we get

$$\Xi_m(\kappa - \kappa\varpi) \leq \Xi_m(\kappa) \leq \Xi_m\left(\frac{\kappa}{\varpi}\right), \text{ for all } \kappa > 0 \text{ and } m \geq 0. \quad (3.2)$$

It follows from (3.1) that for all $m \in \mathbb{N}$ and all $\kappa \geq 0$,

$$\begin{aligned} \Psi_{\top\rho_m\top\rho_{m+1}}(\kappa) &= \Psi_{\Omega_{\rho_{m-1}\sigma_{m-1}\tau_{m-1}v_{m-1}}\Omega_{\rho_m\sigma_m\tau_mv_m}}(\kappa) \\ &\geq \Psi_{\top\rho_{m-1}\top\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\top\sigma_{m-1}\top\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2} \\ &\quad \star \Psi_{\top\tau_{m-1}\top\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\top v_{m-1}\top v_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}; \end{aligned} \quad (3.3)$$

$$\begin{aligned} \Psi_{\top\sigma_m\top\sigma_{m+1}}(\kappa) &= \Psi_{\Omega_{\sigma_{m-1}\tau_{m-1}v_{m-1}\rho_{m-1}}\Omega_{\sigma_m\tau_mv_m\rho_m}}(\kappa) \\ &\geq \Psi_{\top\sigma_{m-1}\top\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\top\tau_{m-1}\top\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2} \\ &\quad \star \Psi_{\top v_{m-1}\top v_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\top\rho_{m-1}\top\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}; \end{aligned} \quad (3.4)$$

$$\begin{aligned}
\Psi_{\neg\tau_m \neg\tau_{m+1}}(\kappa) &= \Psi_{\Omega_{\tau_{m-1} v_{m-1} \rho_{m-1} \sigma_{m-1}} \Omega_{\tau_m v_m \rho_m \sigma_m}}(\kappa) \\
&\geq \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \\
&\quad \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4}; \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
\Psi_{v_m \neg v_{m+1}}(\kappa) &= \Psi_{\Omega_{v_{m-1} \rho_{m-1} \sigma_{m-1} \tau_{m-1}} \Omega_{v_m \rho_m \sigma_m \tau_m}}(\kappa) \\
&\geq \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \\
&\quad \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4}; \tag{3.6}
\end{aligned}$$

It follows from (3.3)-(3.6) and Remark 2.8 that

$$\begin{aligned}
&\Psi_{\neg\rho_m \neg\rho_{m+1}}(\kappa) \\
&\geq \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \\
&\geq \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right) \\
&= \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right);
\end{aligned}$$

$$\begin{aligned}
&\Psi_{\neg\sigma_m \neg\sigma_{m+1}}(\kappa) \\
&\geq \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \\
&\geq \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right) \\
&= \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right);
\end{aligned}$$

$$\begin{aligned}
&\Psi_{\neg\tau_m \neg\tau_{m+1}}(\kappa) \\
&\geq \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \\
&\geq \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right) \\
&= \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right);
\end{aligned}$$

and

$$\begin{aligned}
&\Psi_{v\tau_m \neg v_{m+1}}(\kappa) \\
&\geq \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \\
&\geq \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right) \\
&= \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right).
\end{aligned}$$

This proves that, for all $\kappa > 0$ and all $m \geq 0$,

$$\Psi_{\neg\rho_m \neg\rho_{m+1}}(\kappa), \Psi_{\neg\sigma_m \neg\sigma_{m+1}}(\kappa), \Psi_{\neg\tau_m \neg\tau_{m+1}}(\kappa), \Psi_{v\tau_m \neg v_{m+1}}(\kappa) \geq \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right) \geq \Xi_{m-1}(\kappa). \tag{3.7}$$

Putting $\kappa - \varpi\kappa$ instead of κ , we obtain, for all $\kappa > 0$ and all $m \geq 0$, that

$$\begin{aligned} & \Psi_{\neg\rho_m \neg\rho_{m+1}}(\kappa - \varpi\kappa), \Psi_{\neg\sigma_m \neg\sigma_{m+1}}(\kappa - \varpi\kappa), \Psi_{\neg\tau_m \neg\tau_{m+1}}(\kappa - \varpi\kappa), \Psi_{\neg v_m \neg v_{m+1}}(\kappa - \varpi\kappa) \\ & \geq \Xi_{m-1}(\kappa - \varpi\kappa). \end{aligned} \quad (3.8)$$

Since $\star$ is commutative and $\star \geq .$, by (3.3)-(3.6), we deduce that

$$\begin{aligned} \Xi_m(\kappa) &= \Psi_{\neg\rho_m \neg\rho_{m+1}}(\kappa) \star \Psi_{\neg\sigma_m \neg\sigma_{m+1}} \star \Psi_{\neg\tau_m \neg\tau_{m+1}} \star \Psi_{\neg v_m \neg v_{m+1}} \\ &\geq \left(\Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \right) \\ &\quad \star \left(\Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \right) \\ &\quad \star \left(\Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \right) \\ &\quad \star \left(\Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \right) \\ &= \left(\Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \right) \\ &\quad \star \left(\Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \star \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \right) \\ &\quad \star \left(\Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \right) \\ &\quad \star \left(\Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \right) \\ &\geq \left(\Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \cdot \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \cdot \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \cdot \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \right) \\ &\quad \star \left(\Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \cdot \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \cdot \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \cdot \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \right) \\ &\quad \star \left(\Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \cdot \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \cdot \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \cdot \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \right) \\ &\quad \star \left(\Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_4} \cdot \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1} \cdot \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_2} \cdot \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_3} \right) \\ &= \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1+\ell_2+\ell_3+\ell_4} \\ &\quad \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right)^{\ell_1+\ell_2+\ell_3+\ell_4} \\ &\geq \Psi_{\neg\rho_{m-1} \neg\rho_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\sigma_{m-1} \neg\sigma_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg\tau_{m-1} \neg\tau_m} \left(\frac{\kappa}{\varpi} \right) \star \Psi_{\neg v_{m-1} \neg v_m} \left(\frac{\kappa}{\varpi} \right) \\ &= \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right) \end{aligned}$$

By using (3.2), one can write

$$\Xi_m(\kappa) \geq \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right) \geq \Xi_{m-1}(\kappa) \geq \Xi_{m-1}(\kappa - \kappa\varpi), \quad \forall \kappa > 0, \text{ and } m \geq 1. \quad (3.9)$$

By continuing with the same manner, we have

$$\Xi_m(\kappa) \geq \Xi_{m-1} \left(\frac{\kappa}{\varpi} \right) \geq \Xi_{m-2} \left(\frac{\kappa}{\varpi^2} \right) \geq \dots \geq \Xi_0 \left(\frac{\kappa}{\varpi^m} \right), \quad \forall \kappa > 0, \text{ and } m \geq 1,$$

this leads to for all $\kappa > 0$

$$\lim_{m \rightarrow \infty} \Xi_m(\kappa) \geq \lim_{m \rightarrow \infty} \Xi_0\left(\frac{\kappa}{\varpi^m}\right) = 1 \Rightarrow \lim_{m \rightarrow \infty} \Xi_m(\kappa) = 1. \quad (3.10)$$

From (3.7) and (3.9), we have

$$\Psi_{\neg\rho_m \neg\rho_{m+1}}(\kappa), \Psi_{\neg\sigma_m \neg\sigma_{m+1}}(\kappa), \Psi_{\neg\tau_m \neg\tau_{m+1}}(\kappa), \Psi_{\neg v_m \neg v_{m+1}}(\kappa) \geq \Xi_m(\kappa) \geq \Xi_{m-1}(\kappa - \kappa\varpi). \quad (3.11)$$

After that, we will prove that, for all $\kappa > 0$ and all $m, r \geq 1$,

$$\Psi_{\neg\rho_m \neg\rho_{m+r}}(\kappa), \Psi_{\neg\sigma_m \neg\sigma_{m+r}}(\kappa), \Psi_{\neg\tau_m \neg\tau_{m+r}}(\kappa), \Psi_{\neg v_m \neg v_{m+r}}(\kappa) \geq \star^r \Xi_{m-1}(\kappa - \kappa\varpi). \quad (3.12)$$

We can show it by induction in $r \geq 1$ as follows: Inequality (3.12) holds if $r = 1$, for all $m \geq 1$ and all $\kappa > 0$ by (3.11). Assume that (3.12) is true for all $m \geq 1$ and all $\kappa > 0$ for some r . Now, we prove the relation for $r + 1$. It follows from (3.1), the induction assumption and $\star \geq \cdot$, that

$$\begin{aligned} & \Psi_{\neg\rho_{m+1} \neg\rho_{m+r+1}}(\varpi\kappa) \\ = & \Psi_{\Omega_{\rho_m \sigma_m \tau_m v_m} \Omega_{\rho_{m+r} \sigma_{m+r} \tau_{m+r} v_{m+r}}}(\varpi\kappa) \\ \geq & \Psi_{\neg\rho_m \neg\rho_{m+r}}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma_m \neg\sigma_{m+r}}(\kappa)^{\ell_2} \star \Psi_{\neg\tau_m \neg\tau_{m+r}}(\kappa)^{\ell_3} \star \Psi_{\neg v_m \neg v_{m+r}}(\kappa)^{\ell_4} \\ \geq & (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_1} \star (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_2} \star (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_3} \star (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_4} \\ \geq & (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_1} \cdot (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_2} \cdot (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_3} \cdot (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_4} \\ = & (\star^r \Xi_{m-1}(\kappa - \kappa\varpi))^{\ell_1 + \ell_2 + \ell_3 + \ell_4} \geq \star^r \Xi_{m-1}(\kappa - \kappa\varpi). \end{aligned}$$

Similarly, we arrive to

$$\left(\begin{array}{c} \Psi_{\neg\rho_{m+1} \neg\rho_{m+r+1}}(\varpi\kappa), \Psi_{\neg\sigma_{m+1} \neg\sigma_{m+r+1}}(\varpi\kappa), \\ \Psi_{\neg\tau_{m+1} \neg\tau_{m+r+1}}(\varpi\kappa), \Psi_{\neg v_{m+1} \neg v_{m+r+1}}(\varpi\kappa) \end{array} \right) \geq \star^r \Xi_{m-1}(\kappa - \kappa\varpi).$$

By Definition 2.4 (fms 5), (3.8) and induction assumption, we get

$$\begin{aligned} \Psi_{\neg\rho_{m+1} \neg\rho_{m+r+1}}(\kappa) &= \Psi_{\neg\rho_{m+1} \neg\rho_{m+r+1}}(\kappa - \kappa\varpi + \kappa\varpi) \\ &\geq \Psi_{\neg\rho_m \neg\rho_{m+1}}(\kappa - \kappa\varpi) \star \Psi_{\neg\rho_{m+1} \neg\rho_{m+r+1}}(\kappa\varpi) \\ &\geq \Xi_{m-1}(\kappa - \kappa\varpi) \star (\star^r \Xi_{m-1}(\kappa - \kappa\varpi)) \\ &= \star^{r+1} \Xi_{m-1}(\kappa - \kappa\varpi). \end{aligned}$$

Also, the same result holds if we consider $\Psi_{\neg\sigma_{m+1} \neg\sigma_{m+r+1}}(\kappa)$, $\Psi_{\neg\tau_{m+1} \neg\tau_{m+r+1}}(\kappa)$ and $\Psi_{\neg v_{m+1} \neg v_{m+r+1}}(\kappa)$. This leads to (3.12) is true. This allows us to prove that $\{\neg\rho_m\}$ is Cauchy. Assume that $\kappa > 0$ and $\varepsilon \in (0, 1)$ are given. From the assumption, as $\star$ is a κ -norm of Υ -type, there is $\varphi \in (0, 1)$ so that $\star^r \ell_1 > 1 - \varepsilon$ for all $\ell_1 \in (1 - \varphi, 1]$ and for all $r \geq 1$. From (3.10), $\lim_{m \rightarrow \infty} \Xi_m(\kappa) = 1$, so there is $m_0 \in \mathbb{N}$ so that

$$\Xi_m(\kappa - \kappa\varpi) > 1 - \varphi, \quad \forall m \geq m_0.$$

Hence, by (3.12), we have

$$\Psi_{\neg\rho_m \neg\rho_{m+r}}(\kappa), \Psi_{\neg\sigma_m \neg\sigma_{m+r}}(\kappa), \Psi_{\neg\tau_m \neg\tau_{m+r}}(\kappa), \Psi_{\neg v_m \neg v_{m+r}}(\kappa) > 1 - \varepsilon, \quad \forall m \geq m_0 \text{ and } r \geq 1.$$

Thus, $\{\neg\rho_m\}$ is a Cauchy sequence. Similarly $\{\neg\sigma_m\}$, $\{\neg\tau_m\}$ and $\{\neg v_m\}$ are also Cauchy sequences.

St₃. Proving that Ω and $\lceil$ have a QCP. As ζ is complete, there are $\rho\sigma\tau v \in \zeta$ so that

$$\lim_{m \rightarrow \infty} \lceil \rho_m = \rho, \quad \lim_{m \rightarrow \infty} \lceil \sigma_m = \sigma, \quad \lim_{m \rightarrow \infty} \lceil \tau_m = \tau \text{ and } \lim_{m \rightarrow \infty} \lceil v_m = v.$$

The continuity of $\lceil$ implies that

$$\lim_{m \rightarrow \infty} \lceil \lceil \rho_m = \lceil \rho, \quad \lim_{m \rightarrow \infty} \lceil \lceil \sigma_m = \lceil \sigma, \quad \lim_{m \rightarrow \infty} \lceil \lceil \tau_m = \lceil \tau \text{ and } \lim_{m \rightarrow \infty} \lceil \lceil v_m = \lceil v.$$

The commutativity of Ω and $\lceil$ leads to

$$\lceil \lceil \rho_{m+1} = \lceil \Omega(\rho_m, \sigma_m, \tau_m, v_m) = \Omega(\lceil \rho_m, \lceil \sigma_m, \lceil \tau_m, \lceil v_m).$$

By (3.1), we get

$$\begin{aligned} \Psi_{\lceil \lceil \rho_{m+1} \Omega_{\rho\sigma\tau v}}(\kappa\varpi) &= \Psi_{\Omega_{\lceil \rho_m \lceil \sigma_m \lceil \tau_m \lceil v_m} \Omega_{\rho\sigma\tau v}}(\kappa\varpi) \\ &\geq \Psi_{\lceil \lceil \rho_m \lceil \rho}(\kappa)^{\ell_1} \star \Psi_{\lceil \lceil \sigma_m \lceil \sigma}(\kappa)^{\ell_2} \star \Psi_{\lceil \lceil \tau_m \lceil \tau}(\kappa)^{\ell_3} \star \Psi_{\lceil \lceil v_m \lceil v}(\kappa)^{\ell_4} \\ &\geq \Psi_{\lceil \lceil \rho_m \lceil \rho}(\kappa) \star \Psi_{\lceil \lceil \sigma_m \lceil \sigma}(\kappa) \star \Psi_{\lceil \lceil \tau_m \lceil \tau}(\kappa) \star \Psi_{\lceil \lceil v_m \lceil v}(\kappa). \end{aligned} \quad (3.13)$$

As $m \rightarrow \infty$, in (3.13), we obtain that

$$\lim_{m \rightarrow \infty} \lceil \lceil \rho_{m+1} = \Omega_{\rho\sigma\tau v} = \lceil \rho.$$

Similarly, we deduce that $\Omega_{\sigma\tau v \rho} = \lceil \sigma$, $\Omega_{\tau v \rho \sigma} = \lceil \tau$, $\Omega_{v \rho \sigma \tau} = \lceil v$. This show that (ρ, σ, τ, v) is a ccp of Ω and $\lceil$.

$$\lceil \rho = \Omega_{\rho\sigma\tau v}, \quad \lceil \sigma = \Omega_{\sigma\tau v \rho}, \quad \lceil \tau = \Omega_{\tau v \rho \sigma}, \quad \lceil v = \Omega_{v \rho \sigma \tau}. \quad (3.14)$$

St₄. Showing that $\rho = \Omega_{\rho\sigma\tau v}$, $\sigma = \Omega_{\sigma\tau v \rho}$, $\tau = \Omega_{\tau v \rho \sigma}$ and $v = \Omega_{v \rho \sigma \tau}$. By Stipulation (3.1), we get

$$\begin{aligned} &\Psi_{\lceil \rho \lceil \sigma_{m+1}}(\kappa\varpi) \\ &= \Psi_{\Omega_{\rho\sigma\tau v} \Omega_{\sigma_m \lceil \tau_m \lceil v_m \rho_m}}(\kappa\varpi) \\ &\geq \Psi_{\lceil \rho \lceil \sigma_m}(\kappa)^{\ell_1} \star \Psi_{\lceil \sigma \lceil \tau_m}(\kappa)^{\ell_2} \star \Psi_{\lceil \tau \lceil v_m}(\kappa)^{\ell_3} \star \Psi_{\lceil v \lceil \rho_m}(\kappa)^{\ell_4}; \end{aligned} \quad (3.15)$$

$$\begin{aligned} &\Psi_{\lceil \sigma \lceil \tau_{m+1}}(\kappa\varpi) \\ &= \Psi_{\Omega_{\sigma\tau v \rho} \Omega_{\tau_m \lceil v_m \rho_m \sigma_m}}(\kappa\varpi) \\ &\geq \Psi_{\lceil \sigma \lceil \tau_m}(\kappa)^{\ell_1} \star \Psi_{\lceil \tau \lceil v_m}(\kappa)^{\ell_2} \star \Psi_{\lceil v \lceil \rho_m}(\kappa)^{\ell_3} \star \Psi_{\lceil \rho \lceil \sigma_m}(\kappa)^{\ell_4}; \end{aligned} \quad (3.16)$$

$$\begin{aligned} &\Psi_{\lceil \tau \lceil v_{m+1}}(\kappa\varpi) \\ &= \Psi_{\Omega_{\tau v \rho \sigma} \Omega_{v_m \rho_m \sigma_m \tau_m}}(\kappa\varpi) \\ &\geq \Psi_{\lceil \tau \lceil v_m}(\kappa)^{\ell_1} \star \Psi_{\lceil v \lceil \rho_m}(\kappa)^{\ell_2} \star \Psi_{\lceil \rho \lceil \sigma_m}(\kappa)^{\ell_3} \star \Psi_{\lceil \sigma \lceil \tau_m}(\kappa)^{\ell_4}; \end{aligned} \quad (3.17)$$

$$\begin{aligned} &\Psi_{\lceil v \lceil \rho_{m+1}}(\kappa\varpi) \\ &= \Psi_{\Omega_{v \rho \sigma \tau} \Omega_{\rho_m \sigma_m \tau_m v_m}}(\kappa\varpi) \\ &\geq \Psi_{\lceil v \lceil \rho_m}(\kappa)^{\ell_1} \star \Psi_{\lceil \rho \lceil \sigma_m}(\kappa)^{\ell_2} \star \Psi_{\lceil \sigma \lceil \tau_m}(\kappa)^{\ell_3} \star \Psi_{\lceil \tau \lceil v_m}(\kappa)^{\ell_4}. \end{aligned} \quad (3.18)$$

Set $\nabla_m(\kappa) = \Psi_{\neg\rho\neg\sigma_m}(\kappa\varpi) \star \Psi_{\neg\sigma\neg\tau_m}(\kappa\varpi) \star \Psi_{\neg\tau\neg\nu_m}(\kappa\varpi) \star \Psi_{\neg\nu\neg\rho_m}(\kappa\varpi)$ for all $\kappa > 0$ and $m \geq 0$. It follows from (3.15)-(3.18) that

$$\begin{aligned}
\nabla_{m+1}(\kappa\varpi) &= \Psi_{\neg\rho\neg\sigma_{m+1}}(\kappa\varpi) \star \Psi_{\neg\sigma\neg\tau_{m+1}}(\kappa\varpi) \star \Psi_{\neg\tau\neg\nu_{m+1}}(\kappa\varpi) \star \Psi_{\neg\nu\neg\rho_{m+1}}(\kappa\varpi) \\
&\geq \left(\Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \right) \\
&= \left(\Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_4} \star \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_3} \star \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_2} \right) \\
&\quad \star \left(\Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_4} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_3} \right) \\
&\quad \star \left(\Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_2} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_1} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_3} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_2} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_1} \right) \\
&\geq \left(\Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \cdot \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_4} \cdot \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_3} \cdot \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_2} \right) \\
&\quad \star \left(\Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \cdot \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_1} \cdot \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_4} \cdot \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_3} \right) \\
&\quad \star \left(\left(\Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \cdot \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_2} \cdot \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_1} \cdot \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_4} \right) \right) \\
&\quad \star \left(\Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \cdot \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_3} \cdot \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_2} \cdot \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_1} \right) \\
&= \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\quad \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\geq \Psi_{\neg\rho\neg\sigma_m}(\kappa) \star \Psi_{\neg\sigma\neg\tau_m}(\kappa) \star \Psi_{\neg\tau\neg\nu_m}(\kappa) \star \Psi_{\neg\nu\neg\rho_m}(\kappa) = \nabla_m(\kappa).
\end{aligned}$$

This implies that $\nabla_{m+1}(\kappa\varpi) \geq \nabla_m(\kappa)$ for all $m \geq 0$ and all $\kappa > 0$. Repeating this process,

$$\nabla_m(\kappa) \geq \nabla_{m-1}\left(\frac{\kappa}{\varpi}\right) \geq \nabla_{m-2}\left(\frac{\kappa}{\varpi^2}\right) \geq \dots \geq \nabla_0\left(\frac{\kappa}{\varpi^m}\right), \quad \forall \kappa > 0 \text{ and } m \geq 1. \quad (3.19)$$

By (3.15)-(3.19), we conclude that

$$\begin{aligned}
\Psi_{\neg\rho\neg\sigma_{m+1}}(\kappa\varpi) &\geq \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_2} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_3} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0\left(\frac{\kappa}{\varpi^m}\right);
\end{aligned} \quad (3.20)$$

$$\begin{aligned}
\Psi_{\neg\sigma\neg\tau_{m+1}}(\kappa\varpi) &\geq \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_1} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_2} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_3} \star \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0\left(\frac{\kappa}{\varpi^m}\right);
\end{aligned} \quad (3.21)$$

$$\begin{aligned}
\Psi_{\neg\tau\neg\nu_{m+1}}(\kappa\varpi) &\geq \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_1} \star \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_2} \star \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_3} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0\left(\frac{\kappa}{\varpi^m}\right);
\end{aligned} \quad (3.22)$$

$$\begin{aligned}
\Psi_{\neg\nu\neg\rho_{m+1}}(\kappa\varpi) &\geq \Psi_{\neg\nu\neg\rho_m}(\kappa)^{\ell_1} \star \Psi_{\neg\rho\neg\sigma_m}(\kappa)^{\ell_2} \star \Psi_{\neg\sigma\neg\tau_m}(\kappa)^{\ell_3} \star \Psi_{\neg\tau\neg\nu_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0\left(\frac{\kappa}{\varpi^m}\right).
\end{aligned} \quad (3.23)$$

Thus,

$$\Psi_{\lceil\rho\rceil\lceil\sigma_{m+1}\rceil}(\kappa\varpi), \Psi_{\lceil\sigma\rceil\lceil\tau_{m+1}\rceil}(\kappa\varpi), \Psi_{\lceil\tau\rceil\lceil v_{m+1}\rceil}(\kappa\varpi), \Psi_{\lceil v\rceil\lceil\rho_{m+1}\rceil}(\kappa\varpi) \geq \nabla_0\left(\frac{\kappa}{\varpi^m}\right), \forall \kappa > 0 \text{ and } m \geq 1.$$

Taking limit as $m \rightarrow \infty$ in (3.20)-(3.23) and using $\lim_{m \rightarrow \infty} \nabla_0\left(\frac{\kappa}{\varpi^m}\right) = 1$, for all $\kappa > 0$, we get $\lim_{m \rightarrow \infty} \lceil\rho_m\rceil = \lceil v\rceil$, $\lim_{m \rightarrow \infty} \lceil\sigma_m\rceil = \lceil\rho\rceil$, $\lim_{m \rightarrow \infty} \lceil\tau_m\rceil = \lceil\sigma\rceil$ and $\lim_{m \rightarrow \infty} \lceil v_m\rceil = \lceil\tau\rceil$. This shows with (3.14) that

$$\begin{aligned} \Omega_{\rho\sigma\tau v} &= \lceil\rho\rceil = \lim_{m \rightarrow \infty} \lceil\sigma_m\rceil = \sigma, \quad \Omega_{\sigma\tau v\rho} = \lceil\sigma\rceil = \lim_{m \rightarrow \infty} \lceil\tau_m\rceil = \tau, \\ \Omega_{\tau v\rho\sigma} &= \lceil\tau\rceil = \lim_{m \rightarrow \infty} \lceil v_m\rceil = v, \quad \Omega_{v\rho\sigma\tau} = \lceil v\rceil = \lim_{m \rightarrow \infty} \lceil\rho_m\rceil = \rho. \end{aligned}$$

St₅. We shall prove that $\rho = \sigma = \tau = v$. Put $\Pi(\kappa) = \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau v}(\kappa) \star \Psi_{v\rho}(\kappa)$, for all $\kappa > 0$. Then by (3.1), we can write

$$\begin{aligned} \Psi_{\rho\sigma}(\kappa\varpi) &= \Psi_{\Omega_{\rho\sigma\tau v}\Omega_{\sigma\tau v\rho}}(\kappa\varpi) \geq \Psi_{\lceil\rho\rceil\lceil\sigma\rceil}(\kappa)^{\ell_1} \star \Psi_{\lceil\sigma\rceil\lceil\tau\rceil}(\kappa)^{\ell_2} \star \Psi_{\lceil\tau\rceil\lceil v\rceil}(\kappa)^{\ell_3} \star \Psi_{\lceil v\rceil\lceil\rho\rceil}(\kappa)^{\ell_4} \\ &= \Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\tau v}(\kappa)^{\ell_2} \star \Psi_{v\rho}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_4}; \end{aligned} \quad (3.24)$$

$$\begin{aligned} \Psi_{\sigma\tau}(\kappa\varpi) &= \Psi_{\Omega_{\sigma\tau v\rho}\Omega_{\tau v\rho\sigma}}(\kappa\varpi) \geq \Psi_{\lceil\sigma\rceil\lceil\tau\rceil}(\kappa)^{\ell_1} \star \Psi_{\lceil\tau\rceil\lceil v\rceil}(\kappa)^{\ell_2} \star \Psi_{\lceil v\rceil\lceil\rho\rceil}(\kappa)^{\ell_3} \star \Psi_{\lceil\rho\rceil\lceil\sigma\rceil}(\kappa)^{\ell_4} \\ &= \Psi_{\tau v}(\kappa)^{\ell_1} \star \Psi_{v\rho}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4}; \end{aligned} \quad (3.25)$$

$$\begin{aligned} \Psi_{\tau v}(\kappa\varpi) &= \Psi_{\Omega_{\tau v\rho\sigma}\Omega_{v\rho\sigma\tau}}(\kappa\varpi) \geq \Psi_{\lceil\tau\rceil\lceil v\rceil}(\kappa)^{\ell_1} \star \Psi_{\lceil v\rceil\lceil\rho\rceil}(\kappa)^{\ell_2} \star \Psi_{\lceil\rho\rceil\lceil\sigma\rceil}(\kappa)^{\ell_3} \star \Psi_{\lceil\sigma\rceil\lceil\tau\rceil}(\kappa)^{\ell_4} \\ &= \Psi_{v\rho}(\kappa)^{\ell_1} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\tau v}(\kappa)^{\ell_4}; \end{aligned} \quad (3.26)$$

$$\begin{aligned} \Psi_{v\rho}(\kappa\varpi) &= \Psi_{\Omega_{v\rho\sigma\tau}\Omega_{\rho\sigma\tau v}}(\kappa\varpi) \geq \Psi_{\lceil v\rceil\lceil\rho\rceil}(\kappa)^{\ell_1} \star \Psi_{\lceil\rho\rceil\lceil\sigma\rceil}(\kappa)^{\ell_2} \star \Psi_{\lceil\sigma\rceil\lceil\tau\rceil}(\kappa)^{\ell_3} \star \Psi_{\lceil\tau\rceil\lceil v\rceil}(\kappa)^{\ell_4} \\ &= \Psi_{\rho\sigma}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2} \star \Psi_{\tau v}(\kappa)^{\ell_3} \star \Psi_{v\rho}(\kappa)^{\ell_4}. \end{aligned} \quad (3.27)$$

Using the above four inequalities together, we have

$$\begin{aligned}
\Pi(\kappa\varpi) &= \Psi_{\rho\sigma}(\kappa\varpi) \star \Psi_{\sigma\tau}(\kappa\varpi) \star \Psi_{\tau\nu}(\kappa\varpi) \star \Psi_{\nu\rho}(\kappa\varpi) \\
&\geq \left(\Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\tau\nu}(\kappa)^{\ell_2} \star \Psi_{\nu\rho}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\tau\nu}(\kappa)^{\ell_1} \star \Psi_{\nu\rho}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\nu\rho}(\kappa)^{\ell_1} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\tau\nu}(\kappa)^{\ell_4} \right) \\
&\quad \star \left(\Psi_{\rho\sigma}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2} \star \Psi_{\tau\nu}(\kappa)^{\ell_3} \star \Psi_{\nu\rho}(\kappa)^{\ell_4} \right) \\
&= \left(\Psi_{\rho\sigma}(\kappa)^{\ell_4} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_1} \right) \\
&\quad \star \left(\Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2} \right) \\
&\quad \star \left(\Psi_{\tau\nu}(\kappa)^{\ell_2} \star \Psi_{\tau\nu}(\kappa)^{\ell_1} \star \Psi_{\tau\nu}(\kappa)^{\ell_4} \star \Psi_{\tau\nu}(\kappa)^{\ell_3} \right) \\
&\quad \star \left(\Psi_{\nu\rho}(\kappa)^{\ell_3} \star \Psi_{\nu\rho}(\kappa)^{\ell_2} \star \Psi_{\nu\rho}(\kappa)^{\ell_1} \star \Psi_{\nu\rho}(\kappa)^{\ell_4} \right) \\
&\geq \left(\Psi_{\rho\sigma}(\kappa)^{\ell_4} \cdot \Psi_{\rho\sigma}(\kappa)^{\ell_3} \cdot \Psi_{\rho\sigma}(\kappa)^{\ell_2} \cdot \Psi_{\rho\sigma}(\kappa)^{\ell_1} \right) \\
&\quad \star \left(\Psi_{\sigma\tau}(\kappa)^{\ell_1} \cdot \Psi_{\sigma\tau}(\kappa)^{\ell_4} \cdot \Psi_{\sigma\tau}(\kappa)^{\ell_3} \cdot \Psi_{\sigma\tau}(\kappa)^{\ell_2} \right) \\
&\quad \star \left(\Psi_{\tau\nu}(\kappa)^{\ell_2} \cdot \Psi_{\tau\nu}(\kappa)^{\ell_1} \cdot \Psi_{\tau\nu}(\kappa)^{\ell_4} \cdot \Psi_{\tau\nu}(\kappa)^{\ell_3} \right) \\
&\quad \star \left(\Psi_{\nu\rho}(\kappa)^{\ell_3} \cdot \Psi_{\nu\rho}(\kappa)^{\ell_2} \cdot \Psi_{\nu\rho}(\kappa)^{\ell_1} \cdot \Psi_{\nu\rho}(\kappa)^{\ell_4} \right) \\
&= \Psi_{\rho\sigma}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\sigma\tau}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\quad \star \Psi_{\tau\nu}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\nu\rho}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\geq \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\nu}(\kappa) \star \Psi_{\nu\rho}(\kappa) = \Pi(\kappa).
\end{aligned}$$

Thus, $\Pi(\kappa\varpi) \geq \Pi(\kappa)$ leads to $\Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi}) \geq \Pi(\frac{\kappa}{\varpi^2}) \geq \dots \geq \Pi(\frac{\kappa}{\varpi^m})$, for all $\kappa > 0$ and $m \geq 1$.

Applying (3.24)-(3.27), we get

$$\begin{aligned}
\Psi_{\rho\sigma}(\kappa\varpi) &\geq \Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\tau\nu}(\kappa)^{\ell_2} \star \Psi_{\nu\rho}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_4} \\
&\geq \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\nu}(\kappa) \star \Psi_{\nu\rho}(\kappa) \star \Psi_{\rho\sigma}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}), \\
\Psi_{\sigma\tau}(\kappa\varpi) &\geq \Psi_{\tau\nu}(\kappa)^{\ell_1} \star \Psi_{\nu\rho}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4} \\
&\geq \Psi_{\tau\nu}(\kappa) \star \Psi_{\nu\rho}(\kappa) \star \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}), \\
\Psi_{\tau\nu}(\kappa\varpi) &\geq \Psi_{\nu\rho}(\kappa)^{\ell_1} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\tau\nu}(\kappa)^{\ell_4} \\
&\geq \Psi_{\nu\rho}(\kappa) \star \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\nu}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}), \\
\Psi_{\nu\rho}(\kappa\varpi) &\geq \Psi_{\rho\sigma}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2} \star \Psi_{\tau\nu}(\kappa)^{\ell_3} \star \Psi_{\nu\rho}(\kappa)^{\ell_4} \\
&\geq \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\nu}(\kappa) \star \Psi_{\nu\rho}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}).
\end{aligned}$$

As $m \rightarrow \infty$, we have $\lim_{m \rightarrow \infty} \Pi(\frac{\kappa}{\varpi^m}) = 1$ for all $m \geq 1$, this means that $\Psi_{\rho\sigma}(\kappa\varpi) = \Psi_{\sigma\tau}(\kappa\varpi) = \Psi_{\tau\nu}(\kappa\varpi) = \Psi_{\nu\rho}(\kappa\varpi) = 1$, for all $\kappa > 0$, that is $\rho = \sigma = \tau = \nu$. The uniqueness of ρ follows from (3.1). $\square$

Remark 3.3. In Theorem 3.2, the continuity of $\star$ only discussed at $(1, 1)$, that is if $\{\rho_m\}, \{\sigma_m\} \subset [0, 1]$ are sequences so that $\{\rho_m\} \rightarrow 1$ and $\{\sigma_m\} \rightarrow 1$, therefore $\{\rho_m \star \sigma_m\} \rightarrow 1$, this holds because $\{\rho_m \star \sigma_m\} \geq \{\rho_m \cdot \sigma_m\} \rightarrow 1 \times 1 = 1$.

Example 3.4. Assume that $\zeta = \mathbb{R}$ and $(\mathbb{R}, \Psi^e)$ is defined as Example 2.9. Consider $\wp, \hbar > 0$ and $\varpi \in (0, 1)$ are real numbers so that $8\wp \leq \hbar\varpi$, that is $\frac{\wp}{\varpi} \leq \frac{\hbar}{8}$. For all $\rho, \sigma, \tau, v \in \mathbb{R}$, define $\Omega : \mathbb{R}^4 \rightarrow \mathbb{R}$ and $\Upsilon : \mathbb{R} \rightarrow \mathbb{R}$ by $\Omega_{\rho\sigma\tau v} = \frac{\wp}{2}(\rho - \sigma)$ and $\Upsilon(\rho) = \frac{\hbar}{2}\rho$. It is clear that Υ is continuous, Ω and Υ are commuting and $\Omega(\mathbb{R}^4) = \mathbb{R} = \Upsilon(\mathbb{R})$. Moreover, Ψ^e satisfies

$$\begin{aligned} \Psi_{\Omega_{\rho\sigma\tau v} \Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{v}}}^e(\varpi\kappa) &= \left(e^{|\rho - \hat{\rho}| + |\sigma - \hat{\sigma}|} \right)^{-\frac{\wp}{2\varpi\kappa}} \geq \left(e^{-\frac{2 \max\{|\rho - \hat{\rho}|, |\sigma - \hat{\sigma}|\}}{2\kappa}} \right)^{\frac{\wp}{\varpi}} \\ &\geq \left(e^{-\frac{2 \max\{|\rho - \hat{\rho}|, |\sigma - \hat{\sigma}|\}}{2\kappa}} \right)^{\frac{\hbar}{8}} = \left(e^{-\frac{\hbar}{8\kappa}} \right)^{\max\{|\rho - \hat{\rho}|, |\sigma - \hat{\sigma}|\}} \\ &= \min \left\{ e^{-\frac{\hbar|\rho - \hat{\rho}|}{8\kappa}}, e^{-\frac{\hbar|\sigma - \hat{\sigma}|}{8\kappa}} \right\} \\ &\geq \min \left\{ e^{-\frac{\hbar|\rho - \hat{\rho}|}{8\kappa}}, e^{-\frac{\hbar|\sigma - \hat{\sigma}|}{8\kappa}}, e^{-\frac{\hbar|\tau - \hat{\tau}|}{8\kappa}}, e^{-\frac{\hbar|v - \hat{v}|}{8\kappa}} \right\} \\ &= \min \left\{ e^{-\frac{\hbar|\rho - \hat{\rho}|}{2(4\kappa)}}, e^{-\frac{\hbar|\sigma - \hat{\sigma}|}{2(4\kappa)}}, e^{-\frac{\hbar|\tau - \hat{\tau}|}{2(4\kappa)}}, e^{-\frac{\hbar|v - \hat{v}|}{2(4\kappa)}} \right\} \\ &= \min \left\{ \left(\Psi_{\Upsilon_{\rho} \Upsilon_{\hat{\rho}}}^e(\kappa) \right)^{\frac{1}{4}}, \left(\Psi_{\Upsilon_{\sigma} \Upsilon_{\hat{\sigma}}}^e(\kappa) \right)^{\frac{1}{4}}, \left(\Psi_{\Upsilon_{\tau} \Upsilon_{\hat{\tau}}}^e(\kappa) \right)^{\frac{1}{4}}, \left(\Psi_{\Upsilon_v \Upsilon_{\hat{v}}}^e(\kappa) \right)^{\frac{1}{4}} \right\}. \end{aligned}$$

Thus, by Theorem 3.2, we deduce that Ω and Υ have a QCP.

4 Some related results

In this section, the view of (ζ, ω) as a friable FMS $(\zeta, \Psi^o, \min)$ is used. This tactic permits us to deduce some results involved in the metric space from the corresponding result in the fuzzy setting. Furthermore, without partially ordered set, Theorem 4.1 is just a QCP result, similar to Karapinar and Luong [[30], Corollary 12].

Theorem 4.1. Assume that (ζ, ω) is a CMS and $\Omega : \zeta^4 \rightarrow \zeta$, $\Upsilon : \zeta \rightarrow \zeta$ are two mappings so that:

- $\Omega(\zeta^4) \subseteq \Upsilon(\zeta)$;
- Υ is continuous;
- Υ is commuting with Ω ;

If Ω and Υ satisfy some of the conditions below for $\rho, \sigma, \tau, v, \hat{\rho}, \hat{\sigma}, \hat{\tau}, \hat{v} \in \zeta$:

(i) for some $0 < \varpi < 1$,

$$\omega_{\Omega_{\rho\sigma\tau v} \Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{v}}} \leq \varpi \max \{ \omega_{\Upsilon_{\rho} \Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma} \Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau} \Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_v \Upsilon_{\hat{v}}} \}.$$

(ii) for some $0 < \varpi < 1$ and some $\wp_1, \wp_2, \wp_3, \wp_4 \in [0, \frac{1}{3}]$,

$$\omega_{\Omega_{\rho\sigma\tau v} \Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{v}}} \leq \varpi (\wp_1 \omega_{\Upsilon_{\rho} \Upsilon_{\hat{\rho}}} + \wp_2 \omega_{\Upsilon_{\sigma} \Upsilon_{\hat{\sigma}}} + \wp_3 \omega_{\Upsilon_{\tau} \Upsilon_{\hat{\tau}}} + \wp_4 \omega_{\Upsilon_v \Upsilon_{\hat{v}}}).$$

(iii) for some $\wp_1, \wp_2, \wp_3, \wp_4 \in [0, 1)$ with $\wp_1 + \wp_2 + \wp_3 + \wp_4 < 1$,

$$\omega_{\Omega_{\rho\sigma\tau\nu}\Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\nu}}} \leq \wp_1\omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}} + \wp_2\omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}} + \wp_3\omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}} + \wp_4\omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}}.$$

Then there is a unique point $\rho \in \zeta$ so that $\rho = \Upsilon\rho = \Omega_{\rho\rho\rho\rho}$.

Proof. (i) Suppose that Ψ^o defined as Example 2.9. The completeness of (ζ, ω) leads to $(\zeta, \Psi^o, \min)$ is a complete FMS. Fix $\rho, \sigma, \tau, \nu, \hat{\rho}, \hat{\sigma}, \hat{\tau}, \hat{\nu} \in \zeta$ and $\kappa > 0$, and we will achieve (3.1) by taking $\ell_1 = \ell_2 = \ell_3 = \ell_4 = \frac{1}{4}$ and $\star = \min$. If $\Psi_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}^o(\kappa) = 0$ or $\Psi_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}^o(\kappa) = 0$ or $\Psi_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}^o(\kappa) = 0$ or $\Psi_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}}^o(\kappa) = 0$, then (3.1) is clear. Assume that $\Psi_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}^o(\kappa) = 1$, $\Psi_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}^o(\kappa) = 1$, $\Psi_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}^o(\kappa) = 1$ and $\Psi_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}}^o(\kappa) = 1$. This implies that $\omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}} < \kappa$, $\omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}} < \kappa$, $\omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}} < \kappa$ and $\omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} < \kappa$. Therefore $\kappa > \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \}$ and

$$\varpi\kappa > \varpi \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \geq \omega_{\Omega_{\rho\sigma\tau\nu}\Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\nu}}}.$$

Thus, $\Psi_{\Omega_{\rho\sigma\tau\nu}\Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\nu}}}^o(\kappa\varpi) = 1$ and (3.1) holds.

(ii) Here

$$\begin{aligned} \omega_{\Omega_{\rho\sigma\tau\nu}\Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\nu}}} &\leq \varpi (\wp_1\omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}} + \wp_2\omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}} + \wp_3\omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}} + \wp_4\omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}}) \\ &\leq \varpi \left(\frac{1}{4}\omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}} + \frac{1}{4}\omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}} + \frac{1}{4}\omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}} + \frac{1}{4}\omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \right) \\ &= \frac{\varpi}{4} (\omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}} + \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}} + \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}} + \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}}) \\ &\leq \frac{\varpi}{4} \times 4 \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \\ &= \varpi \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \}. \end{aligned}$$

(iii) If $\varpi = \wp_1 + \wp_2 + \wp_3 + \wp_4 < 1$,

$$\begin{aligned} \omega_{\Omega_{\rho\sigma\tau\nu}\Omega_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\nu}}} &\leq \wp_1\omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}} + \wp_2\omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}} + \wp_3\omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}} + \wp_4\omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \\ &\leq \wp_1 \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \\ &\quad + \wp_2 \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \\ &\quad + \wp_3 \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \\ &\quad + \wp_4 \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \\ &= (\wp_1 + \wp_2 + \wp_3 + \wp_4) \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \} \\ &= \varpi \max \{ \omega_{\Upsilon_{\rho}\Upsilon_{\hat{\rho}}}, \omega_{\Upsilon_{\sigma}\Upsilon_{\hat{\sigma}}}, \omega_{\Upsilon_{\tau}\Upsilon_{\hat{\tau}}}, \omega_{\Upsilon_{\nu}\Upsilon_{\hat{\nu}}} \}. \end{aligned}$$

□

Example 4.2. Consider $\zeta = \mathbb{R}$, $\omega(\rho, \sigma) = |\rho - \sigma|$ for all $\rho, \sigma \in \mathbb{R}$, and for all $\ell_1, \ell_2, \ell_3, \ell_4, \xi, \Psi \in \mathbb{R}$ with $\Psi > |\ell_1| + |\ell_2| + |\ell_3| + |\ell_4|$. Define the mappings $\Omega : \mathbb{R}^4 \rightarrow \mathbb{R}$ and $\Upsilon : \mathbb{R} \rightarrow \mathbb{R}$ by $\Omega_{\rho\sigma\tau\nu} = \frac{(\ell_1\rho + \sigma\ell_2 + \tau\ell_3 + \nu\ell_4 + \xi)}{\Psi}$ and $\Upsilon\rho = \rho$ for all $\rho, \sigma, \tau, \nu \in \mathbb{R}$. It is easy to check that the two mappings verify the hypothesis (iii) of Theorem 4.1 and $(\rho_0, \rho_0, \rho_0, \rho_0)$ is a unique QCP of Ω and Υ , where $\rho_0 = \frac{\xi}{\Psi - \ell_1 - \ell_2 - \ell_3 - \ell_4}$, also $\Omega_{\rho_0\rho_0\rho_0\rho_0} = \rho_0$.

Now, we can generalize Theorem 1.7 [13] by obtaining a coupled coincidence point for $\Omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ and Υ . Only take $\ell_1 = \ell_2 = \frac{1}{2}$ as follows:

Corollary 4.3. Assume that $\star$ is a κ -norm of Υ -type so that $\mu \star \kappa \geq \mu\kappa$, for all $\mu, \kappa \in [0, 1]$. Suppose that $(\zeta, \Psi, \star)$ is a complete FMS and $\Omega : \zeta^2 \rightarrow \zeta$, $\Upsilon : \zeta \rightarrow \zeta$ are two mappings so that

- $\Omega(\zeta^2) \subseteq \Upsilon(\zeta)$,
- Υ is continuous,
- Υ is commuting with Ω ,
- for all $\rho, \sigma, \hat{\rho}, \hat{\sigma} \in \zeta$,

$$\Psi_{\Omega_{\rho\sigma}\Omega_{\hat{\rho}\hat{\sigma}}}(\kappa\varpi) \geq \Psi_{\Upsilon\rho\Upsilon\hat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\Upsilon\sigma\Upsilon\hat{\sigma}}(\kappa)^{\ell_2},$$

where $\varpi \in (0, 1)$ and ℓ_1, ℓ_2 are real numbers in $[0, 1]$ so that $\ell_1 + \ell_2 \leq 1$.

Then there exists a unique $\rho \in \zeta$ such that $\rho = \Upsilon\rho = \Omega_{\rho\rho}$.

Proof. Define $\ell_3 = \ell_4 = 0$ and $\Omega^* : \zeta^4 \rightarrow \zeta$ as $\Omega^*_{\rho\sigma\tau v} = \Omega_{\rho\sigma}$ for all $\rho, \sigma, \tau, v \in \zeta$. Then $\Omega^*(\zeta^4) = \Omega(\zeta^2) \subseteq \Upsilon(\zeta)$ and Ω^* is commuting with Υ , that is $\Upsilon\Omega^*_{\rho\sigma\tau v} = \Upsilon\Omega_{\rho\sigma} = \Omega_{\Upsilon\rho\Upsilon\sigma} = \Omega^*_{\Upsilon\rho\Upsilon\sigma\Upsilon\tau\Upsilon v}$. Also, one can write

$$\begin{aligned} \Psi_{\Omega^*_{\rho\sigma\tau v}\Omega^*_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{v}}}(\kappa\varpi) &= \Psi_{\Omega_{\rho\sigma}\Omega_{\hat{\rho}\hat{\sigma}}}(\kappa\varpi) \geq \Psi_{\Upsilon\rho\Upsilon\hat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\Upsilon\sigma\Upsilon\hat{\sigma}}(\kappa)^{\ell_2} \\ &= \Psi_{\Upsilon\rho\Upsilon\hat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\Upsilon\sigma\Upsilon\hat{\sigma}}(\kappa)^{\ell_2} \star 1 \star 1 \\ &\geq \Psi_{\Upsilon\rho\Upsilon\hat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\Upsilon\sigma\Upsilon\hat{\sigma}}(\kappa)^{\ell_2} \star \Psi_{\Upsilon\tau\Upsilon\hat{\tau}}(\kappa)^{\ell_3} \star \Psi_{\Upsilon v\Upsilon\hat{v}}(\kappa)^{\ell_4}. \end{aligned}$$

Hence by Theorem 3.2, there is $\rho \in \zeta$ so that $\Upsilon\rho = \Omega^*_{\rho\sigma\tau v}$. If $\sigma \in \zeta$ satisfies $\Omega_{\sigma\sigma} = \Upsilon\sigma$, then $\Upsilon\sigma = \Omega_{\sigma\sigma} = \Omega^*_{\sigma\sigma\sigma\sigma}$. Thus, $\sigma = \rho$. $\square$

The proof of the corollary below follows immediately by Theorem 4.1.

Corollary 4.4. [13] Assume that (ζ, ω) is a CMS and $\Omega : \zeta^2 \rightarrow \zeta$, $\Upsilon : \zeta \rightarrow \zeta$ are two mappings so that:

- $\Omega(\zeta^2) \subseteq \Upsilon(\zeta)$;
- Υ is continuous;
- Υ is commuting with Ω .

If Ω and Υ satisfy some of the conditions below for $\rho, \sigma, \hat{\rho}, \hat{\sigma} \in \zeta$:

(i) for some $0 < \varpi < 1$,

$$\omega_{\Omega_{\rho\sigma}\Omega_{\hat{\rho}\hat{\sigma}}} \leq \varpi \max \{ \omega_{\Upsilon\rho\Upsilon\hat{\rho}}, \omega_{\Upsilon\sigma\Upsilon\hat{\sigma}} \}.$$

(ii) for some $0 < \varpi < 1$ and some $\wp_1, \wp_2 \in [0, \frac{1}{2}]$,

$$\omega_{\Omega_{\rho\sigma}\Omega_{\hat{\rho}\hat{\sigma}}} \leq \varpi (\wp_1 \omega_{\Upsilon\rho\Upsilon\hat{\rho}} + \wp_2 \omega_{\Upsilon\sigma\Upsilon\hat{\sigma}}).$$

(iii) for some $\wp_1, \wp_2 \in [0, 1]$ with $\wp_1 + \wp_2 + \wp_3 + \wp_4 < 1$,

$$\omega_{\Omega_{\rho\sigma}\Omega_{\hat{\rho}\hat{\sigma}}} \leq \wp_1 \omega_{\Upsilon\rho\Upsilon\hat{\rho}} + \wp_2 \omega_{\Upsilon\sigma\Upsilon\hat{\sigma}}.$$

Then there is a unique point $\rho \in \zeta$ so that $\rho = \Upsilon\rho = \Omega_{\rho\rho}$.

5 Supportive applications

This section is specially prepared to highlight the importance of theoretical results and how to use them in obtaining the existence of the solution to a Lipschitzian and an integral quadruple system.

5.1 Lipschitzian quadruple systems

Assume that $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \mathbb{R} \rightarrow \mathbb{R}$ are LMs and $\wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{R}$ are real numbers. Let $\mathcal{U} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\mathcal{U}(\rho) = \sum_{i=1}^4 \wp_i \Gamma_i(\rho)$ for all $\rho \in \mathbb{R}$, then $\mathcal{U}$ is also LM and $\varpi_{\mathcal{U}} \leq \sum_{i=1}^4 |\wp_i| \varpi_{\Gamma_i}$. It is easy to see that if $\Lambda = \sum_{i=1}^4 |\wp_i| \varpi_{\Gamma_i} < 1$, then $\mathcal{U}$ is a contraction, thus there is a unique $\rho_0 \in \mathbb{R}$ so that $\mathcal{U}_{\rho_0} = \rho_0$. Now, for all $\rho, \sigma, \tau, v \in \mathbb{R}$, define $\Omega : \zeta^4 \rightarrow \zeta$ as

$$\Omega_{\rho\sigma\tau v} = \wp_1 \Gamma_1(\rho) + \wp_2 \Gamma_2(\sigma) + \wp_3 \Gamma_3(\tau) + \wp_4 \Gamma_4(v).$$

It is obvious that for all $\rho \in \mathbb{R}$, $\Omega_{\rho\rho\rho\rho} = \mathcal{U}_{\rho}$. Also, we have

$$\begin{aligned} \omega(\Omega_{\rho_1\rho_2\rho_3\rho_4}, \Omega_{\sigma_1\sigma_2\sigma_3\sigma_4}) &= \sum_{i=1}^4 |\wp_i| |\Gamma_i(\rho_i) - \Gamma_i(\sigma_i)| \\ &\leq \sum_{i=1}^4 |\wp_i| \varpi_{\Gamma_i} |\rho_i - \sigma_i| \leq \Lambda \max_{1 \leq j \leq 4} \omega(\rho_j, \sigma_j). \end{aligned}$$

If $\Lambda < 1$, then Ω satisfies (3.1) with $\mathcal{T}\rho = \rho$ for all $\rho \in \mathbb{R}$.

According to the above results we can state the corollary below.

Corollary 5.1. *Assume that $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \mathbb{R} \rightarrow \mathbb{R}$ are LMs and $\wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{R}$ so that $\sum_{i=1}^4 |\wp_i| \varpi_{\Gamma_i} < 1$, then the system*

$$\begin{cases} \rho = \wp_1 \Gamma_1(\rho) + \wp_2 \Gamma_2(\sigma) + \wp_3 \Gamma_3(\tau) + \wp_4 \Gamma_4(v), \\ \sigma = \wp_1 \Gamma_1(\sigma) + \wp_2 \Gamma_2(\tau) + \wp_3 \Gamma_3(v) + \wp_4 \Gamma_4(\rho), \\ \tau = \wp_1 \Gamma_1(\tau) + \wp_2 \Gamma_2(v) + \wp_3 \Gamma_3(\rho) + \wp_4 \Gamma_4(\sigma), \\ v = \wp_1 \Gamma_1(v) + \wp_2 \Gamma_2(\rho) + \wp_3 \Gamma_3(\sigma) + \wp_4 \Gamma_4(\tau). \end{cases} \quad (5.1)$$

has a unique solution $(\rho_0, \rho_0, \rho_0, \rho_0)$, where ρ_0 is the only real solution of $\rho = \sum_{i=1}^4 \wp_i \Gamma_i(\rho)$.

Example 5.2. Consider the system

$$\begin{cases} 24 \cos \rho - \frac{18}{1+\sigma^2} + 144 = 120\rho + \left(\frac{4}{1+\tau^2}\right)^2 - 15 \arcsin v, \\ 24 \cos \sigma - \frac{18}{1+\tau^2} + 144 = 120\sigma + \left(\frac{4}{1+v^2}\right)^2 - 15 \arcsin \rho, \\ 24 \cos \tau - \frac{18}{1+v^2} + 144 = 120\tau + \left(\frac{4}{1+\rho^2}\right)^2 - 15 \arcsin \sigma, \\ 24 \cos v - \frac{18}{1+\rho^2} + 144 = 120\rho + \left(\frac{4}{1+\sigma^2}\right)^2 - 15 \arcsin \tau, \end{cases} \quad (5.2)$$

If we select $\Gamma_1(\rho) = 6 + \cos \rho$, $\Gamma_2(\rho) = \frac{1}{1+\rho^2}$, $\Gamma_3(\rho) = \left(\frac{1}{1+\rho^2}\right)^2$ and $\Gamma_4(\rho) = \arcsin \rho$. Then $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are LMs, and $\varpi_{\Gamma_1} = \varpi_{\Gamma_4} = 1$, $\varpi_{\Gamma_2} = \frac{3\sqrt{3}}{8}$ and $\varpi_{\Gamma_3} = \frac{27}{64}$. Let $\wp_1 = \frac{1}{5}$, $\wp_2 = -\frac{3}{20}$, $\wp_3 = \frac{2}{15}$ and $\wp_4 = \frac{1}{8}$. Then $\sum_{i=1}^4 |\wp_i| \varpi_{\Gamma_i} = 0.479 < 1$. Because system (5.2) is a special case of system (5.1). So the problem (5.2) has a unique solution $(\rho_0, \rho_0, \rho_0, \rho_0)$, where ρ_0 represent a unique solution of

$$24 \cos \rho - \frac{18}{1 + \rho^2} + 144 = 120\rho + \left(\frac{4}{1 + \rho^2}\right)^2 - 15 \arcsin \rho.$$

By program of matlab or mathematica or the bisection method, we can approximate value $\rho_0 = 1.26624$.

5.2 An integral quadruple system

Assume that $\ell_1, \ell_2 \in \mathbb{R}$ with $\ell_1 < \ell_2$ and set $\varphi = [\ell_1, \ell_2]$. Let $\zeta = L^1(\varphi)$ equipped with $\omega_1(\Gamma, \aleph) = \int_{\varphi} |\Gamma(\kappa), \aleph(\kappa)| \omega \kappa$, where $\int$ is the Lebesgue integral. It is clear that $(L^1(\varphi), \omega_1)$ is a CMS. Suppose that $\varpi, \wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{R}$ are real numbers and $\beth : \mathbb{R}^4 \rightarrow \mathbb{R}$ is a mapping satisfying $\beth(0, 0, 0, 0) = 0$ and

$$|\beth_{\rho_1 \rho_2 \rho_3 \rho_4} - \beth_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}| \leq \varpi \sum_{i=1}^4 \wp_i |\rho_i - \sigma_i|, \quad \forall (\rho_1, \rho_2, \rho_3, \rho_4), (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{R}^4.$$

If $B \in \mathbb{R}$, we want to find the functions $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in L^1(\varphi)$ so that

$$\Gamma_i(\rho) = B + \int_{[\ell_1, \rho]} \beth(\Gamma_i(\kappa), \Gamma_{i+1}(\kappa), \Gamma_{i+2}(\kappa), \Gamma_{i+3}(\kappa)) \omega \kappa, \quad (5.3)$$

is fulfilled for all $\rho \in \varphi$, $i = 1, 2, 3, 4$.

For $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in L^1(\varphi)$ and all $\rho \in \varphi$, define the mapping Ω by

$$\Omega_{\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4}(\rho) = B + \int_{[\ell_1, \rho]} \beth(\Gamma_1(\kappa), \Gamma_2(\kappa), \Gamma_3(\kappa), \Gamma_4(\kappa)) \omega \kappa. \quad (5.4)$$

According to (5.3) and (5.4) we see that $\Omega_{\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4} \in L^1(\varphi)$, hence $\Omega : L^1(\varphi)^4 \rightarrow L^1(\varphi)$ is well defined.

In addition to,

$$\begin{aligned} & \omega_1(\Omega_{\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4}, \Omega_{\aleph_1 \aleph_2 \aleph_3 \aleph_4}) \\ &= \int_{\varphi} |\Omega_{\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4}(\rho) - \Omega_{\aleph_1 \aleph_2 \aleph_3 \aleph_4}(\rho)| \omega \rho \\ &= \int_{\varphi} \left(\int_{[\ell_1, \rho]} |\beth(\Gamma_1(\kappa), \Gamma_2(\kappa), \Gamma_3(\kappa), \Gamma_4(\kappa)) - \beth(\aleph_1(\kappa), \aleph_2(\kappa), \aleph_3(\kappa), \aleph_4(\kappa))| \omega \kappa \right) \omega \rho \\ &\leq \int_{\varphi} \left(\int_{[\ell_1, \rho]} \varpi \sum_{i=1}^4 \wp_i |\Gamma_i - \aleph_i| \omega \kappa \right) \omega \rho \\ &\leq \varpi \sum_{i=1}^4 \wp_i \int_{\varphi} \left(\int_{\varphi} |\Gamma_i - \aleph_i| \omega \kappa \right) \omega \rho \\ &= \varpi \sum_{i=1}^4 \wp_i \int_{\varphi} \omega_1(\Gamma_i, \aleph_i) \omega \rho = \varpi(\ell_2 - \ell_1) \sum_{i=1}^4 \wp_i \omega_1(\Gamma_i, \aleph_i). \end{aligned}$$

If we take $\varpi(\ell_2 - \ell_1) \sum_{i=1}^4 \wp_i = \Lambda < 1$, then Ω justify (3.1) with $\mathfrak{T}(\Gamma) = \Gamma$, for all $\Gamma \in L^1(\varphi)$. We conclude from the above results that system (5.3) has a unique solution $(\Gamma_0, \Gamma_0, \Gamma_0, \Gamma_0)$, where Γ_0 is a unique solution of the equation

$$\Gamma_0(\rho) = B + \int_{[\ell_1, \rho]} \mathfrak{Z}(\Gamma_0(\kappa), \Gamma_0(\kappa), \Gamma_0(\kappa), \Gamma_0(\kappa)) \omega \kappa,$$

for $\Gamma_0 \in L^1(\varphi)$ and all $\rho \in \varphi$. This is a simple application of the Banach contraction principle.

Availability of data and material

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Competing interests

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Author Contributions

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