Evolution behaviour of kink breathers and lump- M -solitons ($M \rightarrow \infty$) for the (3+1)-dimensional Hirota-Satsuma-Ito-like equation

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Abstract In this paper, some novel lump solutions and interaction phenomenon between lump and kink M -soliton are investigated. Firstly, we study the evolution and degeneration behaviour of kink breather wave solution with different forms for the (3+1)-dimensional Hirota-Satsuma-Ito-like equation by symbolic computation and Hirota bilinear form. In the process of degeneration of breather waves, some novel lump solutions are derived by the limit method. In addition, M -fissionable soliton and the interaction phenomenon between lump solutions and kink M -solitons (lump- M -solitons) are investigated, the theorem and corollary about the conditions for the existence of the interaction phenomenon are given and proved further. The lump- M -solitons with different types is studied to illustrate the correctness and availability of the given theorem and corollary, such as lump-cos type, lump-cosh-exponential type, lump-cosh-cos-cosh type. Several three-dimensional figures are drawn to better depict the nonlinear dynamic behaviours including the oscillation of breather wave, the emergence of lump, the evolution behaviour of fission and fusion of lump- M -solitons and so on.

Keywords: Hirota bilinear form; Breather wave; Lump- M -soliton; Interaction phenomenon

1 Introduction

It is well known that nonlinear partial differential equations (PDEs) involved both space and time variables, are fundamental models in nonlinear science, especially in nonlinear dynamics. The wave phenomenon of nonlinear PDEs are playing an essential role in studying various areas of physics and mathematics, such as nonlinear optics, plasmas, oceanography and Bose-Einstein condensates [1-6]. Solitons, breathers, lumps and rogue waves are typical wave models to study nonlinear scientific issues [7,8,9]. Lump wave is a special kind of rational function wave with the characteristics of energy concentration and localization property in the space [10-15]. Therefore, a series of methods has been quickly developed to obtain lump wave including the first mathematical description of lump wave [16], the long wave limit method [17,18], the direct method [19], the parameter limit method [20], etc. An effective method was

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proposed to study the M -lump solutions of the integrable systems in [18], this positive method has attracted a lot of researchers' attention. Many valuable and interesting results have sprung up, lump waves of the (2+1)-dimensional nonlinear equations [21-26]. Lump solutions of the (3+1)-dimensional nonlinear systems, such as potential Yu-Toda-Sasa-Fukuyama equation [27], Sharma-Tasso-Olver-like equation [28], B-type Kadomtsev-Petviashvili-Boussinesq equation [29,30]. Lump waves of the (4+1)-dimensional nonlinear equations [20]. Inspired by the above researches, we mainly focus on the (3+1)-dimensional Hirota-Satsuma-Ito-like (HSII) to study the evolution and degeneration behaviour of breathers and interaction solutions with kink M -solitons in this work. By applying the Hirota bilinear method [31,32] and other systematic method, the structures of solutions for this equation are analyzed.

The (3+1)-dimensional Hirota-Satsuma-Ito-like equation is a new nonlinear wave model which is firstly proposed in [33], it is generated from (2+1)-dimensional Hirota-Satsuma-Ito equation [34] and Hirota-Satsuma equation [35], which reflects abundant physical meaning in nonlinear shallow waves. The equation reads

$$\alpha u_{yt} + u_{xxx} + 3u_x u_{xt} + 3u_{xx} u_t + \beta u_{zt} = 0 \quad (1)$$

where α and β are real nonzero constants, $u(x, y, z, t)$ describes the unidirectional propagation of shallow water waves. The lump solutions are obtained and some interaction phenomenon are discussed in [33]. Based on the previous research, we mainly focuses on investigating evolution and deformation of kink breather waves, the emergence of lump solutions, interaction phenomenon of lump- M -soliton and analyzing dynamical behaviour of each kinds of solutions.

However, to the best of our knowledge, Eq.(1) has many properties and exact solutions that have not been studied, which deserve further study and discussion. The evolution and degradation of breather wave solutions, and the superposition between lump and kink M -soliton ($M \rightarrow \infty$) with different kinds has not been studied thoroughly. In this manuscript, we aim to construct breather wave solutions firstly, as degeneration of breather waves, lump waves emerge. Then the interaction between lump and M -soliton solutions of Eq.(1) is given. The structure as follows: In section 2, Based on Hirota bilinear form, we construct kink breather waves with different forms, lump waves will be derived from breather waves by the limit method. In section 3, M -fissionable soliton and sufficient conditions of the existence for interaction between lump and kink M -solitons will be obtained, theoretical proof of superposition behaviour and examples will be given. Finally some conclusions are concluded in the last section.

2 Evolution and degeneration from breather to Lump solution

In this section, based on the Hirota bilinear method, the breather wave solutions of Eq.(1) with different structural forms are constructed. Under the action of oscillation of breather wave, some lump or lump-type solutions are found out by using parameter limit method [9,21]. Through Painlevé analysis,

we assume

$$u(x, y, z, t) = 2(\ln f)_x. \quad (2)$$

where f is an unknown real-valued function about x, y, z, t . Eq.(1) can be converted to the bilinear operator D as follows

$$(D_x^3 D_t + \alpha D_y D_t + \beta D_z D_t) f \cdot f = 0 \quad (3)$$

The Hirota bilinear operator $D_x^m D_t^n$ are defined by [31] ($n, m \geq 0$)

$$D_x^l D_y^m D_z^n D_t^r f(x, t) \cdot g(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^r f(x, y, z, t) g(x', y', z', t')|_{x'=x, y'=y, z'=z, t'=t}.$$

On the basis of [8,9], we choose the following test function

$$f(x, y, z, t) = \delta_1 \cosh \xi_1 + \delta_2 \cos \xi_2 + \delta_3 \cosh \xi_3 \quad (4)$$

where $\xi_i = k_i(a_i x + b_i y + c_i z + d_i t + \gamma_i)$, $k_i, a_i, b_i, c_i, d_i, \gamma_i$ ($i = 1, 2, 3$) are free real parameters to be determined. Substituting Eq.(4) into Eq.(3) with computer algebra software Mathematic 12, we get two sets of parameter relations:

Case 1:

$$\begin{cases} a_1 = \frac{a_3 k_3}{k_1}, a_2 = \frac{a_3 d_3 k_3^2 (\delta_1^2 - \delta_3^2)}{d_2 \delta_2^2 k_2^2}, d_1 = -\frac{d_3 k_3}{k_1}, b_1 = -\frac{\beta c_1}{\alpha} + a_3^3 \left(-\frac{k_3^3}{\alpha k_1} + \frac{3d_3^2 k_3^5 (\delta_1^2 - \delta_3^2)^2}{\alpha d_2^2 \delta_2^4 k_1 k_2^2}\right), \\ b_2 = -\frac{\beta c_2}{\alpha} + a_3^3 d_3 k_3^4 (\delta_1^2 - \delta_3^2) \left(-\frac{3}{\alpha d_2 k_2^2 \delta_2^2} + \frac{d_3^2 k_3^2 (\delta_1^2 - \delta_3^2)^2}{\alpha d_2^2 \delta_2^2 k_2^4}\right), b_3 = -\frac{\beta c_3}{\alpha} + a_3^3 \left(-\frac{k_3^2}{\alpha} + \frac{3d_3^2 k_3^4 (\delta_1^2 - \delta_3^2)^2}{\alpha d_2^2 \delta_2^2 k_2^2}\right). \end{cases} \quad (5)$$

where $\delta_i k_i d_i \neq 0$ ($i = 1, 2, 3$). A multi-breather solitary solution is constructed by inserting Eq.(4) and Eq.(5) into Eq.(2) as follows

$$u(x, y, z, t) = \frac{2(a_1 k_1 \delta_1 \sinh \xi_1 + a_2 k_2 \delta_2 \sin \xi_2 + a_3 k_3 \delta_3 \sinh \xi_3)}{\delta_1 \cosh \xi_1 + \delta_2 \cos \xi_2 + \delta_3 \cosh \xi_3} \quad (6)$$

where

$$\begin{cases} \xi_1 = k_1 \left(\frac{a_3 k_3}{k_1} x + \left(-\frac{\beta c_1}{\alpha} + a_3^3 \left(-\frac{k_3^3}{\alpha k_1} + \frac{3d_3^2 k_3^5 (\delta_1^2 - \delta_3^2)^2}{\alpha d_2^2 \delta_2^4 k_1 k_2^2} \right) \right) y + c_1 z - \frac{d_3 k_3}{k_1} t + \gamma_1, \right. \\ \xi_2 = k_2 \left(\left(\frac{a_3 d_3 k_3^2 (\delta_1^2 - \delta_3^2)}{d_2 \delta_2^2 k_2^2} \right) x + \left(-\frac{\beta c_2}{\alpha} + a_3^3 d_3 k_3^4 (\delta_1^2 - \delta_3^2) \left(-\frac{3}{\alpha d_2 k_2^2 \delta_2^2} + \frac{d_3^2 k_3^2 (\delta_1^2 - \delta_3^2)^2}{\alpha d_2^2 \delta_2^2 k_2^4} \right) \right) y + c_2 z + d_2 t + \gamma_2, \right. \\ \left. \xi_3 = k_3 \left(a_3 x + \left(-\frac{\beta c_3}{\alpha} + a_3^3 \left(-\frac{k_3^2}{\alpha} + \frac{3d_3^2 k_3^4 (\delta_1^2 - \delta_3^2)^2}{\alpha d_2^2 \delta_2^2 k_2^2} \right) \right) y + c_3 z + d_3 t + \gamma_3. \right. \end{cases} \quad (7)$$

The solution u expressed in Eq.(6) is a kind of double breather wave solution, which takes on the characteristics of double solitary waves with the variable ξ_1 and ξ_3 , and the characteristics of periodic solitary waves with the variable ξ_2 . Besides, under the action of the function $\cos \xi_2$, u also shows the characteristics of local oscillation. Therefore, the breather wave solutions with these characteristics are named double breathers [36,37]. The oscillation behaviour of double breathers are presented in Fig.1(a,b,c), when we choose the parameters as $\alpha = 20, \beta = 1, \delta_1 = 2, \delta_2 = 4, \delta_3 = \frac{1}{4}, a_3 = \frac{1}{4}, k_1 = 2, k_2 = \frac{3}{2}, k_3 = \frac{1}{4}, c_1 = c_2 = c_3 = 1, d_2 = 4, d_3 = \frac{1}{2}, z = 0$.

Fig. 1: Spatial oscillation of double breathers in Eq.(6) as (a) $\gamma_i = -2$, (b) $\gamma_i = 0$, (c) $\gamma_i = 2 (i = 1, 2, 3)$, at $t = 0$.

In order to construct the lump solutions from Eq.(6), the parameters $\delta_i (i = 1, 2, 3)$ must satisfy:

$$\lim_{k_i \rightarrow 0} (\delta_1 + \delta_2 + \delta_3) = 0 \quad (8)$$

so we take $k_2 = rk_1$, $k_3 = sk_1$, $\delta_1 = \mu^2 \cosh(lk_1)$, $\delta_2 = -(\mu^2 + \nu^2) \cosh(mk_1)$, $\delta_3 = \nu^2 \cosh(nk_1)$, where μ, ν are real numbers, m, n, l, r, s are real number or pure imaginary numbers. Letting $k_1 \rightarrow 0$, lump solution are given by

$$u(x, y, z, t) = \frac{4(s\mu^2 a_3 \theta_1 + \frac{2s^2(\mu^2 - \nu^2 d_2) a_3 d_3}{(\mu^2 + \nu^2) d_2} \theta_2 + s^2 \nu^2 a_3 \theta_3)}{\mu^2 \theta_1^2 + r^2 (\mu^2 + \nu^2) \theta_2^2 + s^2 \nu^2 \theta_3^2 + (m^2 + l^2) \mu^2 + (m^2 + n^2) \nu^2} \quad (9)$$

where

$$\lim_{k_i \rightarrow 0} \frac{\xi_i}{k_i} = \theta_i (i = 1, 2, 3).$$

$$\theta_1 = sa_3 x - \frac{\beta c_1}{\alpha} y + c_1 z - sd_3 t + \gamma_1, \theta_2 = \frac{s^2(\mu^2 - \nu^2) a_3 d_3}{r^2(\mu^2 + \nu^2) d_2} x - \frac{\beta c_2}{\alpha} y + c_2 z + d_2 t + \gamma_2, \theta_3 = a_3 x - \frac{\beta c_3}{\alpha} y + c_3 z + d_3 t + \gamma_3.$$

When $k_1 \rightarrow 0$, the characteristics of period and breather of solution Eq.(6) are degenerated as a rational function solution in Eq.(9). Besides, the solution u in Eq.(9) goes to 0 when $|x|, |y|, |z|, |t|$ tend to ∞ in any direction, and shows the characteristic of polynomial attenuation on all spatial variables. Meanwhile, we know that the solution is non-singular as the parameters satisfy $(m^2 + l^2)\mu^2 + (m^2 + n^2)\nu^2 > 0$ from

the expression of Eq.(9). From Fig.2, the lump solution u has a upward peak and a downward valley, the lump solution with this structures is called lump or lump-type solution [12,14].

Fig. 2: Spatial structure of lump solution in Eq.(9) as $\alpha = \beta = 1, l = n = 0, s = 1, r = 2, m = 1, \mu = \frac{1}{4}, \nu = 1, a_3 = 1, c_1 = 0, c_2 = -3, c_3 = -1, d_2 = 0, d_3 = 1$ and (a) $\gamma_i = -2$, (b) $\gamma_i = 0$, (c) $\gamma_i = 2(i = 1, 2, 3)$, at $z = 0, t = 0$.

Case 2:

$$\delta_3 = 0, b_1 = \frac{-\beta c_1 - a_1^3 k_1^2 + 3a_1 a_2^2 k_2^2}{\alpha}, b_2 = \frac{-\beta c_2 + a_2^3 k_2^2 - 3a_1^2 a_2 k_1^2}{\alpha}, d_2 = -\frac{a_1 d_1 k_1^2 \delta_1^2}{a_2 k_2^2 \delta_2^2} \quad (10)$$

so we get

$$f(x, y, z, t) = \delta_1 \cosh(k_1(a_1 x + b_1 y + c_1 z + d_1 t + \gamma_1)) + \delta_2 \cos(k_2(a_2 x + b_2 y + c_2 z + d_2 t + \gamma_2)) \quad (11)$$

An exact one breather-wave solution of HSII equation is obtained by combining Eq.(10), (11) and (2), the spatial structure is shown in Fig.3.

Similarly, we take $k_2 = r k_1, \delta_1 = \cosh(l k_1), b_2 = -\cos(m k_1)$, and let $k_1 \rightarrow 0$. New type lump solution is obtained (Fig.4).

$$u(x, y, z, t) = \frac{4(a_1 \zeta_1 + r a_2 \zeta_2)}{\zeta_1^2 + \zeta_2^2 + l^2 + m^2}. \quad (12)$$

where $\zeta_1 = a_1x - \frac{\beta c_1}{\alpha}y + c_1z + d_1t + \gamma_1$, $\zeta_2 = ra_2x - \frac{\beta rc_2}{\alpha}y + rc_2z - \frac{a_1d_1}{ra_2}t + \gamma_2$, $ra_2 \neq 0$, and $l^2 + m^2 > 0$. When $a_2 = 0$, lump solution as follows:

$$u(x, y, z, t) = \frac{4(a_1\zeta_1 + ra_2\zeta_2)}{\zeta_1^2 + \zeta_2^2 + l^2 + m^2}.$$

where $\zeta_1 = a_1x - \frac{\beta c_1}{\alpha}y + c_1z + \gamma_1$, $\zeta_2 = -\frac{\beta rc_2}{\alpha}y + rc_2z + rd_2t + \gamma_2$.

Fig. 3: Spatial behaviour of breather solitary solution at $t = 0$ with the parameters $\alpha = 20, \beta = 1, \delta_1 = 2, \delta_2 = -1, a_1 = \frac{1}{6}, a_2 = \frac{1}{4}, k_1 = \frac{1}{2}, k_2 = \frac{3}{4}, c_1 = 1, c_2 = -2, d_2 = 1, \gamma_i = 0, (i = 1, 2, 3)$, at $z = 0, t = 0$.

Fig. 4: Spatial degradation behaviour from breather solitary solution to lump solution at $t = 0$ as $\alpha = \beta = 1, r = 1, s = 1, l = 0, m = 2, a_1 = 1, a_2 = 1, c_1 = 0, c_2 = -3, c_3 = -1, d_2 = 1, \gamma_i = 0 (i = 1, 2, 3)$ and (a) $z = -5$, (b) $z = 0$, (c) $z = 5$.

3 Interaction between lump solutions and kink M -solitons

3.1 Kink M -soliton solutions

We choose the multi-soliton function as follows

$$f(x, y, z, t) = 1 + \sum_{j=1}^M \delta_j e^{(p_j x + q_j y + r_j z + s_j t + \sigma_j)}. \quad (13)$$

Based on the transformation Eq.(2) and bilinear form Eq.(3), then $s_j = 0, q_j = -\frac{p_j^3 + \beta r_j}{\alpha}$ for $j = 1, 2, \dots, M$. To better observe the evolution behaviour of kink M -soliton, Fig.5 provides the spatial structure of superposition of the kink M -soliton as the number of solitons increases. This constructed M -soliton shown in Fig.5 is also named M -fissionable wave solution, [38] described the dynamic properties and asymptotic behaviour particularly.

Fig. 5: Spatial superposition behavior of interaction Eq.(19) as $\alpha = \beta = -3, \delta_i = 1, r_i = 1 (i = 1, 2, 3, 4), z = 1$ (a) $N = 1, p_1 = \frac{1}{2}$, (b) $N = 2, p_1 = \frac{1}{2}, p_2 = 1$, (c) $N = 3, p_1 = \frac{1}{2}, p_2 = 1, p_3 = \frac{3}{2}$, (d) $N = 4, p_1 = \frac{1}{2}, p_2 = 1, p_3 = \frac{3}{2}, p_4 = 2$.

3.2 Theoretical proof of superposition behaviour

In this section, the lump M -solitons will be investigated, which are also called interaction solutions by

some reserchers [34,39,40,41]. and a theorem is given and prove it completely. Now, we assume a test function consisting of quadratic and exponential functions of sum type as follows:

$$F(x, y, z, t) = a_0 + \sum_{i=1}^N (a_i x + b_i y + c_i z + d_i t + \gamma_i)^2 + \sum_{j=1}^M \delta_j e^{(p_j x + q_j y + r_j z + s_j t + \sigma_j)} \stackrel{def}{=} \Phi(x, y, z, t) + \Psi(x, y, z, t). \quad (14)$$

where $a_0, a_i, b_i, c_i, d_i (i = 1, 2, 3, \dots, N)$ and $\delta_j, p_j, q_j, r_j, s_j, \sigma_j (j = 1, 2, 3, \dots, M)$ are some free real parameters, $\Phi(x, y, z, t)$ represents a quadratic polynomial function, and $\Psi(x, y, z, t)$ is a muliti-soliton function. Substituting Eq.(14) into Eq.(2), some new interaction solutions of (3+1)-dimensional HSII equation can be obtained.

Theorem: Assume the function Φ is a solution of Eq.(3), and the parameters satisfy the relations $\sum_{i=1}^N a_i d_i = 0$, $\sum_{i=1}^N (\alpha b_i + \beta c_i) d_i = 0$, $s_j = 0$ and $q_j = -\frac{p_j^3 + \beta r_j}{\alpha}$, then $F(x, y, z, t)$ is also a solution of Eq.(3).

Proof: By the Hirota bilinear operator D_- , we have

$$P(D_x, D_y, D_z, D_t)F \cdot F = ((D_x^3 + \alpha D_y + \beta D_z)D_t)(\Phi \cdot \Phi + \Phi \cdot \Psi + \Psi \cdot \Phi + \Psi \cdot \Psi). \quad (15)$$

Thanks to $\Phi(x, y, z, t)$ is a solution of Eq.(3) and $s_j = 0, \Psi_t = 0$, we have

$$P(D_x, D_y, D_z, D_t)(\Phi \cdot \Phi) = 0, P(D_x, D_y, D_z, D_t)(\Psi \cdot \Psi) = 0, \quad (16)$$

so that

$$\begin{aligned} P(D_x, D_y, D_z, D_t)F \cdot F &= ((D_x^3 + \alpha D_y + \beta D_z)D_t)(\Phi \cdot \Phi + \Phi \cdot \Psi + \Psi \cdot \Phi + \Psi \cdot \Psi) \\ &= 2((D_x^3 + \alpha D_y + \beta D_z)D_t)(\Phi \cdot \Psi) = 2(D_x^3 + \alpha D_y + \beta D_z)(\Phi_t \Psi - \Phi \Psi_t) \\ &= 4[(\Phi_{xxx} \Psi - 3\Phi_{xxt} \Psi_x + 3\Phi_{xt} \Psi_{xx} - \Phi_t \Psi_{xxx}) + \alpha(\Phi_{yt} \Psi - \Phi_y \Psi_t) + \beta(\Phi_{zt} \Psi - \Phi_z \Psi_t)] \\ &= 4[(\Phi_{xxxt} + \alpha \Phi_{yt} + \beta \Phi_{zt})\Psi - \Phi_t(\Psi_{xxx} + \alpha \Psi_y + \Phi_z) + 3\Phi_{xt} \Psi_{xx} - 3\Phi_{xxt} \Psi_x]. \end{aligned} \quad (17)$$

According to the form of Φ and Ψ , some relationship of parameters as follows:

$$\sum_{i=1}^N a_i d_i = 0, \sum_{i=1}^N (\alpha b_i + \beta c_i) d_i = 0, s_j = 0, q_j = -\frac{p_j^3 + \beta r_j}{\alpha}, \quad (18)$$

and

$$\Psi_{xxx} + \alpha \Psi_y + \beta \Phi_z = \sum_{j=1}^M \delta_j p_j^3 e^{\eta_j} + \alpha \sum_{j=1}^M \delta_j q_j e^{\eta_j} + \beta \sum_{j=1}^M \delta_j r_j e^{\eta_j} = 0. \quad (19)$$

where $\eta_j = p_j x + q_j y + r_j z + s_j t + \sigma_j$. Substituting Eq.(18) and (19) into Eq.(17), then the theorem is proved completely.

Corollary: Assume

$$g(x, y, z, t) = a_0 + \sum_{i=1}^N (a_i x + b_i y + c_i z + d_i t + \gamma_i)^2 + \Phi(x, y, z, t). \quad (20)$$

The function $\Phi(x, y, z, t)$ satisfies $\Phi_t = 0$ and $\Phi_{xxx} + \alpha\Phi_y + \beta\Phi_z = 0$, and the parameters satisfy the relations $\sum_{i=1}^N a_i d_i = 0$, $\sum_{i=1}^N (\alpha b_i + \beta c_i) d_i = 0$, then $g(x, y, z, t)$ is also a solution of Eq.(3).

3.3 Superposition behaviour of Lump- M -soliton

In this subsection, we will give some examples to illustrate the effectiveness of above theorem and corollary, the evolution behaviour of spatio-temporal structure of lump- M -solitons with the change of soliton number M and time t will be studied.

When $N = 2, M = 0$ in Eq.(14), substituting Eq.(14) into Eq.(3), we obtain the following two solutions with the aid of Mathematica:

$$a_1 = -\frac{a_2 d_2}{d_1}, c_1 = -\frac{\alpha b_1}{\beta}, c_2 = -\frac{\alpha b_2}{\beta}. \quad (21)$$

$$a_2 = -\frac{a_1 d_1}{d_2}, b_1 = 0, b_2 = -\frac{\alpha c_2}{\beta}, c_1 = 0. \quad (22)$$

$$a_1 = 0, b_1 = -\frac{\alpha c_1}{\beta}, b_2 = -\frac{\alpha c_2}{\beta}, d_2 = 0. \quad (23)$$

$$a_2 = 0, b_1 = -\frac{\alpha c_1}{\beta}, b_2 = -\frac{\alpha c_2}{\beta}, d_1 = 0. \quad (24)$$

Firstly, we give the application of the theorem, the spatio-temporal evolution behaviour of lump- M -solitons with the change of soliton number M by drawing three-dimensional plots (Fig.6,7,8). Combining Eq.(2), (14) and Eq.(23), the interaction solutions are given by

$$u(x, y, z, t) = \frac{4a_2(a_2x - \frac{\beta c_2}{\alpha}y + c_2z + \gamma_2) + 2\sum_{j=1}^M \delta_j p_j e^{\eta_j}}{a_0 + (-\frac{\beta c_1}{\alpha}y + c_1z + d_1t + \gamma_1)^2 + (a_2x - \frac{\beta c_2}{\alpha}y + c_2z + \gamma_2)^2 + \Psi} \quad (25)$$

where $\eta_j = p_j x - \frac{p_j^3 + \beta r_j}{\alpha}y + r_j z + \sigma_j$.

From Fig.6,7,8, the fission behaviours of the spatio-temporal structure of lump- M -soliton with the increase of the number M are studied, we take lump-1-soliton, lump-2-soliton and lump-3-soliton as the main research and presentation objects. In fact, the lump- M -soliton ($M \rightarrow +\infty$) solutions have similar evolution structures. The lump generates from the kink wave and then separates from it with the increase of the time. By the change process of u , we can see that the lump always exists as $t \rightarrow +\infty$, their amplitude including lump and kink wave remain constant. The parameters in Fig.6,7,8 are taken as $\alpha = -3, \beta = 2, a_0 = 4, a_2 = 2, c_1 = 1, c_2 = 2, d_1 = -1, \gamma_1 = \gamma_2 = 0, \delta_1 = \delta_2 = \delta_3 = 1, p_1 = 1, p_2 = -1, p_3 = -\frac{1}{2}, r_1 = -1, r_2 = -2, r_3 = -3, \sigma_1 = \sigma_2 = \sigma_3 = 0, z = 1$.

Fig. 6: Spatial superposition behavior of Lump- M -soliton as $M = 1$, (a) $t = -10$, (b) $t = 0$, (c) $t = 15$. (d) $t = 35$.

Fig. 7: Spatial superposition behavior of Lump- M -soliton as $M = 2$, (a) $t = -10$, (b) $t = 0$, (c) $t = 15$. (d) $t = 35$.

Fig. 8: Spatial superposition behavior of Lump- M -soliton as $M = 3$, (a) $t = -10$, (b) $t = 0$, (c) $t = 15$. (d) $t = 35$.

Secondly, we investigate the application of the corollary. Actually, there are so many functions $\Phi(x, y, z, t)$ that satisfy $\Phi_t = 0$ and $\Phi_{xxx} + \alpha\Phi_y + \beta\Phi_z = 0$, we will give two examples as follows:

Example 1: We study a new interaction phenomenon, Combining Eq.(23) and the corollary. Here, the test function is written as

$$f(x, y, z, t) = a_0 + \left(-\frac{\beta c_1}{\alpha}y + c_1z + d_1t + \gamma_1\right)^2 + \left(a_2x - \frac{\beta c_2}{\alpha}y + c_2z + \gamma_2\right)^2 + \sum_{l=1}^L \varrho_l \cosh(p_lx - \frac{p_l^3 + \beta r_l}{\alpha}y + r_lz + s_l). \quad (26)$$

Substituting Eq.(26) into Eq.(2), interaction solutions of lump- L -cosh ($L \rightarrow +\infty$) type are obtained. In order to better demonstrate the behaviour of spatial structure with the increase of the number of cosh type L . In Fig.9, the lump generates from the kink wave and it is gradually drowned or swallowed by the kink wave. When $t = 0$, the amplitude gets to the extreme point, the collision between a lump and 1-cosh kink wave is completely nonelastic. The fusion process shown in Fig.10, the lump generates from the of the intersection of 2-cosh kink wave, then it vanishes as the amplitude changes rapidly. This phenomenon is similar to the rogue wave, it means a kind of rogue wave will appear based on the interaction between a lump and 2-cosh kink wave. The lump generates from the of the intersection of 3-cosh kink wave in Fig.11. As the time goes by, and the lump wave is gradually drowned or swallowed by the kink wave. Figures are drawn with the parameters taken as $\alpha = -3, \beta = 2, a_0 = 4, a_2 = 8, c_1 = 1, c_2 = 2, d_1 = -1, \gamma_1 = \gamma_2 = 0, \delta_1 = \delta_2 = \delta_3 = 1, p_1 = 1, p_2 = \frac{1}{2}, p_3 = \frac{1}{4}, r_1 = -1, r_2 = 1, r_3 = -2, \sigma_1 = \sigma_2 = \sigma_3 = 0, z = 1$.

Fig. 9: Spatial superposition behavior of Lump- L -soliton as $L = 1$, (a) $t = -10$, (b) $t = 0$, (c) $t = 15$. (d) $t = 35$.

Fig. 10: Spatial superposition behavior of Lump- L -soliton as $L = 2$, (a) $t = -10$, (b) $t = 0$, (c) $t = 15$. (d) $t = 35$.

Fig. 11: Spatial superposition behavior of Lump- L -soliton as $L = 3$, (a) $t = -10$, (b) $t = 0$, (c) $t = 15$. (d) $t = 35$.

Example 2: We study the interaction phenomenon between lump and M -solitons with different types, Combining Eq.(23) and the corollary. Assume the test function is written as

$$f(x, y, z, t) = a_0 + \sum_{i=1}^N (a_i x + b_i y + c_i z + d_i z + \gamma_i)^2 + \sum_{j=1}^M \delta_j e^{p_j x - \frac{p_j^3 + \beta r_j}{\alpha} y + r_j z + s_j} + \sum_{k=1}^K \lambda_k \cos(q_k x - \frac{q_k^3 + \beta h_k}{\alpha} y + h_k z + \omega_k) + \sum_{l=1}^L \varrho_l \cosh(\varphi_l x - \frac{\varphi_l^3 + \beta \psi_l}{\alpha} y + \psi_l z + v_l). \quad (27)$$

Some new interaction solutions are obtained, the spatial structures of the lump and M -solitons with different types are given in Fig.12.

Fig. 12: Spatial evolution behavior of lump and M -solitons with different types.

In Fig.12, different type of interaction as follows:

(a) Interaction of lump-cos type as $\alpha = -1, \beta = 1, a_0 = 4, a_2 = 1, c_1 = 1, c_2 = 0, d_1 = -1, \gamma_1 = \gamma_2 = 0, \lambda_1 = 1, q_1 = 1, h_1 = -1, \omega_1 = 0, z = 1$ at time $t = 0$.

(b) Interaction of lump-cos-exponential type as $\alpha = -1, \beta = 1, a_0 = 4, a_2 = 1, c_1 = 1, c_2 = 0, d_1 = -1, \gamma_1 = \gamma_2 = 0, \lambda_1 = 1, q_1 = 1, h_1 = -1, \omega_1 = 0, \delta_1 = 1, p_1 = 1, r_1 = -1, s_1 = 0, z = 1$ at time $t = 0$.

(c) Interaction of lump-cosh-cos-cosh type as $\alpha = -1, \beta = 1, a_0 = 4, a_2 = 4, c_1 = 1, c_2 = 0, d_1 = -1, \gamma_1 = \gamma_2 = 0, \lambda_1 = 1, q_1 = \frac{1}{2}, h_1 = 1, \omega_1 = 0, \varrho_1 = \frac{1}{100}, \varphi_1 = \frac{1}{2}, \psi_1 = 0, v_l = 0, \varrho_2 = \frac{1}{100}, \varphi_2 = \frac{1}{4}, \psi_2 = -1, v_2 = 0, z = 1$ at time $t = 0$.

4 Conclusion

Based on symbolic computation and Hirota bilinear method, the evolution and degradation behaviours of breather wave solutions with different kinds are investigated, some new lump and lump-type solutions are obtained from the breather wave solutions via using the parameter limit method. Besides, inspired by the previous researches on the study of interaction phenomenon, we have studied the superposition behaviour between lump solution and different types of M -solitons ($M \rightarrow \infty$) for (3+1)-dimensional HSII equation, the theorem proof has been proposed firstly in this paper. Meanwhile, several three-dimensional spatio-temporal structure figures of breather waves and the interaction solutions are drawn to better reflect the oscillation of solitary wave and the evolution of interaction behaviour, including the oscillation and fusion-fission phenomenon, superposition and evolution behaviour of lump- M -solitons and so on. Fig.1 and Fig.3 show the oscillation of kink breather wave including double breather and breather solitary wave. In Fig.2 and Fig.4, lump solutions are generated from kink breather wave, it is also kink lump waves. M -soliton solutions are known as M -fissionable wave which are described in Fig.5. Some diverse interaction phenomenon shown in the following Figures have great significance to the nonlinear waves in fluid mechanics. In this paper, the methods used to obtain new exact solutions also can be

extended to solve other nonlinear partial differential equations. For some important classical mathematical and physical models, the methods of studying solutions are very significant.

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Declarations

Conflict of interest The authors declare that there are no conflicts of interests with publication of this work.

Ethical standards The authors ensure the compliance with ethical standards for this work.

Data availability statement

The authors declare that data supporting the findings of this study are available within the article, the figures are concrete expression.

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