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Temperature estimation of an entangled pair of trapped ions

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ABSTRACT

We apply estimation theory to a system formed by two interacting trapped ions. By using the Fisher matrix formalism, we introduce a simple scheme for estimation of the temperature of the longitudinal vibrational modes of the ions. We use the ions interaction to effectively infer the temperature of the individual ions, by optimising the interaction time evolution and by measuring only over one of the ions. We also investigate the effect of a non-thermal reservoir over the inference approach. The non-classicality of one of the ions vibrational modes, introduced due to a squeezed thermal reservoir, improves the indirect inference of the individual temperatures.

Introduction

In the quantum mechanical context, the temperature of a system is a nonlinear function of the density operator, so it cannot directly correspond to a quantum observable - one has to indirectly estimate its value by measuring another observable. This indirect procedure for temperature estimation implies in an additional uncertainty for the measured value, which should be handled appropriately. Therefore, any strategy aimed to determine the temperature of a quantum system reduces to a parameter estimation problem¹⁻³. Quantum theory of estimation (QET) provides a formal framework in order to optimise the inference procedure by minimising the additional uncertainty^{4,5}.

Techniques of quantum parameter estimation have been devoted to estimate the temperature in the context of quantum thermodynamics⁶⁻⁹ and also for technological applications in many different branches of Science - ranging from Material Sciences to Biology and Medicine^{10,11}. A central purpose is to employ a controlled quantum system (with view to applications in low temperature measurement¹²) to explore the thermodynamics in the regime of small-scale physics¹³⁻¹⁶, where quantum effects become predominant¹⁷⁻¹⁹. Several quantum systems have been used to estimate very low temperature such as Bose-Einstein condensates²⁰, ultracold lattice gases²¹, trapped ions²²⁻²⁴, single-qubit⁶, to mention a few.

In this work, we address the issue of the vibrational degrees of freedom temperature estimation in a system of two interacting trapped ions. Our interest is to investigate how precisely we are allowed to estimate the local differences of temperatures of vibrational degrees of freedom of the two ions simultaneously. To this purpose, we employ the multiparameter Fisher information

$$\mathcal{F}_{\alpha\beta} = \sum_{r_i} \mathcal{P}(r_i) \left(\frac{\partial \ln \mathcal{P}(r_i)}{\partial \theta_\alpha} \right) \left(\frac{\partial \ln \mathcal{P}(r_i)}{\partial \theta_\beta} \right), \quad (1)$$

to investigate the parameter estimation accuracy²⁵⁻²⁷, where θ_i denotes the outcome of a measurement, $\mathcal{P}(r_i) = p(r_i|\theta_i)$ is the conditional probability of measuring r_i if the value of the parameter under consideration is θ_i . From the practice standpoint, increasing the Fisher information of the system tends to increase the maximum precision that can be obtained by an estimation scheme. Mathematically, this relationship is quantified by the Cramér-Rao lower bound^{28,29},

$$\text{Var}(\hat{\theta}_i) \geq \frac{1}{\mathcal{F}_{ii}}, \quad (2)$$

where $\hat{\theta}_i$ is the estimator of the unknown parameter θ_i . The Cramér-Rao lower bound is saturated asymptotically by the optimization of the elements of the Fisher information matrix via a suitable choice of all its dependent parameters. Quantum

parameter estimation purpose in quantum thermometry is to use a measurement over a quantum probe system, to infer the temperature of the reservoir it is immersed. For that purpose, an optimisation over the measurement operation is usually employed in order to find the maximal overall Fisher information (and consequently the measurement with the minimal dispersion). However, the procedures employed, are sometimes cumbersome and quite generally do not bring any information on how and which observable is to be measured¹². Such a difficulty is even worse in the situation when systems of continuous variables are involved^{16,30,31}. Here we take a more pragmatical approach, instead of optimising the measurement operator we employ measurements accessible in actual ion experiments^{32,33} for the calculation of the Fisher information. We employ the detection of one of the ions vibrational mode phonon number, which is accessed experimentally, though the observation of the asymmetry between the red and blue motional sidebands of hyperfine Raman transitions^{34,35}. The phonon number detection on one of the ions (say ion 1) is then employed for simultaneous inference of the temperature of both ions.

In what follows, firstly we introduce the theoretical model and we show how to describe the ionic vibrational modes. We propose a joint estimation scheme, where a single probe state is used to estimate the temperatures of the two ions (T_1 and T_2) with a single projective measurement corresponding to the number of excitations in the first ion vibrational degree of freedom. In particular, we calculate the Fisher information as a function of the temperature of the ions and the interaction parameter g . We analyse the Fisher information of both ions in two scenarios – In the first one, we consider ions with distinct temperatures. Secondly, to infer how other changes in the vibrational state of one ion affects the performance of our approach, we consider a non-thermal bath in equilibrium with one of the ions.

1 Two Interacting Charges

We consider two interacting trapped ions having the same charge q and mass m . Each ion is confined in a potential well along of coordinate x and separated by a distance d , as shown in Fig. (1), accordingly with refs.^{32,33}

Figure 1. Scheme of two interacting trapped ions with the same charge q . Each ion is confined in a potential well along of coordinate x and separated by a distance d . Schematic diagram based on Ref.³².

Considering only the coordinates along the longitudinal separation of the ions, the Hamiltonian that describes this system reads^{32,33}

$$H = \sum_{j=1}^2 \left(\frac{p_j^2}{2m} + \frac{1}{2} m \omega^2 x_j^2 \right) + H_e, \quad (3)$$

where ω is the frequency of oscillation of each ion due to the trapping potential, and p_i and x_i are the momentum and position of the ion $i = 1, 2$, respectively. The term H_e is the ions electrostatic interaction energy given by³⁶

$$H_e = \frac{1}{4\pi\epsilon_0} \frac{q^2}{d + x_2 - x_1}, \quad (4)$$

where $(4\pi\epsilon_0)^{-1} = 9 \times 10^9 \text{ Nm}^2/\text{C}^2$. By expanding (4) in powers of $\frac{x_2 - x_1}{d}$, and considering small relative displacements we can rewrite the Coulomb interaction energy H_e approximately as

$$H_e(d) \approx \frac{q^2}{4\pi\epsilon_0 d} \left[1 - \frac{x_2 - x_1}{d} + \frac{(x_2 - x_1)^2}{d^2} \right]. \quad (5)$$

Introducing the operators

$$x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i^\dagger + a_i), \quad i = 1, 2, \quad (6)$$

and

$$p_i = i\sqrt{\frac{\hbar m\omega}{2}}(a_i^\dagger - a_i), \quad i = 1, 2, \quad (7)$$

the total system Hamiltonian becomes

$$\begin{aligned} H = & \hbar\Omega a_1^\dagger a_1 + \hbar\Omega a_2^\dagger a_2 + \sqrt{\frac{\hbar k^2 q^4}{2m\omega d^4}}(a_1 + a_1^\dagger) - \sqrt{\frac{\hbar k^2 q^4}{2m\omega d^4}}(a_2 + a_2^\dagger) \\ & + \frac{\hbar k q^2}{2m\omega d^3}(a_1^2 + a_1^{\dagger 2}) + \frac{\hbar k q^2}{2m\omega d^3}(a_2^2 + a_2^{\dagger 2}) - \frac{\hbar k q^2}{m\omega d^3}(a_1 a_2^\dagger + a_2 a_1^\dagger + a_1 a_2 + a_1^\dagger a_2^\dagger), \end{aligned} \quad (8)$$

where $\Omega = \omega + (kq^2/m\omega d^3)$ are the ions trap frequencies deviated by a small frequency shift due the ions interaction, and $k = 1/4\pi\epsilon_0$. In the interaction picture obtained by the unitary transformation $U = e^{-\frac{i}{\hbar}H_o t}$, where $H_o = \hbar\Omega a_1^\dagger a_1 + \hbar\Omega a_2^\dagger a_2$ corresponds to the trapped ions free evolution, the Hamiltonian becomes

$$\begin{aligned} \tilde{H} = & \hbar\sqrt{\frac{\hbar k^2 q^4}{2m\omega d^4}}(a_1 e^{-i\Omega t} + a_1^\dagger e^{i\Omega t}) - \sqrt{\frac{\hbar k^2 q^4}{2m\omega d^4}}(a_2 e^{-i\Omega t} + a_2^\dagger e^{i\Omega t}) + \frac{\hbar k q^2}{2m\omega d^3}(a_1^2 e^{-2i\Omega t} + a_1^{\dagger 2} e^{2i\Omega t}) \\ & + \frac{\hbar k q^2}{2m\omega d^3}(a_2^2 e^{-2i\Omega t} + a_2^{\dagger 2} e^{2i\Omega t}) - \frac{\hbar k q^2}{m\omega d^3}(a_1 a_2^\dagger + a_2 a_1^\dagger + a_1 a_2 e^{-2i\Omega t} + a_1^\dagger a_2^\dagger e^{2i\Omega t}), \end{aligned} \quad (9)$$

and the terms oscillating at frequencies Ω and 2Ω are averaged to zero for typical scales of time and can be disregard. The ions interaction Hamiltonian reduces to the simple form

$$\tilde{H} = \hbar g (a_1 a_2^\dagger + a_1^\dagger a_2), \quad (10)$$

with the coupling constant $g = -kq^2/m\omega d^3$. This coupling Hamiltonian is similar in form to that one for two radiation modes interacting through a beam-splitter. It is well known that a beam-splitter can only entangle two light modes in at least one of them is non-classical³⁷⁻³⁹. Therefore, it is interesting to investigate whether or not the non-classicality of one of the states affects the process of temperature inference. For that, in the next section we develop a formalism that allows to tackle the dynamics of the two ions vibrational modes independently of the initial equilibrium conditions.

2 Bipartite Gaussian States

All states easily accessed experimentally for our purposes, are Gaussian. Therefore, here we develop the formalism more appropriate to investigate the properties of the ionic vibrational degrees of freedom. A two-mode bipartite quantum state ρ is Gaussian if its symmetric characteristic function⁴⁰ is given by $C(\eta) = \text{Tr}[D(\eta)\rho] = e^{-\frac{1}{2}\eta^\dagger \mathbf{V} \eta}$, where $D(\eta) = e^{-\eta^\dagger \mathbf{E} \mathbf{v}}$ is a displacement operator in the four-vector η -space: $\eta^\dagger = (\eta_1^*, \eta_1, \eta_2^*, \eta_2)$, $\mathbf{v}^\dagger = (a_1^\dagger, a_1, a_2^\dagger, a_2)$, and

$$\mathbf{E} = \begin{pmatrix} \mathbf{Z} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (11)$$

where a_1 ($a_1^\dagger$) and a_2 ($a_2^\dagger$) are annihilation (creation) operators for party 1 and 2, respectively. $\mathbf{V}$ is a 4×4 covariance matrix with elements $V_{ij} = (-1)^{i+j} \langle \{v_i, v_j^\dagger\} \rangle / 2$, which can be decomposed in four block 2×2 matrices,

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{V}_2 \end{pmatrix}, \quad (12)$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are 2×2 Hermitian matrices containing only local elements, while $\mathbf{C}$ is a 2×2 matrix representing the correlation between the two parties and explicitly are written as

$$\mathbf{V}_i = \begin{pmatrix} n_i & m_i \\ m_i^* & n_i \end{pmatrix}, i = 1, 2, \quad \mathbf{C} = \begin{pmatrix} m_s & m_c \\ m_c^* & m_s^* \end{pmatrix}. \quad (13)$$

Positivity and separability for bipartite Gaussian quantum states have been largely investigated^{41–43}. Besides the requirement of the uncertainty principle

$$\mathbf{V} + \frac{1}{2}\mathbf{E} \geq 0, \quad (14)$$

there is a necessary and sufficient condition, which must be satisfied for separable Gaussian states⁴³

$$\tilde{\mathbf{V}} + \frac{1}{2}\mathbf{E} \geq 0, \quad (15)$$

under a partial phase space mirror reflection, $\tilde{\mathbf{V}} = \mathbf{T}\mathbf{V}\mathbf{T} : \mathbf{T}\mathbf{v}^\dagger = \mathbf{v}^\dagger_T = (a_1^\dagger, a_1, a_2, a_2^\dagger)$, with

$$\mathbf{T} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix}, \text{ and } \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (16)$$

The physical positivity criterion (14) applies only if⁴³

$$n_1 \geq \sqrt{|m_1|^2 + \frac{1}{4}}, \quad (17)$$

$$n_2 \geq \frac{s}{d} + \sqrt{\frac{1}{4} \left[\frac{||m_c|^2 - |m_s|^2|}{d} - 1 \right]^2 + |m_2 - c|^2}, \quad (18)$$

respectively, with $s = n_1 (|m_c|^2 + |m_s|^2) - m_c m_s m_1^* - m_c^* m_s^* m_1$, $c = 2n_1 m_s^* m_c - m_c^2 m_1^* - (m_s^*)^2 m_1$, and $d = n_1^2 - \frac{1}{4} - |m_1|^2$. Similarly the separability condition (15) writes explicitly into (17) and

$$n_2 \geq \frac{s}{d} + \sqrt{\frac{1}{4} \left[\frac{||m_c|^2 - |m_s|^2|}{d} + 1 \right]^2 + |m_2 - c|^2}. \quad (19)$$

The ions vibrational modes are assumed as initially uncoupled, so that in (12) $\mathbf{C} = \mathbf{0}$, and they prepared in special local states to be discussed later. However, the effect of the interaction (10) is to correlate the two modes. This can be seen as the following Bogoliubov operation B :

$$\rho_{out} = B \rho_{in} B^\dagger, \quad (20)$$

$$B \mathbf{v} B^\dagger = \mathbf{M} \mathbf{v}, \quad (21)$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{R} & \mathbf{S} \\ -\mathbf{S}^* & \mathbf{R}^* \end{pmatrix}, \mathbf{R} = \cos \theta \begin{pmatrix} e^{i\phi_0} & 0 \\ 0 & e^{-i\phi_0} \end{pmatrix}, \mathbf{S} = \sin \theta \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_1} \end{pmatrix}, \quad (22)$$

where ρ_{out} is the density operator for the joint output state. The output symmetric characteristic function is given by

$$C_{out}(\eta) = Tr[D(\eta)\rho_{out}] = Tr[B^\dagger D(\eta)B \rho_{in}]. \quad (23)$$

Now with the help of Eqs.(11-13), $B^\dagger D(\eta)B = e^{-\eta^\dagger \mathbf{E} \mathbf{M}^{-1} \mathbf{v}} \equiv D(\zeta)$, with $\zeta = \mathbf{M} \eta$, since $\mathbf{M} \mathbf{E} \mathbf{M}^{-1} = \mathbf{E}$. Thus

$$C_{out}(\eta) = C_{in}(\zeta) = e^{-\frac{1}{2} \zeta^\dagger \mathbf{V} \zeta} = e^{-\frac{1}{2} \eta^\dagger \mathbf{V}' \eta}, \quad (24)$$

where $\mathbf{V}' = \mathbf{M}^{-1} \mathbf{V} \mathbf{M}$, and analogously to (12), $\mathbf{V}'$ can be block decomposed with

$$\mathbf{V}'_1 = \mathbf{R}^* \mathbf{V}_1 \mathbf{R} + \mathbf{S} \mathbf{V}_2 \mathbf{S}^*, \quad (25)$$

$$\mathbf{V}'_2 = \mathbf{S}^* \mathbf{V}_1 \mathbf{S} + \mathbf{R} \mathbf{V}_2 \mathbf{R}^*, \quad (26)$$

$$\mathbf{C}' = \mathbf{R}^* \mathbf{V}_1 \mathbf{S} - \mathbf{S} \mathbf{V}_2 \mathbf{R}^*. \quad (27)$$

For the specific unitary transformation induced by the time-indepented two-modes coupling (10) $\phi_0 = \phi_1 = 0$, and $\theta = gt$, where t is the time variable.

Figure 2. Entanglement of the ions longitudinal vibrational as a function of the squeezing parameter of ion 2, and the interaction $\theta = gt$. Only for $2(\bar{n}_2 + 1/2) \leq e^{2r}$, is that the vibrational mode 2 is non-classical, allowing that the evolution (32) entangle the ions (represented by the negative values of the vertical axis). Maximal entanglement occurs at $\theta = \pi/4$ (or $3\pi/4$), corresponding to a 50:50 beam-splitter.

3 Fisher Information

The fluctuations in the trap parameters may introduce perturbations in the equilibrium state of each distinct ion. Typically, the ions coupling parameter g is very small, and therefore the temperature of the system must reach cryostatic regimes of a few kelvins. Even so, in³², e.g., a thermal heating rate of 1,885 quanta per second was observed. Therefore there may be an inhomogeneity of the ions equilibrium temperatures. Further experimental limitations constrains the direct access to all ions in the trap³², as was suggested by the inference approach in²⁴. Therefore, our purpose here is to use the phonon number detection (over one of the ions only) as a way to infer simultaneously the temperature differences of both ions.

As mentioned before, we shall consider the case where the ions are initially uncorrelated so the total density operator factorises as

$$\rho_{12} = \rho_1 \otimes \rho_2, \quad (28)$$

where ρ_1 and ρ_2 are the reduced density operators of ions 1 and 2, respectively. We assume that the ion 1 is initially in equilibrium with a thermal reservoir at temperature T_1 , while ion 2 can be in equilibrium with a thermal or non-thermal (for future purposes) reservoir at temperature T_2 . For that, we introduce a controllable squeezing parameter so that the ion 2 can be prepared in a thermal squeezed state, respectively, by

$$\rho_1 = \int d^2\alpha \frac{e^{-\frac{|\alpha|^2}{\bar{n}_1}}}{\pi\bar{n}_1} |\alpha\rangle\langle\alpha|, \quad \rho_2 = \int d^2\beta \frac{e^{-\frac{|\beta|^2}{\bar{n}_2}}}{\pi\bar{n}_2} S|\beta\rangle\langle\beta|S^\dagger, \quad (29)$$

where

$$S = e^{\frac{r}{2}(a_2^2 + a_2^{\dagger 2})},$$

is the one-mode squeezing operator, and the average thermal photon number given by

$$\bar{n}_1 = \frac{1}{e^{\frac{\hbar\Omega}{k_B T_1}} - 1}, \quad \bar{n}_2 = \frac{1}{e^{\frac{\hbar\Omega}{k_B T_2}} - 1}, \quad (30)$$

where T_1 and T_2 correspond to the equilibrium temperature for the ion-1 and ion-2, respectively.

The time evolution of the density operator with Hamiltonian (10) occurs with the following expression

$$\rho_{12}(\theta) = U(\theta)\rho_{12}U^\dagger(\theta), \quad (31)$$

where ($\theta = gt$)

$$U(\theta) = \exp[-i\theta(a_1 a_2^\dagger + a_1^\dagger a_2)]. \quad (32)$$

To understand the correlation induced by (32), we plot in Fig. 2 the bound (19) as a function of θ and the squeezing parameter r for the ion 2 for a fixed temperature of $T_1 = T_2 = 2.8 \times 10^{-5}$ K. We see that only for $2(\bar{n}_2 + 1/2) \leq e^{2r}$, is that the vibrational mode 2 is non-classical, allowing that the evolution (32) entangle the ions. as it is expected the maximal entanglement occurs at $\theta = \pi/4$ (or $3\pi/4$), which corresponds to a 50:50 beam-splitter.

By applying estimation techniques to the state $\hat{\rho}_{12}(t)$ we can achieve the ultimate bound of parameter T_1 or T_2 using measurement schemes feasible with current technology. In particular, the estimation of the temperatures of the ions can be implemented by choosing the projective measurement corresponding to the number of phonons in the longitudinal vibrational mode of the ion 1, as described in Fig. 3. It shows our estimation scheme, where the dynamics of two trapped ions is governed by the unitary operation (31). We can calculate the Fisher information corresponding to the parameters T_1 and T_2 via the number of excitations $\hat{\Pi}_1 = |k\rangle\langle k|$ in the longitudinal vibrational degree of freedom of ion 1.

Figure 3. Scheme for the indirect estimation of the temperatures of two trapped ions interacting via an electromagnetic interaction. After some interaction time between the two ions, a energy measurement corresponding to the number of particles is performed on the 1-ion, followed by post-processing for the estimation of parameters T_1 and T_2 .

For the number of phonon measurements, the Fisher information for ion 1 and ion 2 is computed as

$$F_{11} = \sum_{k_1} P_1(k) \left(\frac{\partial \ln P_1(k)}{\partial T_1} \right)^2, \quad F_{22} = \sum_k P_1(k) \left(\frac{\partial \ln P_1(k)}{\partial T_2} \right)^2, \quad F_{12} = F_{21} = \sum_k P_1(k) \left(\frac{\partial \ln P_1(k)}{\partial T_1} \right) \left(\frac{\partial \ln P_1(k)}{\partial T_2} \right), \quad (33)$$

where $P_1(k)$ is the probability distribution of the occupation number obtained by measuring the energy of Ion 1.

The probability distribution of k phonons is given by

$$P_1(k) = \frac{1}{k! \sqrt{\det(\mathbf{V}' - \frac{1}{2}\mathbf{I})}} \int d^2\gamma |\gamma|^{2k} e^{\frac{1}{2}\gamma^\dagger \mathbf{Z}(\mathbf{V}' - \frac{1}{2}\mathbf{I})\mathbf{Z}\gamma - |\gamma|^2}, \quad (34)$$

where

$$\mathbf{V}' - \frac{1}{2}\mathbf{I} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad (35)$$

with $a = \bar{n}_1 \cos^2(gt) + [(\bar{n}_2 + 1/2) \cosh 2r - 1/2]$, and $b = (\bar{n}_2 + 1/2) \sinh(2r) \sin^2(gt)$.

The probability distribution given by Eq. 34 involves a Gaussian integral that can be evaluated exactly so that the probability distribution is given as follows

$$P_1(k) = \frac{(-1)^k}{k! \sqrt{a^2 - b^2}} \frac{\partial^k}{\partial A^k} \left(\frac{1}{\sqrt{A^2 - B^2}} \right), \quad (36)$$

where

$$A^2 = \left(1 + \frac{a}{a^2 - b^2} \right)^2, \quad (37)$$

and

$$B^2 = \frac{b^2}{(a^2 - b^2)^2}. \quad (38)$$

Figure 4. Fisher information F_{11} , F_{22} and F_{12} as a function of parameter $\theta = gt$, for and $T_1 = 2.8 \times 10^{-5}K$, and $T_2 = 2.08 \times 10^{-5}K$. Here we have fixed $\Omega = 4$ MHz.⁴⁴.

Now using the series expansion

$$\frac{1}{\sqrt{A^2 - B^2}} = \frac{1}{A} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \left(\frac{B}{A}\right)^{2n}, \quad (39)$$

the derivatives in relation to A are easily evaluated, so the probability distribution reduces to

$$P_1(k) = \frac{1}{A^{k+1} \sqrt{a^2 - b^2}} \sum_{n=0}^{\infty} \frac{(2n+k)!}{k!(2^n n!)^2} \left(\frac{B}{A}\right)^{2n}, \quad (40)$$

The series in Eq. 40 can be identified with the Hypergeometric function ${}_2F_1\left(\frac{1+k}{2}, \frac{2+k}{2}, 1, \left(\frac{B}{A}\right)^2\right)$ ⁴⁵ so we can write

$$P_1(k) = \frac{{}_2F_1\left(\frac{1+k}{2}, \frac{2+k}{2}, 1, \left(\frac{B}{A}\right)^2\right)}{A^{k+1} \sqrt{a^2 - b^2}}. \quad (41)$$

In the case that $r = 0$, with both modes in thermal states, the probability distribution reduces to the simple expression

$$P_1(k) = \frac{[\bar{n}_1 \cos^2(gt) + \bar{n}_2 \sin^2(gt)]^k}{[1 + \bar{n}_1 \cos^2(gt) + \bar{n}_2 \sin^2(gt)]^{k+1}}. \quad (42)$$

For the calculations of the elements of the Fisher information we set $k = 200$ in order to perform the numerical calculations.

In Fig. 4, the Fisher information F_{11} , F_{22} and F_{12} evolution with $\theta = gt$, is represented for two thermal reservoirs (for $r = 0$) at $T_1 = 2.8 \times 10^{-5}K$, and $T_2 = 2.08 \times 10^{-5}K$ and for a non-thermal reservoir (for $r \neq 0$). We can see by the corresponding curves of F_{11} that the temperature T_1 is best inferred at the instants where $\theta = n\pi$, for $n = 0, 1, 2, \dots$. This occurs because at those instants the initial equilibrium state of the ion 1 has recurred⁴⁶. In contrast, as can be seen in F_{22} , at $\theta = m\frac{\pi}{2}$, for

Figure 5. Fisher information F_{11} , F_{22} and F_{12} as a function of parameter T_2 for $\theta = \pi/4$ and $T_1 = 2.8 \times 10^{-5} K$. Here we have fixed $\Omega = 4$ MHz⁴⁴.

$m = 1, 3, 5, \dots$ the state of ions 1 and 2 has been swapped, by the characteristic of evolution (32), and therefore is not surprising that those are the best instants for inference of temperature of the vibrational mode of ion 2 – it occurs because when the states are interchanged⁴⁶, when detection of the ion 1 vibrational mode is performed, in fact, we are effectively measuring the state of ion 2. Simultaneous inference of both ions temperatures occurs when $\theta = m\pi/4$, for $m = 1, 3, 5, \dots$, as depicted by F_{12} . Even when a non-thermal state is involved, the approach is successful for the inference of the ions temperature, as we can see in the same figures by varying r , the same characteristics hold. The effect of squeezing, however is to improve the entanglement and therefore it turns out that the distribution of points where the inference is optimal is spreaded, meaning that there is an advantage in using entanglement as a way to improve the accessibility at distinct instants of time for inference.

In Fig. (5), the Fisher information F_{11} , F_{22} and F_{12} are depicted as functions of the temperature T_2 for $\theta = \pi/4$ (similar results are observed for odd multiples of $\theta = \pi/4$), and different values of r , for a fixed temperature of ion 1, $T_1 = 2.8 \times 10^{-5} K$. This result indicates a precision loss for the estimation of temperature of the 1-ion, when we increase the r (this is reproduced in the graphs of F_{11} and F_{12} in Fig. (5)). The same cannot be said for estimating only the temperature T_2 , which improves with an increase in r (see graphs of F_{22} in Fig. (5)) under the same conditions, working as a super probe in the non-classical regime $2(\bar{n}_2 + 1/2) \leq e^{2r}$, for this maximally entangled situation for $\theta = \pi/4$. In fact, the maximal precision is still observed at $\theta = \pi/2$, as depicted in Fig. 6, in contrast to F_{11} and F_{12} , which are null for any T_2 we see that F_{22} reaches optimal values, which however is disturbed by the non-thermal reservoir feature.

4 Concluding Remarks

In summary, we have considered a simple electrostatic interaction architecture between two trapped ions, in which this interaction is approximated to a beam-splitter interaction. For Entanglement analysis and temperature estimation, we take ion 1 with a thermal initial state, and ion 2 with the initial state to be thermal and non-thermal (due to quadrature squeezing). We use the interaction to effectively infer the temperature of the individual ions, by optimising the interaction time evolution. It was observed that the estimation of the temperature of ion 2 is improved when the initial state is non-classical. Simultaneous measurements are often difficult to implement, in particular when associated with motional degrees of ions. The most practical way is to estimate the temperature by measuring each ion. Here, we chose to measure the ion that does not contain a possible squeezing, making it as a classical channel probe, contrary to what it has in the literature, where they use detectors with quantum properties for estimating parameters of classical systems^{7, 47–49}.

Figure 6. Fisher information F_{22} as a function of parameter T_2 for $\theta = \pi/2$ and $T_1 = 2.8 \times 10^{-5} \text{K}$. Here we have fixed $\Omega = 4 \text{ MHz}$ ⁴⁴. At $\theta = \pi/2$, both F_{11} and F_{12} are null for any T_2 .

References

1. Paris, M. G. Quantum estimation for quantum technology. *International Journal Quantum Information* **7**, 125–137 (2009).
2. Paris, M. & Rehacek, J. *Quantum state estimation*, vol. 649 (Springer Science & Business Media, 2004).
3. Helstrom, C. W. *Quantum detection and estimation theory* (Academic press, 1976).
4. Holevo, A. S. *Statistical structure of quantum theory*, vol. 67 (Springer Science & Business Media, 2003).
5. Giovannetti, V., Lloyd, S. & Maccone, L. Quantum metrology. *Physical review letters* **96**, 010401 (2006).
6. Razavian, S., Benedetti, C., Bina, M., Akbari-Kourbolagh, Y. & Paris, M. G. Quantum thermometry by single-qubit dephasing. *The European Physical Journal Plus* **134**, 284 (2019).
7. Campbell, S., Mehboudi, M., De Chiara, G. & Paternostro, M. Global and local thermometry schemes in coupled quantum systems. *New Journal Physics* **19**, 103003 (2017).
8. Farajollahi, B., Jafarzadeh, M., Jahromi, H. R. & Amniat-Talab, M. Estimation of temperature in micromaser-type systems. *Quantum Information Processing* **17**, 1–23 (2018).
9. Hofer, P. P., Brask, J. B., Perarnau-Llobet, M. & Brunner, N. Quantum thermal machine as a thermometer. *Physical review letters* **119**, 090603 (2017).
10. Yang, J.-M., Yang, H. & Lin, L. Quantum dot nano thermometers reveal heterogeneous local thermogenesis in living cells. *ACS nano* **5**, 5067–5071 (2011).
11. Kucsko, G. *et al.* Nanometre-scale thermometry in a living cell. *Nature* **500**, 54 (2013).
12. De Pasquale, A. & Stace, T. M. *Quantum Thermometry*, 503–527 (Springer International Publishing, Cham, 2018).
13. Brunelli, M., Olivares, S. & Paris, M. G. Qubit thermometry for micromechanical resonators. *Physical Review A* **84**, 032105 (2011).
14. Marzolino, U. & Braun, D. Erratum: Precision measurements of temperature and chemical potential of quantum gases [phys. rev. a 88, 063609 (2013)]. *Physical Review A* **91**, 039902 (2015).
15. Salvatori, G., Mandarino, A. & Paris, M. G. Quantum metrology in lipkin-meshkov-glick critical systems. *Physical Review A* **90**, 022111 (2014).
16. Correa, L. A., Mehboudi, M., Adesso, G. & Sanpera, A. Individual quantum probes for optimal thermometry. *Phys. Rev. Lett.* **114**, 220405, DOI: [10.1103/PhysRevLett.114.220405](https://doi.org/10.1103/PhysRevLett.114.220405) (2015).
17. Allahverdyan, A. & Nieuwenhuizen, T. M. Testing the violation of the clausius inequality in nanoscale electric circuits. *Physical Review B* **66**, 115309 (2002).
18. Hilt, S. & Lutz, E. System-bath entanglement in quantum thermodynamics. *Physical Review A* **79**, 010101 (2009).

19. Williams, N. S., Le Hur, K. & Jordan, A. N. Effective thermodynamics of strongly coupled qubits. *Journal Physics A: Mathematical Theoretical* **44**, 385003 (2011).
20. Sabín, C., White, A., Hackermuller, L. & Fuentes, I. Impurities as a quantum thermometer for a bose-einstein condensate. *Scientific reports* **4**, 6436 (2014).
21. Mehboudi, M., Moreno-Cardoner, M., De Chiara, G. & Sanpera, A. Thermometry precision in strongly correlated ultracold lattice gases. *New Journal Physics* **17**, 055020 (2015).
22. Turchette, Q. *et al.* Heating of trapped ions from the quantum ground state. *Physical Review A* **61**, 063418 (2000).
23. Roßnagel, J., Tolazzi, K. N., Schmidt-Kaler, F. & Singer, K. Fast thermometry for trapped ions using dark resonances. *New Journal Physics* **17**, 045004 (2015).
24. Ivanov, P. A. Quantum thermometry with trapped ions. *Optics Communications* **436**, 101–107 (2019).
25. Kay, S. M. *Fundamentals of statistical signal processing* (Prentice Hall PTR, 1993).
26. Liu, J., Yuan, H., Lu, X.-M. & Wang, X. Quantum fisher information matrix and multiparameter estimation. *Journal Physics A: Mathematical Theoretical* **53**, 023001 (2019).
27. Ataman, S. Single-versus two-parameter fisher information in quantum interferometry. *Physical Review A* **102**, 013704 (2020).
28. Scheffé, H. *et al.* H. cramér, mathematical methods of statistics. *Bulletin American Mathematical Society* **53**, 733–735 (1947).
29. Rao, C. R., Rao, C. R., Statistiker, M., Rao, C. R. & Rao, C. R. *Linear statistical inference and its applications*, vol. 2 (Wiley New York, 1973).
30. Šafránek, D., Lee, A. R. & Fuentes, I. Quantum parameter estimation using multi-mode gaussian states. *New Journal Physics* **17**, 073016, DOI: [10.1088/1367-2630/17/7/073016](https://doi.org/10.1088/1367-2630/17/7/073016) (2015).
31. Šafránek, D. Estimation of gaussian quantum states. *Journal Physics A: Mathematical Theoretical* **52**, 035304, DOI: [10.1088/1751-8121/aaf068](https://doi.org/10.1088/1751-8121/aaf068) (2018).
32. Brown, K. R. *et al.* Coupled quantized mechanical oscillators. *Nature* **471**, 196–199, DOI: [10.1038/nature09721](https://doi.org/10.1038/nature09721) (2011).
33. Harlander, M., Lechner, R., Brownnutt, M., Blatt, R. & Hänsel, W. Trapped-ion antennae for the transmission of quantum information. *Nature* **471**, 200–203 (2011).
34. Monroe, C. *et al.* Resolved-sideband raman cooling of a bound atom to the 3d zero-point energy. *Phys. Rev. Lett.* **75**, 4011–4014, DOI: [10.1103/PhysRevLett.75.4011](https://doi.org/10.1103/PhysRevLett.75.4011) (1995).
35. Meekhof, D. M., Monroe, C., King, B. E., Itano, W. M. & Wineland, D. J. Generation of nonclassical motional states of a trapped atom. *Phys. Rev. Lett.* **76**, 1796–1799, DOI: [10.1103/PhysRevLett.76.1796](https://doi.org/10.1103/PhysRevLett.76.1796) (1996).
36. Prytz, K. *Electrodynamics: The Field Free Approach*, vol. 11 (Springer, 2015).
37. Kim, M. S., Son, W., Bužek, V. & Knight, P. L. Entanglement by a beam splitter: Nonclassicality as a prerequisite for entanglement. *Phys. Rev. A* **65**, 032323, DOI: [10.1103/PhysRevA.65.032323](https://doi.org/10.1103/PhysRevA.65.032323) (2002).
38. Xiang-bin, W. Theorem for the beam-splitter entangler. *Phys. Rev. A* **66**, 024303, DOI: [10.1103/PhysRevA.66.024303](https://doi.org/10.1103/PhysRevA.66.024303) (2002).
39. de Oliveira, M. & Munro, W. Nonclassicality and information exchange in deterministic entanglement formation. *Physics Letters A* **320**, 352–359, DOI: <https://doi.org/10.1016/j.physleta.2003.11.037> (2004).
40. de Oliveira, M. C. Characterization and quantification of symmetric gaussian-state entanglement through a local classicality criterion. *Physical Review A* **72**, 012317 (2005).
41. Englert, B.-G. & Wódkiewicz, K. Tutorial notes on one-party and two-party gaussian states. *International Journal Quantum Information* **01**, 153–188, DOI: [10.1142/S0219749903000206](https://doi.org/10.1142/S0219749903000206) (2003). <https://doi.org/10.1142/S0219749903000206>.
42. Simon, R. Peres-horodecki separability criterion for continuous variable systems. *Phys. Rev. Lett.* **84**, 2726–2729, DOI: [10.1103/PhysRevLett.84.2726](https://doi.org/10.1103/PhysRevLett.84.2726) (2000).
43. de Oliveira, M. C. P-representable subset of all bipartite gaussian separable states. *Phys. Rev. A* **70**, 034303, DOI: [10.1103/PhysRevA.70.034303](https://doi.org/10.1103/PhysRevA.70.034303) (2004).
44. Seidelin, S. *et al.* Microfabricated surface-electrode ion trap for scalable quantum information processing. *Physical review letters* **96**, 253003 (2006).

45. Weber, H. J. & Arfken, G. B. *Essential mathematical methods for physicists*, ISE (Elsevier, 2003).
46. de Oliveira, M. C., Mizrahi, S. S. & Dodonov, V. V. Information transfer in the course of a quantum interaction. *Journal Optics B: Quantum Semiclassical Optics* **1**, 610–617, DOI: [10.1088/1464-4266/1/5/319](https://doi.org/10.1088/1464-4266/1/5/319) (1999).
47. Latune, C., Escher, B., de Matos Filho, R. & Davidovich, L. Quantum limit for the measurement of a classical force coupled to a noisy quantum-mechanical oscillator. *Physical Review A* **88**, 042112 (2013).
48. Caves, C. M. Quantum-mechanical radiation-pressure fluctuations in an interferometer. *Physical Review Letters* **45**, 75 (1980).
49. Aasi, J. *et al.* Enhanced sensitivity of the ligo gravitational wave detector by using squeezed states of light. *Nature Photonics* **7**, 613–619 (2013).