Lump Solutions, Multi Lump Solutions and More Soliton Solutions of a Novel (2+1)-dimensional Nonlinear Evolution Equation

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Lump solutions, multi lump solutions and more soliton solutions of a novel (2+1)-dimensional nonlinear evolution equation

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Abstract

Exact solutions of a new (2+1)-dimensional nonlinear evolution equation are studied. Through the Hirota bilinear method, the test function method and the improved tanh-coth and tan-cot method, with the assistance of symbolic operations, one can obtain the lump solutions, multi lump solutions and more soliton solutions. Finally, by determining different parameters, we draw the three-dimensional plots and density plots at different times.

Mathematics Subject Classification. 35G20, 35C05, 35C07.

Keywords: test functions; lump solutions; multi lump solutions; more soliton solutions; the improved tanh-coth and tan-cot methods

1 Introduction

Nonlinear evolution equation is considered as a basic model to study many nonlinear phenomena of various complexities, and it is produced in many areas of physics, such as plasma physics, solid state physics and others. From the past to the present, great progress has been made in the research for nonlinear evolution equations, and a great deal of effective methods have been put forward to calculate their exact solutions, for instance, the Hirota's bilinear method [5], the extended homoclinic test function method [22], the inverse scattering and the Darboux transformation [1].

In 1977, lump waves [18] confined to all directions in space have been found for the first time, which has given rise to the attention of numerous scholars. The lump solutions can be obtained by direct bilinear method. Specifically, the auxiliary function for bilinear form is set as a quadratic function. A large number of lump solutions are solved. For example, the specific process of finding lump solutions is shown in the Kadomtsev–Petviashvili (KP) equation [15], the Boussinesq Equation [12], the BKP equation [6], the generalized Bogoyavlensky-Konopelchenko equation [4], the reduced pgbKP and pgKP equations [16] and so on [9–11, 13, 14]. Compared with soliton solutions, the lump solutions [19, 24] are rational function solutions, which decays polynomially in all directions of space, but the lump-type solution [17, 25] is localized in mainly all directions of space. In the form of breathing, the multi lump solution [8, 23] is a straight

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line. Using some trigonometric functions and hyperbolic functions, single soliton solutions and multi soliton solutions are obtained by the improved tan-cot and tanh-coth method [3] in the (3+1)-dimensional extended Date– Jimbo–Kashiwara–Miwa(DJKM) equation.

A novel nonlinear evolution equation in (2+1)-dimensional is proposed by [2]

$$u_{xxxx} + c_1 u_{tt} + c_2 u_{xt} + c_3 u_{xy} + c_4 (u^2)_{xx} = 0. \quad (1)$$

where coefficients $c_i (1 \leq i \leq 4)$ are arbitrary constants.

If $c_2 = 0, c_3 = 0$, one can get

$$u_{xxxx} + c_1 u_{tt} + c_4 (u^2)_{xx} = 0. \quad (2)$$

Under the condition $\frac{c_4}{c_1} = \alpha, c_1 = \frac{1}{\beta}$, equation (2) becomes Benjamin-Ono equation [21]

$$u_{tt} + \beta u_{xxxx} + \alpha (u^2)_{xx} = 0.$$

If $y = x, c_2 = 0$, one can get

$$u_{xxxx} + c_1 u_{tt} + c_3 u_{xx} + c_4 (u^2)_{xx} = 0. \quad (3)$$

Under the condition $\frac{c_4}{c_1} = \alpha, c_1 = \frac{1}{\beta}, \frac{c_3}{c_1} = \gamma$, equation (3) becomes the Boussinesq equation [20]

$$u_{tt} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \gamma u_{xx} = 0.$$

Taking

$$u = 6(\ln f)_{xx}. \quad (4)$$

and $c_4 = 1$, the bilinear form for equation (1) is as follows:

$$(D_x^4 + c_1 D_t^2 + c_2 D_x D_t + c_3 D_x D_y) f \cdot f = 0, \quad (5)$$

where function f is arbitrary with regard to variables x, y and t , and the Hirota's bilinear derivative operators D_x, D_y and D_t [7] are constructed by

$$D_x^k D_y^i D_t^j f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^k \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^i \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^j f(x, y, t) \\ \cdot g(x', y', t')|_{x'=x, y'=y, t'=t}.$$

The m-lump solution is derived from equation (1) by the long wave limit method [2]. This paper intends to research lump solutions, multi lump solutions and more soliton solutions on the basis of bilinear equation(5).

The structure of the paper is divided into five parts. The first section is the background introduction. In the second section, the lump solutions are obtained by quadratic function, and the lump solutions in many cases are discussed. In the third section, the multi lump solution is obtained by exponential function and trigonometric function, and its density diagram is like a pipeline. In Section 4, the more soliton solution is obtained by the improved tanh-coth and tan-cot method. The fifth section is the conclusion.

2 Lump solution of equation

For the purpose of gaining the lump solution for equation (1), we suppose that the quadratic function form of $f(x, y, t)$ is

$$f(x, y, t) = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + b_1y + a_6t + b_2)^2 + (b_3x + a_7y + a_8t + b_4)^2 + a_9. \quad (6)$$

Substituting the quadratic function form (6) into bilinear equation (5), and making the polynomial coefficients of variables x, y, t are equal to zero. Obviously, we are able to gain a series of algebraic equations about $b_j (1 \leq j \leq 4)$ and $a_i (1 \leq i \leq 9)$. With the help of mathematical software Maple, by solving the above series of algebraic equations, one can get the following cases:

Case 1.

$$\begin{aligned} a_1 &= a_1, a_2 = \frac{a_1c_1(a_6^2 - a_3^2) - a_3c_2(a_1^2 + a_5^2) - 2a_3a_5a_6c_1}{c_3(a_1^2 + a_5^2)}, a_3 = a_3, a_4 = 0, \\ a_5 &= a_5, a_6 = a_6, a_7 = 0, a_8 = 0, a_9 = -\frac{3(a_1^6 + 3a_1^4a_5^2 + 3a_1^2a_5^4 + a_5^6)}{c_1(a_1a_6 - a_3a_5)^2}, \\ b_1 &= \frac{a_5c_1(a_3^2 - a_6^2) - a_6c_2(a_1^2 + a_5^2) - 2a_1a_3a_6c_1}{c_3(a_1^2 + a_5^2)}, b_i = 0 (2 \leq i \leq 4). \end{aligned} \quad (7)$$

where $a_1a_5 \neq 0, a_1a_6 - a_3a_5 \neq 0$.

Therefore, by comparing equation (6) and limiting conditions (7), one can write a type of quadratic function solutions of equation (5)

$$f_1 = (a_1x + a_2y + a_3t)^2 + (a_5x + b_1y + a_6t)^2 + a_9. \quad (8)$$

Then, substituting quadratic function (8) into transformation (4), we can obtain

$$\begin{aligned} u_1 &= \frac{12(a_1^2 + a_5^2)}{(a_1x + a_2y + a_3t)^2 + (a_5x + b_1y + a_6t)^2 + a_9} \\ &\quad - \frac{24[(a_1x + a_2y + a_3t)a_1 + (a_5x + b_1y + a_6t)a_5]^2}{[(a_1x + a_2y + a_3t)^2 + (a_5x + b_1y + a_6t)^2 + a_9]^2}. \end{aligned} \quad (9)$$

Fig. 1 gives three dimensional plots of function (9) and its density plots with a series of parameters $c_1 = -\frac{15}{32}, c_2 = \frac{45}{32}, c_3 = \frac{75}{32}, a_1 = -\frac{3}{5}, a_3 = 0, a_5 = \frac{11}{10}, a_6 = 5$ when $t = -1, 0, 1$.

Fig. 2 gives three dimensional plots of function (9) and its density plots with a series of parameters $c_1 = -1, c_2 = 2, c_3 = -\frac{1}{10}, a_1 = 1, a_3 = -2, a_5 = 3, a_6 = 2$ when $t = -6, 0, 6$.

Case 2.

$$\begin{aligned} a_1 &= -\frac{2a_3c_1}{c_2}, a_2 = -\frac{a_3c_2}{c_3}, a_3 = a_3, a_4 = a_4, a_5 = \frac{c_1(a_3^2 - a_6^2)}{c_2a_6}, \\ a_6 &= a_6, a_7 = 0, a_8 = 0, a_9 = -\frac{3(a_3^2 + a_6^2)^4c_1^3}{a_6^4c_2^4a_3^2}, b_i = 0 (1 \leq i \leq 4). \end{aligned} \quad (10)$$

where $a_3a_6 \neq 0$.

Therefore, by comparing equation (6) and limiting conditions (10), we can write a type of quadratic function solutions of equation (5)

$$f_2 = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6t)^2 + a_9. \quad (11)$$

Figure 1: Three-dimensional plots and density plots of (9) with $c_1 = -\frac{15}{32}, c_2 = \frac{45}{32}, c_3 = \frac{75}{32}, a_1 = -\frac{3}{5}, a_3 = 0, a_5 = \frac{11}{10}, a_6 = 5$ when $t = -1$ in (a) and (d), $t = 0$ in (b) and (e), $t = 1$ in (c) and (f), respectively.

Figure 2: Three-dimensional plots and density plots of (9) with $c_1 = -1, c_2 = 2, c_3 = -\frac{1}{10}, a_1 = 1, a_3 = -2, a_5 = 3, a_6 = 2$ when $t = -6$ in (a) and (d), $t = 0$ in (b) and (e), $t = 6$ in (c) and (f), respectively.

Then, substituting quadratic function (11) into transformation (4), we can obtain

$$u_2 = \frac{12(a_1^2 + a_5^2)}{(a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6t)^2 + a_9} - \frac{24[(a_1x + a_2y + a_3t + a_4)a_1 + (a_5x + a_6t)a_5]^2}{[(a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6t)^2 + a_9]^2}. \quad (12)$$

Fig. 3 gives three dimensional plots of function (12) and its density plots with a series of parameters $c_1 = 1.3, c_2 = 2.1, c_3 = 1.5, a_3 = 0.5, a_4 = -3, a_6 = 1$ when $t = -1, 0, 1$.

Figure 3: Three-dimensional plots and density plots of (12) with $c_1 = 1.3, c_2 = 2.1, c_3 = 1.5, a_3 = 0.5, a_4 = -3, a_6 = 1$ when $t = -1$ in (a) and (d), $t = 0$ in (b) and (e), $t = 1$ in (c) and (f), respectively.

Case 3.

$$a_1 = a_1, a_2 = -\frac{a_1a_3c_2 + c_1(a_3^2 - a_8^2)}{c_3a_1}, a_3 = a_3, a_4 = a_4, a_5 = 0, a_6 = 0, \\ a_7 = -\frac{a_8(a_1c_2 + 2a_3c_1)}{c_3a_1}, a_8 = a_8, a_9 = -\frac{3a_1^4}{c_1a_8^2}, b_i = 0(1 \leq i \leq 4). \quad (13)$$

where $a_1 \neq 0, a_8 \neq 0$.

Therefore, by comparing equation (6) and limiting conditions (13), we can write a type of quadratic function solutions of equation (5)

$$f_3 = (a_1x + a_2y + a_3t + a_4)^2 + (a_7y + a_8t)^2 + a_9. \quad (14)$$

Then, substituting quadratic function (14) into transformation (4), we can obtain

$$u_3 = \frac{12a_1^2[(a_1x + a_2y + a_3t + a_4)^2 + (a_7y + a_8t)^2 + a_9] - 24(a_1x + a_2y + a_3t + a_4)^2a_1^2}{[(a_1x + a_2y + a_3t + a_4)^2 + (a_7y + a_8t)^2 + a_9]^2}. \quad (15)$$

Fig. 4 gives three dimensional plots of function (15) and its density plots with a series of parameters $c_1 = 1, c_2 = 2, c_3 = 1.8, a_1 = 4, a_3 = 0.6, a_4 = -5, a_7 = 1$ when $t = -6, 0, 6$.

Figure 4: Three-dimensional plots and density plots of (15) with $c_1 = 1, c_2 = 2, c_3 = 1.8, a_1 = 4, a_3 = 0.6, a_4 = -5, a_7 = 1$ when $t = -6$ in (a) and (d), $t = 0$ in (b) and (e), $t = 6$ in (c) and (f), respectively.

Case 4.

$$\begin{aligned}
 a_1 &= a_1, a_2 = a_2, a_3 = 0, a_4 = \sqrt{\frac{3c_1^2 a_1^2 a_6^2 - a_1 a_2 a_9 c_2^2 c_3 + a_2^2 a_9 c_1 c_3^2}{a_1 a_2^3 c_3^3 + a_1 a_2 a_6^2 c_2^2 c_3 - a_2^2 a_6^2 c_1 c_3^2}} a_6, \\
 a_5 &= -\frac{a_1 a_6 c_2}{c_3 a_2}, a_6 = a_6, a_7 = \frac{-a_1 a_2^2 c_3^2 + a_6^2 (a_2 c_1 c_3 - a_1 c_2^2)}{\sqrt{-a_1^2 (a_2^2 c_3^2 + a_6^2 c_2^2) + a_1 a_2 a_6^2 c_1 c_3}}, a_8 = 0, \\
 a_9 &= a_9, b_1 = 0, b_2 = 0, b_3 = \frac{\sqrt{-a_1^2 (a_2^2 c_3^2 + a_6^2 c_2^2) + a_1 a_2 a_6^2 c_1 c_3}}{c_3 a_2}, b_4 = 0.
 \end{aligned} \tag{16}$$

where $\sqrt{-a_1^2 (a_2^2 c_3^2 + a_6^2 c_2^2) + a_1 a_2 a_6^2 c_1 c_3} > 0, a_2 \neq 0$.

Therefore, by comparing equation (6) and limiting conditions (16), we can write a type of quadratic function solutions of equation (5)

$$f_4 = (a_1 x + a_2 y + a_4)^2 + (a_5 x + a_6 t)^2 + (b_3 x + a_7 y)^2 + a_9. \tag{17}$$

Then, substituting quadratic function (17) into transformation (4), we can obtain

$$\begin{aligned}
 u_4 &= \frac{12(a_1^2 + a_5^2 + b_3^2)}{(a_1 x + a_2 y + a_4)^2 + (a_5 x + a_6 t)^2 + (b_3 x + a_7 y)^2 + a_9} \\
 &\quad - \frac{24[(a_1 x + a_2 y + a_4)a_1 + (a_5 x + a_6 t)a_5 + (b_3 x + a_7 y)b_3]^2}{[(a_1 x + a_2 y + a_4)^2 + (a_5 x + a_6 t)^2 + (b_3 x + a_7 y)^2 + a_9]^2}.
 \end{aligned} \tag{18}$$

Figure 5: Three-dimensional plots and density plots of (18) with $c_1 = -5, c_2 = -2, c_3 = 1, a_1 = 1.5, a_2 = 0.6, a_6 = -5, a_9 = 1$ when $t = -11$ in (a) and (d), $t = 0$ in (b) and (e), $t = 11$ in (c) and (f), respectively.

Fig. 5 gives three dimensional plots of function (18) and its density plots with a series of parameters $c_1 = -5, c_2 = -2, c_3 = 1, a_1 = 1.5, a_2 = 0.6, a_6 = -5, a_9 = 1$ when $t = -11, 0, 11$.

3 Multi lump solutions of equation

Now for equation (1), one can consider its multi lump solutions and function f is set to [8]

$$f(x, y, t) = e^{-\xi_1} + a_1 e^{\xi_1} + a_2 \sin(\xi_2). \quad (19)$$

with $\xi_1 = k_1(x + r_1 y - w_1 t), \xi_2 = k_2(r_2 y - w_2 t)$.

Substituting (19) into bilinear form (5), we are able to get algebraic systems for $a_1, a_2, k_1, r_1, w_1, k_2, r_2, w_2$, and the relationship is as below:

$$\begin{aligned} a_1 &= \frac{k_2^2 a_2^2 [-3(k_1^2 + k_2^2)^2 + c_1(w_1 - w_2)^2 k_1^2]}{4k_1^2 [3(k_1^2 + k_2^2)^2 + c_1(w_1 - w_2)^2 k_2^2]}, a_2 = a_2, k_1 = k_1, k_2 = k_2, w_1 = w_1, \\ w_2 &= w_2, r_1 = \frac{3k_2^4 + [2k_1^2 + (-2c_1 w_2 + c_2)w_1 + c_1 w_2^2]k_2^2 - k_1^4 + (-c_1 w_1^2 + c_2 w_1)k_1^2}{c_3(k_1^2 + k_2^2)}, \\ r_2 &= \frac{-3k_1^4 + [-2k_2^2 + (-2c_1 w_1 + c_2)w_2 + c_1 w_1^2]k_1^2 + k_2^4 + (-c_1 w_2^2 + c_2 w_2)k_2^2}{c_3(k_1^2 + k_2^2)}. \end{aligned} \quad (20)$$

where $k_1 k_2 \neq 0, w_1 w_2 \neq 0$.

Therefore, by comparing equation (19) and limiting conditions (20), we can write a type of multi lump solutions of equation (5)

$$f_5 = e^{-[k_1(x+r_1 y-w_1 t)]} + a_1 e^{k_1(x+r_1 y-w_1 t)} + a_2 \sin[k_2(x + r_2 y - w_2 t)]. \quad (21)$$

Then, substituting exponential trigonometric function (21) into transformation (4), we can obtain

$$u_5 = \frac{6[k_1^2 e^{-\xi_1} + a_1 k_1^2 e^{\xi_1} - a_2 k_2^2 \sin(\xi_2)]}{e^{-\xi_1} + a_1 e^{\xi_1} + a_2 \sin(\xi_2)} - \frac{6[-k_1 e^{-\xi_1} + a_1 k_1 e^{\xi_1} + a_2 k_2 \cos(\xi_2)]^2}{[e^{-\xi_1} + a_1 e^{\xi_1} + a_2 \sin(\xi_2)]^2}. \quad (22)$$

Fig. 6 gives the plots of (22) and its density plots for the parameters $c_1 = -15.75, c_2 = -660.5, c_3 = 1386, k_1 = 1, k_2 = -1, a_2 = 2, w_1 = -2, w_2 = 2$ when $t = -2, 0, 2$.

Figure 6: Three-dimensional plots and density plots of (18) with parameters $c_1 = -15.75, c_2 = -660.5, c_3 = 1386, k_1 = 1, k_2 = -1, a_2 = 2, w_1 = -2, w_2 = 2$, when $t = -2$ in (a) and (d), $t = 0$ in (b) and (e), $t = 2$ in (c) and (f), respectively.

4 More soliton solution

In order to find some new soliton solutions for equation(1), we will set the function u in the form of trigonometric function and hyperbolic function [3].

4.1 the improved tanh-coth method

Through calculating equation (1), one can find the equality for solving the value of n is $n + 4 = n + n + 2$, so $n = 2$.

Taking

$$u = a_0 + a_1 \tanh(\xi) + a_2 [\tanh(\xi)]^2 + a_3 [\tanh(\xi)]^{-3} + a_4 [\tanh(\xi)]^{-2}, \quad (23)$$

with $\xi = kx + ry - wt$. Substitute trigonometric hyperbolic function (23) into equation (1) and collect all coefficients of $(\tanh(\xi))^j$, where $j = -1, -2, -3, -4, -5, -6, -7, -8$ and $0, 1, 2, 3, 4, 5, 6$. Then making these coefficients are equal to zero, and creating an algebraic

system about $a_0, a_1, a_2, a_3, a_4, w$. Finally, we can get

$$\begin{aligned} k &= k, r = r, w = w, a_0 = \frac{8k^4 - krc_3 + kwc_2 - c_1w^2}{2c_4k^2}, \\ a_1 &= 0, a_2 = -\frac{6k^2}{c_4}, a_3 = 0, a_4 = -\frac{6k^2}{c_4}. \end{aligned} \quad (24)$$

Therefore, we can derive the solution

$$u_6 = a_0 + a_2[\tanh(kx + ry - wt)]^2 + a_4[\tanh(kx + ry - wt)]^{-2}. \quad (25)$$

Fig. 7 gives the plots of (25) and its density plots for the parameters $c_1 = -4, c_2 = 2, c_3 = -1, c_4 = 6, k = 0.1, r = 0.2, w = -1$ when $t = -4, 0, 4$.

Figure 7: Three-dimensional plots and density plots of (25) with $c_1 = -4, c_2 = 2, c_3 = -1, c_4 = 6, k = 0.1, r = 0.2, w = -1$ while $t = -4$ in (a) and (d), $t = 0$ in (b) and (e), $t = 4$ in (c) and (f), respectively.

4.2 the improved tah-cot method

In a like manner, we can assume the function u in the form

$$u = a_0 + a_1 \tan(\xi) + a_2 [\tan(\xi)]^2 + a_3 [\tan(\xi)]^{-3} + a_4 [\tan(\xi)]^{-2}, \quad (26)$$

where $\xi = kx + ry - wt$.

Substitute trigonometric hyperbolic function (26) into equation (1) and collect all coefficients of $(\tan(\xi))^j$, where $j = -1, -2, -3, -4, -5, -6, -7, -8$ and $0, 1, 2, 3, 4, 5, 6$. Then making these coefficients are equal to zero, and creating an algebraic system about $a_0, a_1, a_2, a_3, a_4, w$. Finally, we can get

$$k = k, r = -\frac{8k^4 + 2c_4k^2a_0 - kwc_2 + c_1w^2}{kc_3}, w = w,$$

$$a_0 = a_0, a_1 = 0, a_2 = -\frac{6k^2}{c_4}, a_3 = 0, a_4 = -\frac{6k^2}{c_4}. \quad (27)$$

Therefore, we can derive the solution

$$u_7 = a_0 + a_2[\tan(kx + ry - wt)]^2 + a_4[\tan(kx + ry - wt)]^{-2}. \quad (28)$$

Fig. 8 gives the plots of (28) and its density plots for the parameters $c_1 = -4, c_2 = 2, c_3 = -1.08, c_4 = 6, k = 0.1, w = -1, a_0 = 31.84$ when $t = -4, 0, 4$.

Figure 8: Three-dimensional plots and density plots of (28) with parameters $c_1 = -4, c_2 = 2, c_3 = -1.08, c_4 = 6, k = 0.1, w = -1, a_0 = 31.84$ while $t = -4$ in (a) and (d), $t = 0$ in (b) and (e), $t = 4$ in (c) and (f), respectively.

5 Conclusion

To sum up, we discuss the (2+1) nonlinear evolution equation by making use of bilinear method, test function, improved tan-cot and tanh-coth method. At the same time, through using the powerful computing power of software Maple, we give lump solutions, multi lump solutions and more soliton solutions with non-zero real number c_1, c_2, c_3, c_4 . By determining different parameters, the three-dimensional diagram and density diagram of each solution obtained in this paper are drawn in Figure 1-8, which is proved that this technology is effective in dealing with nonlinear evolution equations.

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