Integrability and Multisoliton Solutions of the Reverse Space or/and Time Nonlocal Fokas-Lenells (FL) Equation

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Integrability and multisoliton solutions of the reverse space or/and time nonlocal Fokas-Lenells (FL) equation

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Abstract

This paper studies reverse space or/and time nonlocal Fokas-Lenells (FL) equation, which describes the propagation of nonlinear light pulses in monomode optical fibers when certain higher-order nonlinear effects are considered, by Hirota bilinear method. Firstly, variable transformations from reverse space nonlocal FL equation to reverse time and reverse space-time nonlocal FL equations are constructed. Secondly, the one-, two- and three-soliton solutions of the reverse space nonlocal FL equation are derived through Hirota bilinear method, and the soliton solutions of reverse time and reverse space-time nonlocal FL equations are given through variable transformations. Dynamical behaviors of the multisoliton solutions are discussed in detail by analyzing their wave structures. Thirdly, asymptotic analysis of two- and three-soliton solutions of reverse space nonlocal FL equation is used to investigated the elastic interaction and inelastic interaction. At last, the Lax integrability and conservation laws of three types of nonlocal FL equations is studied. The results obtained in this paper possess new properties that different from the ones for FL equation, which are useful in exploring novel physical phenomena of nonlocal systems in nonlinear media.

Keywords: Nonlocal Fokas-Lenells equation; Soliton solutions; Hirota bilinear method; Asymptotic analysis; Conservation laws.

1 Introduction

The Fokas-Lenells (FL) equation was derived as an integrable generalization of the nonlinear Schrödinger (NLS) equation using bi-Hamiltonian methods [1], which is a completely integrable nonlinear partial differential equation (here means it admits a Lax pair). The FL equation

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describes the propagation of nonlinear light pulses in monomode optical fibers when certain higher-order nonlinear effects are taken into account [2, 3], and it contains a lot of physical features in the solitary waves theory and optical fibers phenomena [4–6]. For constructing the soliton solutions, the bright/dark solitons and rouge waves of the FL equation, there are a large number of research literatures. The inverse scattering transformation was established by Fokas and Lenells in their original paper [2]. The theta function representations of algebro-geometric solutions were constructed in [7]. In [8] multi-Hamiltonian structure and infinitely many conservation laws were established for the vector Kaup-Newell hierarchy of the positive and negative orders. Some other methods, such as Darboux transformation method [5, 8–10], Hirota bilinear method [11–13], Riemann-Hilbert problem [14–16], Bäcklund transformation [17] and trial equation method [18] could be found in references.

Recently, the nonlocal systems have been done by Ablowitz and Musslimani when they propose the reverse space nonlocal Schrödinger equation with parity-time (PT) symmetry [19]. The solution's evolution at location x depends not only on the local solution at x, but also on the nonlocal solution at the distant position $-x$ [20]. Since the nonlocal NLS equation was found, a large number of nonlocal integrable systems have been studied, such as nonlocal modified Korteweg-de Vries (KdV) equation [21], reverse space-time nonlocal Fokas-Lenells equation [22] and so on. Like the local case, the nonlocal systems also have integral properties, and some methods in local system are still applicable in the nonlocal systems. For instance, Gürses and Pekcan investigated the nonlocal Schrödinger equation and modified KdV equation, and found their soliton solutions by Hirota bilinear method [23, 24]. Yang et al. proposed the localized wave solutions of the reverse space nonlocal Lakshmanan-Porsezian-Daniel equation by the Darboux transformations [25]. He, Fan and Xu studied the Cauchy problem with decaying initial data for the reverse space-time nonlocal modified KdV equation by Riemann-Hilbert method [26]. Feng et al. [27] considered a nonlocal nonlinear Schrödinger equation with PT-symmetry for both zero and nonzero boundary conditions via the combination of Hirota's bilinear method and the Kadomtsev-Petviashvili hierarchy reduction method. Peng et al. [28] investigated the fully PT-symmetric inverse space nonlocal (2+1)-dimensional nonlinear NLS equation by using Hirota's bilinear method. Liu et al. [29] studied the nonlocal Gross-Pitaevskii equation with a parabolic potential employing the reduction approach on double Wronskians. The main purpose of this paper is to focus on nonlocal FL equations and research the applicatory of the Hirota bilinear method to find their multisoliton solutions. In these new types of nonlocal equations, in addition to the terms at the space-time point (x, t), there are terms at the points $(-x, t)$, $(x, -t)$ and $(-x, -t)$. It's distinctly different from local FL equation that these nonlocal FL equations have their novel spatial and/or temporal coupling, which could give rise to new physical effects and novel physical applications.

Here we consider the reverse space nonlocal Fokas-Lenells equation

$$u_{xt}(x,t) - iu(x,t) + 2iu(x,t)u^*(-x,t)u_x(x,t) = 0, \tag{1}$$

where $u(x,t)$ is a complex valued function for the independent spatial variable x and temporal variable t, and $u^*(-x,t)$ denotes complex conjugate of $u(x,t)$. The subscript x (or t) denotes partial derivative with respect to x (or t). Through the method in [20], the variable transformations from reverse space nonlocal FL equation to reverse time and reverse space-time nonlocal FL equation can be derived as follows

$$a)x \to -ix, t \to it, \tag{2}$$

$$b)x \to -x, t \to it. \tag{3}$$

Through these variable transformations, reverse time and reverse space-time nonlocal FL equation are presented subsequently

$$u_{xt}(x,t) - iu(x,t) - 2u(x,t)u^*(x,-t)u_x(x,t) = 0, \tag{4}$$

$$u_{xt}(x,t) - u(x,t) - 2u(x,t)u^*(-x,-t)u_x(x,t) = 0, \tag{5}$$

where $u = u(x,t)$ is a complex-valued function of x and t, and the $*$ denotes complex conjugation. In this paper, we use the Hirota bilinear method to get one-, two- and three-soliton solutions of the reverse space nonlocal FL equation (1), then study multisoliton solutions of the reverse time and inverse space-time nonlocal FL equations through variable transforms. Asymptotic analysis is used to investigate the elastic interactions and inelastic interactions of the two solitons and the three solitons solutions, and dynamical behaviors of the multisoliton solutions are investigated by analyzing their wave structures. Finally, the Lax pairs and conservation laws of three types of nonlocal FL equations are obtained.

The outline of this paper is presented as follows. In Section 2, the one-, two- and three-soliton solutions of three types of nonlocal FL equations are obtained by using Hirota bilinear method and the variable transformations (2) and (3). And some figures are given to describe the dynamic characteristics of these soliton solutions. In Section 3, the asymptotic analysis on two- and three- soliton solutions of the reverse space nonlocal FL equation is given. In Section 4, we exhibit the Lax pairs of three types of nonlocal FL equations. Meanwhile, based on the Lax pairs, the infinitely many conservation laws of these equations (1), (4) and (5) are derived. Finally, the conclusions of this paper are stated in Section 5.

2 Multisoliton solutions of three types of nonlocal FL equations

2.1 One-soliton solutions of three types of nonlocal FL equations

In order to receive one-soliton solution of reverse space nonlocal FL equation, the Hirota bilinear method [30–33] and symbolic computation are used. By introducing the dependent variable transformations

$$u(x,t)=\frac{G(x,t)}{F(x,t)},\ u^*(-x,t)=\frac{G^*(-x,t)}{F^*(-x,t)}, \tag{6}$$

where $G(x,t)$, $G^*(-x,t)$, $F(x,t)$ and $F^*(-x,t)$ are complex functions, the nonlocal FL equation (1) converts into the following bilinear equation

$$\frac{1}{F^2}(D_xD_tG\cdot F-iGF)+\frac{G}{F^3}(-D_xD_tF\cdot F+2i\frac{G^*D_xG\cdot F}{F^*})=0. \tag{7}$$

This equation can be decoupled into the following system of bilinear equations for the functions F and G,

$$D_xD_tG\cdot F=iGF, \tag{8}$$
$$D_xD_tF\cdot F=2i\frac{G^*D_xG\cdot F}{F^*}, \tag{9}$$

where the D_x and D_t are bilinear operators. These operators defined as

$$D_x^mD_t^n(GF)=(\frac{\partial}{\partial x}-\frac{\partial}{\partial x_1})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t_1})^nG(x,t)F(x_1,t_1)|_{(x=x_1,t=t_1)}, \tag{10}$$

where m and n are non-negative integers.

Solving the above series of bilinear equations (8)-(9) and combining (6), some soliton solutions can be obtained. We expand the unknown functions (x,t), $G^*(-x,t)$, $F(x,t)$ and $F^*(-x,t)$as a polynomial of small parameter ϵ as follows

$$\begin{aligned}
&G(x,t)=\epsilon G_1+\epsilon^3G_3+\epsilon^5G_5+\cdots,\\
&G^*(-x,t)=\epsilon G_1^*+\epsilon^3G_3^*+\epsilon^5G_5^*+\cdots,\\
&F(x,t)=1+\epsilon^2F_2+\epsilon^4F_4+\epsilon^6F_6+\cdots,\\
&F^*(-x,t)=1+\epsilon^2F_2^*+\epsilon^4F_4^*+\epsilon^6F_6^*+\cdots,
\end{aligned} \tag{11}$$

where the G_1, F_2, etc. are functions with spatial variable x and temporal variable t, the functions G_1^*, F_2^*, etc. with variables $-x$ and t. Substituting the above expansions into Eqs. (8)-(9), and comparing the coefficients of ϵ, the unknown functions $G(x,t)$, $G^*(-x,t)$, $F(x,t)$ and $F^*(-x,t)$ can be obtained by selecting appropriate functions G_1, G_1^*, F_2, F_2^*.

In this section, the unknown functions $G(x,t)$, $G^*(-x,t)$, $F(x,t)$ and $F^*(-x,t)$ are expanded in terms of a small parameter ϵ as follows

$$\begin{aligned}
&G(x,t)=\epsilon G_1,\\
&G^*(-x,t)=\epsilon G_1^*,\\
&F(x,t)=1+\epsilon^2F_2,\\
&F^*(-x,t)=1+\epsilon^2F_2^*,
\end{aligned} \tag{12}$$

Substituting (12) into bilinear equation (8)-(9), we obtain a set of equations by comparing the coefficients of same powers of ϵ to zero

$$G_{1xt} = iG_1, \tag{13}$$
$$F_{2xt} = iG_1^* G_{1x}, \tag{14}$$

where G_1, G_1^*, F_2 and F_2^* are given rise to as follows

$$\begin{aligned} G_1 &= e^{\eta_1}, \\ G_1^* &= e^{\eta_1^*}, \\ F_2 &= A_1 e^{\eta_1+\eta_1^*}, \\ F_2^* &= A_1^* e^{\eta_1+\eta_1^*}. \end{aligned} \tag{15}$$

We suppose that $\eta_1 = k_1 x - \omega_1 t + \eta_{10}$, $\eta_1^* = -k_1^* x - \omega_1^* t + \eta_{10}^*$, and k_1, k_1^* are arbitrary complex constants. Form Eqs. (13)-(14), the relations about ω_1, A_1 and k_1 are given as follows

$$\omega_1 = -\frac{i}{k_1}, \tag{16}$$
$$A_1 = \frac{ik_1}{(k_1 - k_1^*)(-\omega_1 - \omega_1^*)}. \tag{17}$$

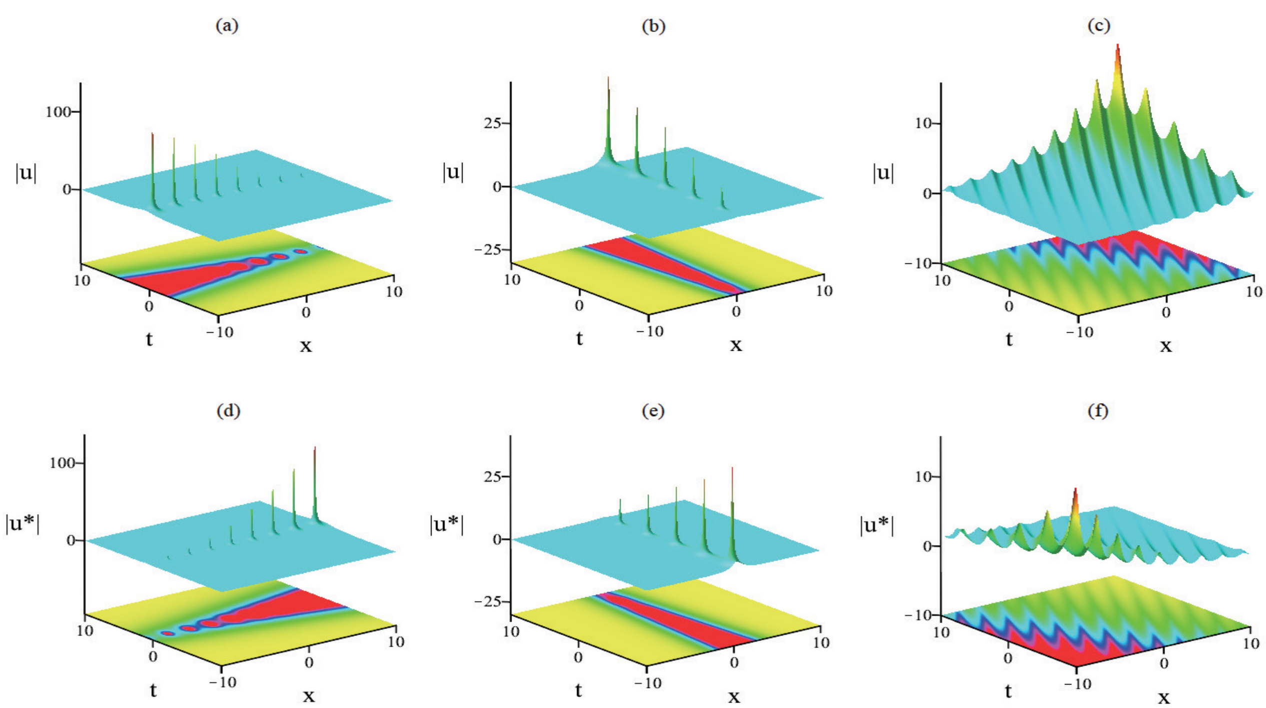

Figure 1: One soliton solutions of reverse space or/and time nonlocal FL equations (with parameters: $k_1 = -0.15 + 1.3i$, $k_1^* = -0.15 - 1.3i$, $\eta_{10} = \eta_{10}^* = 0$). (a) and (d) describe the reverse space FL equation; (b) and (e) describe the reverse time FL equation; (c) and (f) describe the reverse space-time FL equation.

Since the ω_1^* is the complex conjugate of ω_1 and the A_1^* is the complex conjugate of A_1, the expressions for ω_1^* and A_1^* are presented as follows

$$\omega_1^* = \frac{i}{k_1^*}, \tag{18}$$

$$A_1^* = \frac{-ik_1^*}{(k_1^* - k_1)(-\omega_1^* - \omega_1)}. \tag{19}$$

Then, the general one-soliton solution of the reverse space nonlocal FL equation (1) is

$$u(x,t) = \frac{e^{\eta_1}}{1 + A_1 e^{\eta_1+\eta_1^*}}. \tag{20}$$

According to the bilinear form of parity transformed complex conjugate equation, the parity transformed complex conjugate field is derived in the form

$$u^*(-x,t) = \frac{e^{\eta_1^*}}{1 + A_1^* e^{\eta_1+\eta_1^*}}. \tag{21}$$

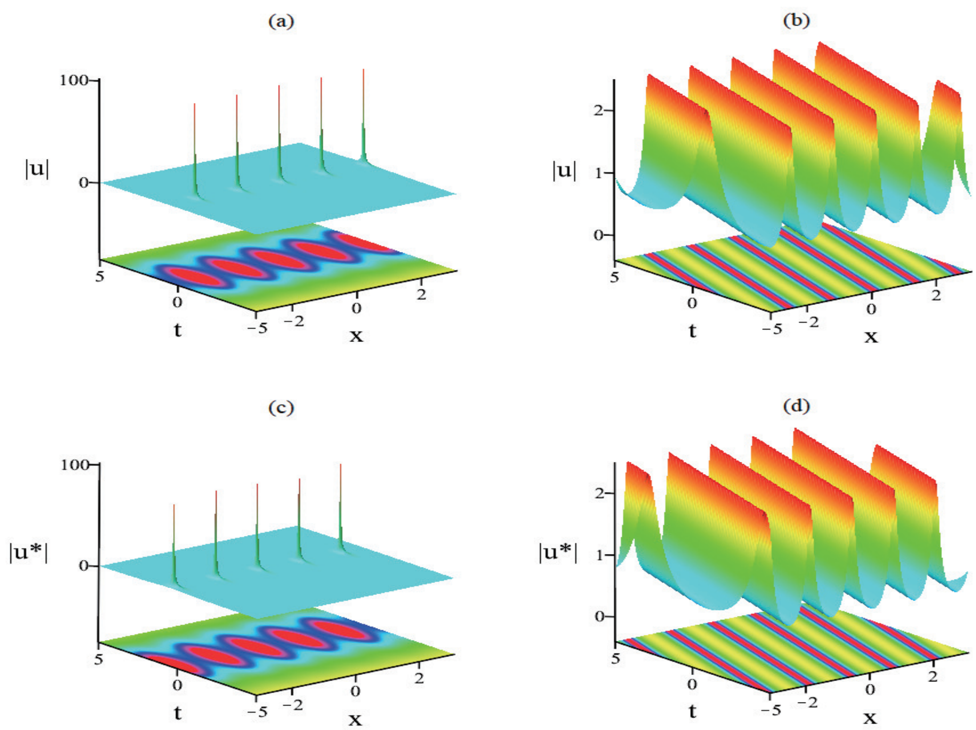

Figure 2: One soliton solutions of reverse space and reverse space-time nonlocal FL equations (with parameters: $k_1 = 2.4i$, $k_1^* = -2.4i$, $\eta_{10} = \eta_{10}^* = 0$). (a) and (c) describe the reverse space FL equation; (b) and (d) describe the reverse space-time FL equation.

Substituting the variable transformations Eqs. (2)-(3) into one-soliton solutions Eqs. (20)-(21) of the reverse space nonlocal FL equation, then one-soliton solutions of the reverse time and the reverse space-time nonlocal FL equation are given

$$a)\quad u(x,t)=\frac{e^{\xi_1}}{1+A_1e^{\xi_1+\xi_1^*}}, \tag{22}$$

$$u^*(x,-t)=\frac{e^{\xi_1^*}}{1+A_1^*e^{\xi_1+\xi_1^*}}, \tag{23}$$

$$b)\quad u(x,t)=\frac{e^{\zeta_1}}{1+A_1e^{\zeta_1+\zeta_1^*}}, \tag{24}$$

$$u^*(-x,-t)=\frac{e^{\zeta_1^*}}{1+A_1^*e^{\zeta_1+\zeta_1^*}}, \tag{25}$$

where $\xi_1=-ik_1x-i\omega_1t+\eta_{10}$, $\xi_1^*=ik_1^*x-i\omega_1^*t+\eta_{10}^*$, $\zeta_1=-k_1x-i\omega_1t+\eta_{10}$, $\zeta_1^*=k_1^*x-i\omega_1^*t+\eta_{10}^*$.

Here, some figures are provided to describe the one-soliton solutions Eqs. (20)-(25) of three types of nonlocal FL equations (see Fig.1-Fig.3). In Fig.1, (a), (b) and (c) are the profiles of $|u|$, and (d), (e) and (f) are the profiles of $|u^*|$. The results show that the solutions of three types of FL equations are periodic wave, and the periodic oscillations have exponential growth trend. It is obvious that $|u|$ and $|u^*|$ of the reverse space/time nonlocal FL equation have the same shapes as spatial/time evolution, but their enhancing shapes are antipodal. In order to intuitively observe one-soliton solutions' difference between the reverse space/time nonlocal FL equation and the reverse space-time nonlocal FL equation, more figures (Fig.2, Fig.3) are provided. These figures have the same parameters k_1, k_1^*, η_{10}, η_{10}^* for different equations.

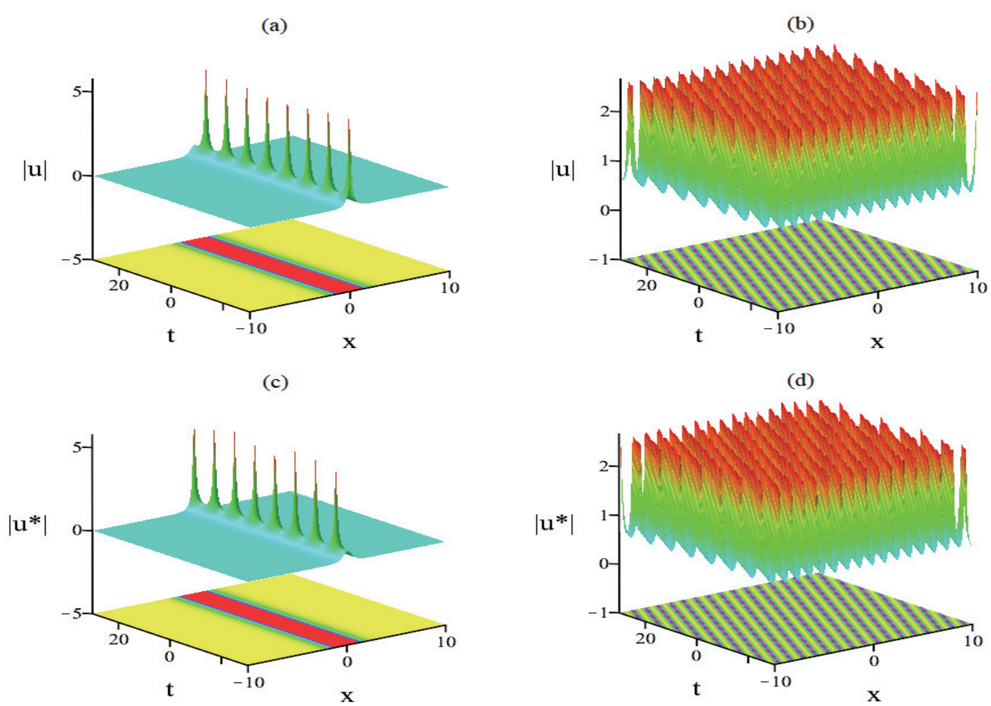

Figure 3: One soliton solutions of reverse time and reverse space-time nonlocal FL equations (with parameters: $k_1=2.5i$, $k_1^*=-2.5i$, $\eta_{10}=\eta_{10}^*=0$). (a) and (c) describe the reverse time FL equation; (b) and (d) describe the reverse space-time FL equation.

2.2 Two-soliton solutions of three types of nonlocal FL equations

The two-soliton solution of the reverse space nonlocal FL equation (1) can also be obtained with Hirota bilinear method. We consider the truncating of the following expansions $G(x,t) = \epsilon G_1 + \epsilon^3 G_3$, $G^*(-x,t) = \epsilon G_1^* + \epsilon^3 G_3^*$, $F(x,t) = 1 + \epsilon^2 F_2 + \epsilon^4 F_4$, $F^*(-x,t) = 1 + \epsilon^2 F_2^* + \epsilon^4 F_4^*$.

Substituting these expansions into the bilinear equations (8)-(9), and equating the coefficients of same powers of ϵ to zero, a set of equations can be derived

$$G_{1xt} = iG_1, \tag{26}$$

$$G_{1xt}F_2 + G_{3xt} - G_{1t}F_{2x} - G_{1x}F_{2t} + G_1F_{2xt} = i(G_1F_2 + G_3), \tag{27}$$

$$F_{2xt} = iG_{1x}G_1^*, \tag{28}$$

$$F_{4xt} + F_2F_{2xt} - F_{2x}F_{2t} + F_2^*F_{2xt} = iG_1^*(G_{1x}F_2 + G_{3x} - G_1F_{2x}) + iG_{1x}G_3^*, \tag{29}$$

where G_1, G_1^*, F_2 and F_2^* are given rise to as follows

$$\begin{aligned}
&G_1 = e^{\eta_1} + e^{\eta_2},\\
&G_1^* = e^{\eta_1^*} + e^{\eta_2^*},\\
&F_2 = A_1 e^{\eta_1+\eta_1^*} + A_2 e^{\eta_1+\eta_2^*} + A_3 e^{\eta_2+\eta_1^*} + A_4 e^{\eta_2+\eta_2^*},\\
&F_2^* = A_1^* e^{\eta_1+\eta_1^*} + A_2^* e^{\eta_1^*+\eta_2} + A_3^* e^{\eta_1+\eta_2^*} + A_4^* e^{\eta_2+\eta_2^*}.
\end{aligned} \tag{30}$$

In the above expressions, $\eta_1 = k_1x - \omega_1 t + \eta_{10}$, $\eta_1^* = -k_1^*x - \omega_1^*t + \eta_{10}^*$, $\eta_2 = k_2x - \omega_2t + \eta_{20}$, $\eta_2^* = -k_2^*x - \omega_2^*t + \eta_{20}^*$, and k_1, k_1^*, k_2 and k_2^* are arbitrary complex constants. Form Eqs. (26)-(28), we know

$$\begin{aligned}
&\omega_1 = -\frac{i}{k_1}, \omega_1^* = \frac{i}{k_1^*},\\
&\omega_2 = -\frac{i}{k_2}, \omega_2^* = \frac{i}{k_2^*},
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
&A_1 = \frac{ik_1}{(k_1 - k_1^*)(-\omega_1 - \omega_1^*)}, A_1^* = \frac{-ik_1^*}{(k_1^* - k_1)(-\omega_1^* - \omega_1)},\\
&A_2 = \frac{ik_1}{(k_1 - k_2^*)(-\omega_1 - \omega_2^*)}, A_2^* = \frac{-ik_1^*}{(k_1^* - k_2)(-\omega_1^* - \omega_2)},\\
&A_3 = \frac{ik_2}{(-k_1^* + k_2)(-\omega_1^* - \omega_2)}, A_3^* = \frac{-ik_2^*}{(-k_1 + k_2^*)(-\omega_1 - \omega_2^*)},\\
&A_4 = \frac{ik_2}{(k_2 - k_2^*)(-\omega_2 - \omega_2^*)}, A_4^* = \frac{-ik_2^*}{(k_2^* - k_2)(-\omega_2^* - \omega_2)}.
\end{aligned} \tag{32}$$

Thus, a set of equations for unknown functions $G_1(x,t)$, $G_1^*(-x,t)$, $F_2(x,t)$ and $F_2^*(-x,t)$ are obtained. Substituting the expressions for G_1 and F_2 into the Eq. (27), the function G_3 and its parity transformed complex conjugate G_3^* are given in the form

$$G_3 = B_1 e^{\eta_1+\eta_2+\eta_1^*} + B_2 e^{\eta_1+\eta_2+\eta_2^*}, \tag{33}$$

$$G_3^* = B_1^* e^{\eta_1^*+\eta_2^*+\eta_1} + B_2^* e^{\eta_1^*+\eta_2^*+\eta_2}, \tag{34}$$

where

$$B_1 = -\frac{(k_1-k_2)^2 k_1^{*3}}{(k_2-k_1^*)^2(k_1-k_1^*)^2}, B_2 = -\frac{(k_1-k_2)^2 k_2^{*3}}{(k_2-k_2^*)^2(k_1-k_2^*)^2},$$
$$B_1^* = -\frac{(k_1^*-k_2^*)^2 k_1^{3}}{(k_2^*-k_1)^2(k_1^*-k_1)^2}, B_2^* = -\frac{(k_1^*-k_2^*)^2 k_2^{3}}{(k_2^*-k_2)^2(k_1^*-k_2)^2}.$$

Then substituting the expressions of G_1, G_1^*, G_3, G_3^*, F_2 and F_2^* into Eq. (29), the functions F_4 and F_4^* are derived as follows

$$F_4 = C_1 e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*},\ F_4^* = C_1^* e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}, \tag{35}$$

where

$$C_1 = \frac{k_1^2 k_2^2 (k_1-k_2)^2 k_1^* k_2^* (k_1^*-k_2^*)^2}{(k_1-k_1^*)^2(k_1-k_2^*)^2(k_2-k_1^*)^2(k_2-k_2^*)^2},$$
$$C_1^* = \frac{k_1 k_2 (k_1-k_2)^2 k_1^{*2} k_2^{*2} (k_1^*-k_2^*)^2}{(k_1-k_1^*)^2(k_1-k_2^*)^2(k_2-k_1^*)^2(k_2-k_2^*)^2}.$$

The general nonlocal two-soliton solution of the reverse space nonlocal FL equation (1) is given as follows

$$u(x,t) = \frac{G_1+G_3}{1+F_2+F_4}. \tag{36}$$

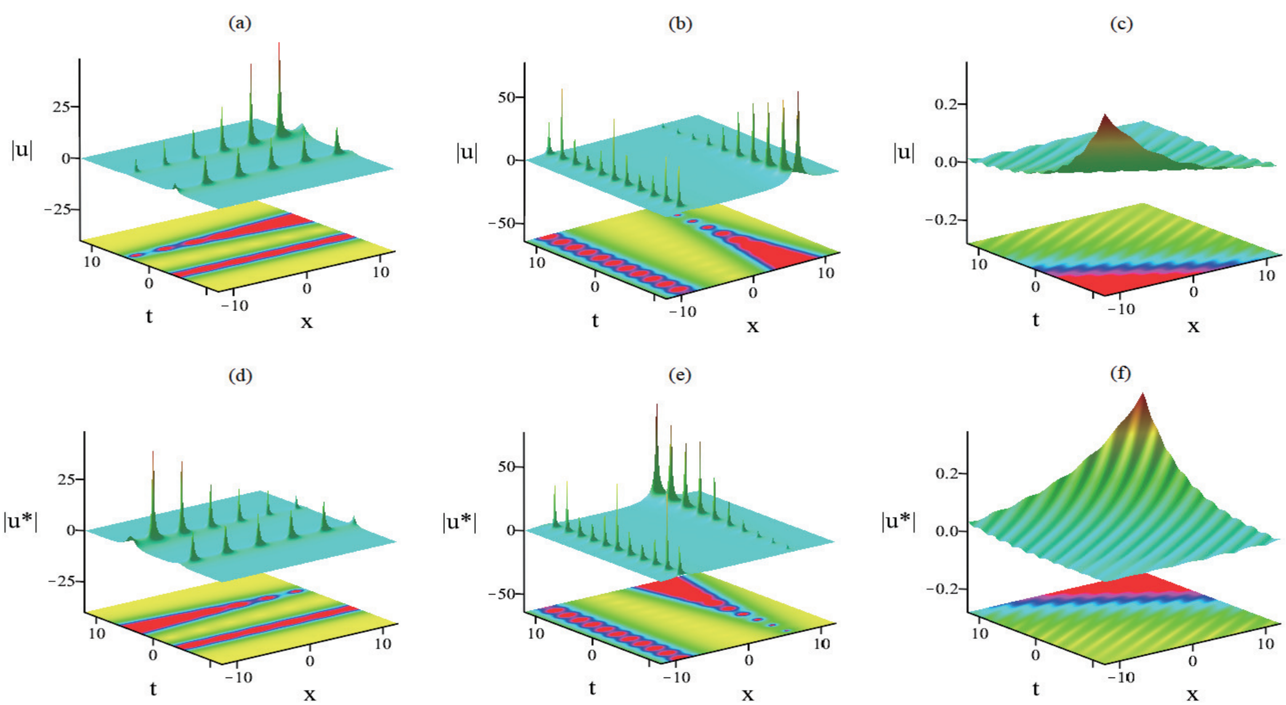

Figure 4: Two solitons solutions of reverse space or/and time nonlocal FL equations (with parameters: $k_1 = 0.7i$, $k_1^* = -0.7i$, $k_2 = 0.1-0.8i$, $k_2^* = 0.1+0.8i$, $\eta_{10} = \eta_{10}^* = 2.5$, $\eta_{20} = \eta_{20}^* = 1$). (a) and (d) describe the reverse space FL equation; (b) and (e) describe the reverse time FL equation; (c) and (f) describe the reverse space-time FL equation.

According to the bilinear form of parity transformed complex conjugate equation, the parity transformed complex conjugate field is derived in the form

$$u^*(-x,t)=\frac{G_1^*+G_3^*}{1+F_2^*+F_4^*}. \tag{37}$$

Through the transformations $x=-i\hat{x}, t=i\hat{t}$ and $x=-\hat{x}, t=i\hat{t}$, the two-soliton solutions (36)-(37) of reverse space nonlocal FL equation transform into two-soliton solutions of the reverse time nonlocal FL equation (4) and the reverse space-time nonlocal FL equation (5). The solutions are presented as follows

$$a)\ u(x,t)=\frac{e^{\xi_1}+e^{\xi_2}+B_1e^{\xi_1+\xi_2+\xi_1^*}+B_2e^{\xi_1+\xi_2+\xi_2^*}}{1+A_1e^{\xi_1+\xi_1^*}+A_2e^{\xi_1+\xi_2^*}+A_3e^{\xi_2+\xi_1^*}+A_4e^{\xi_2+\xi_2^*}+C_1e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}}, \tag{38}$$

$$u^*(x,-t)=\frac{e^{\xi_1^*}+e^{\xi_2^*}+B_1^*e^{\xi_1^*+\xi_2^*+\xi_1}+B_2^*e^{\xi_1^*+\xi_2^*+\xi_2}}{1+A_1^*e^{\xi_1+\xi_1^*}+A_2^*e^{\xi_1^*+\xi_2}+A_3^*e^{\xi_2^*+\xi_1}+A_4^*e^{\xi_2+\xi_2^*}+C_1^*e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}}, \tag{39}$$

$$b)\ u(x,t)=\frac{e^{\zeta_1}+e^{\zeta_2}+B_1e^{\zeta_1+\zeta_2+\zeta_1^*}+B_2e^{\zeta_1+\zeta_2+\zeta_2^*}}{1+A_1e^{\zeta_1+\zeta_1^*}+A_2e^{\zeta_1+\zeta_2^*}+A_3e^{\zeta_2+\zeta_1^*}+A_4e^{\zeta_2+\zeta_2^*}+C_1e^{\zeta_1+\zeta_2+\zeta_1^*+\zeta_2^*}}, \tag{40}$$

$$u^*(-x,-t)=\frac{e^{\zeta_1^*}+e^{\zeta_2^*}+B_1^*e^{\zeta_1^*+\zeta_2^*+\zeta_1}+B_2^*e^{\zeta_1^*+\zeta_2^*+\zeta_2}}{1+A_1^*e^{\zeta_1+\zeta_1^*}+A_2^*e^{\zeta_1^*+\zeta_2}+A_3^*e^{\zeta_2^*+\zeta_1}+A_4^*e^{\zeta_2+\zeta_2^*}+C_1^*e^{\zeta_1+\zeta_2+\zeta_1^*+\zeta_2^*}}, \tag{41}$$

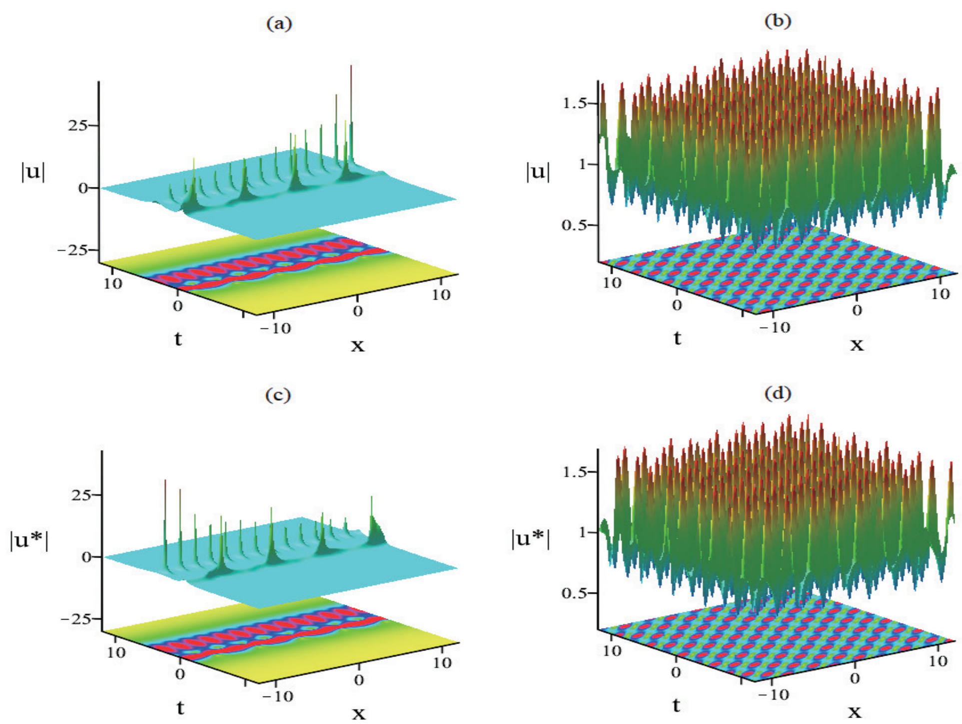

Figure 5: Two-soliton solutions of reverse space and reverse space-time nonlocal FL equations (with parameters: $k_1=1.74i$, $k_1^*=-1.74i$, $k_2=-0.5i$, $k_2^*=0.5i$, $\eta_{10}=\eta_{10}^*=-1.5$, $\eta_{20}=\eta_{20}^*=0$). (a) and (c) describe the reverse space FL equation; (b) and (d) describe the reverse space-time FL equation.

where

$$\xi_1 = -ik_1x - i\omega_1 t + \eta_{10}, \xi_1^* = ik_1^*x - i\omega_1^* t + \eta_{10}^*,$$
$$\xi_2 = -ik_2x - i\omega_2 t + \eta_{20}, \xi_2^* = ik_2^*x - i\omega_2^* t + \eta_{20}^*,$$
$$\zeta_1 = -k_1x - i\omega_1 t + \eta_{10}, \zeta_1^* = k_1^*x - i\omega_1^* t + \eta_{10}^*,$$
$$\zeta_2 = -k_2x - i\omega_2 t + \eta_{20}, \zeta_2^* = k_2^*x - i\omega_2^* t + \eta_{20}^*.$$

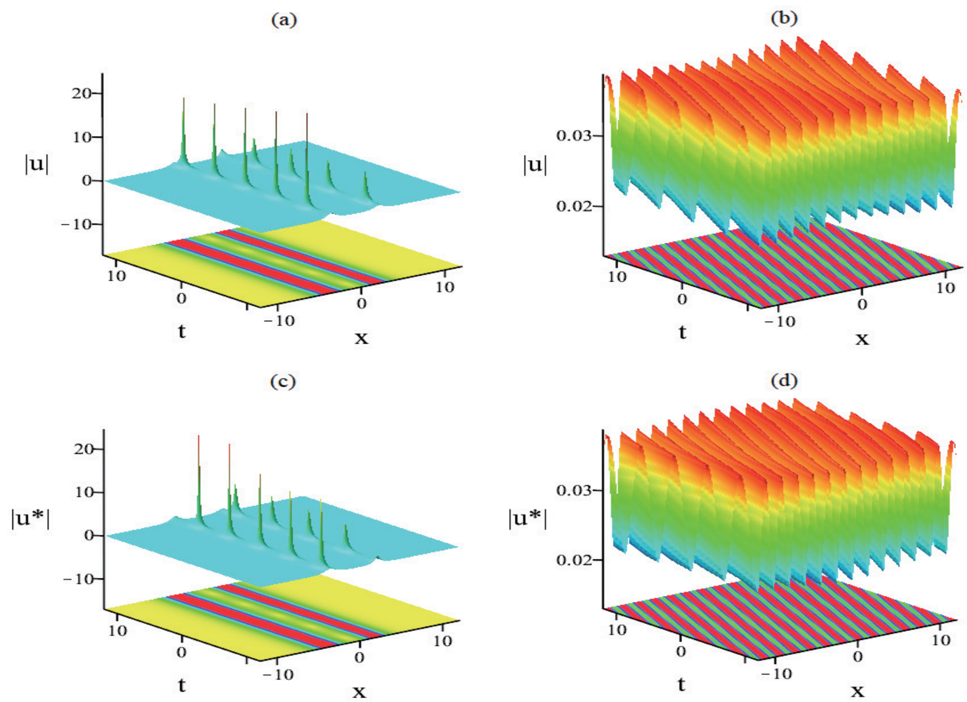

Figure 6: Two-soliton solutions of reverse time and reverse space-time nonlocal FL equations (with parameters: $k_1 = 1.5i$, $k_1^* = -1.5i$, $k_2 = -1.8i$, $k_2^* = 1.8i$, $\eta_{10} = \eta_{10}^* = 1$, $\eta_{20} = \eta_{20}^* = -0.5$). (a) and (c) describe the reverse time FL equation; (b) and (d) describe the reverse space-time FL equation.

Then some figures are given to describe the two-soliton solutions (36)-(41) of three types of nonlocal FL equations (see Fig.4-Fig.6). In Fig.4, (a), (b) and (c) are the profiles of $|u|$, and (d), (e) and (f) are the profiles of $|u^*|$. Profiles of the reverse space nonlocal FL equation and the reverse time nonlocal FL equation present two breather-like solitons, while the two-soliton solution of the reverse space-time nonlocal FL equation are periodic wave, and the periodic oscillations have exponential growth trend. It is obvious that $|u|$ and $|u^*|$ of the reverse space/time FL equation have the same shapes as spatial/time evolution, but their enhancing shapes are antipodal. Figure 5 shows the comparison of the reverse space FL equation and the reverse space-time FL equation, and figure 6 shows the difference between the reverse time FL equation and the reverse space-time FL equation. These figures have the same parameters k_1, k_1^*, k_2, k_2^*,

η_{10}, η_{10}^*, η_{20}, η_{20}^* for different equations. Through these pictures, we could observe two-soliton solutions' differences intuitively. The shapes of two-soliton solutions of the reverse space/time FL equation are parallel with the x or t axis, however two-soliton solution of the reverse space-time FL equation is not parallel with neither x axis nor t axis, which can be viewed as a parallel superposition of time and space local solitons.

2.3 Three-soliton solutions of three types of nonlocal FL equations

Through Hirota bilinear method, the three-soliton solution of the reverse space nonlocal FL equation (1) can be obtained. The truncating expansions of $G(x,t)$, $G^*(-x,t)$, $F(x,t)$ and $F^*(-x,t)$ are given as follows

$$\begin{aligned}
&G(x,t)=\epsilon G_1+\epsilon^3 G_3+\epsilon^5 G_5,\\
&G^*(-x,t)=\epsilon G_1^*+\epsilon^3 G_3^*+\epsilon^5 G_5^*,\\
&F(x,t)=1+\epsilon^2 F_2+\epsilon^4 F_4+\epsilon^6 F_6,\\
&F^*(-x,t)=1+\epsilon^2 F_2^*+\epsilon^4 F_4^*+\epsilon^6 F_6^*.
\end{aligned}$$

Substituting these expansions into the bilinear equations (8)-(9) and equating the coefficients of same powers of ϵ to zero, a set of equations can be derived

$$D_xD_tG_1\cdot 1=iG_1, \tag{42}$$

$$D_xD_t(G_1\cdot F_2+G_3\cdot 1)=i(G_1F_2+G_3), \tag{43}$$

$$D_xD_t(G_1\cdot F_4+G_3\cdot F_2+G_5\cdot 1)=i(G_1F_4+G_3F_2+G_5), \tag{44}$$

$$D_xD_t1\cdot F_2=iG_1^*D_xG_1\cdot 1, \tag{45}$$

$$\begin{aligned}
&D_xD_t1\cdot F_4+\frac{1}{2}D_xD_tF_2\cdot F_2+F_2^*D_xD_t1\cdot F_2=iG_1^*D_x(G_1\cdot F_2+G_3\cdot 1)\\
&+iG_3^*D_xG_1\cdot 1,
\end{aligned} \tag{46}$$

$$\begin{aligned}
&D_xD_t(1\cdot F_6+F_2\cdot F_4)+\frac{1}{2}F_2^*D_xD_t(F_2\cdot F_2)+F_2^*D_xD_t1\cdot F_4+F_4^*D_xD_t1\cdot F_2\\
&=iG_1^*D_x(G_1\cdot F_4+G_3\cdot F_2+G_5\cdot 1)+iG_3^*D_x(G_1\cdot F_2+G_3\cdot 1)+iG_5^*D_x(G_1\cdot 1),
\end{aligned} \tag{47}$$

where G_1, G_1^*, F_2 and F_2^* are given rise to as follows

$$\begin{aligned}
&G_1=e^{\eta_1}+e^{\eta_2}+e^{\eta_3},\\
&G_1^*=e^{\eta_1^*}+e^{\eta_2^*}+e^{\eta_3^*},\\
&F_2=A_1e^{\eta_1+\eta_1^*}+A_2e^{\eta_1+\eta_2^*}+A_3e^{\eta_2+\eta_1^*}+A_4e^{\eta_2+\eta_2^*}+A_5e^{\eta_1+\eta_3^*}+A_6e^{\eta_2+\eta_3^*}+A_7e^{\eta_3+\eta_1^*}\\
&+A_8e^{\eta_3+\eta_2^*}+A_9e^{\eta_3+\eta_3^*},\\
&F_2^*=A_1^*e^{\eta_1+\eta_1^*}+A_2^*e^{\eta_1^*+\eta_2}+A_3^*e^{\eta_2^*+\eta_1}+A_4^*e^{\eta_2+\eta_2^*}+A_5^*e^{\eta_1^*+\eta_3}+A_6^*e^{\eta_2^*+\eta_3}+A_7^*e^{\eta_3^*+\eta_1},\\
&+A_8^*e^{\eta_3^*+\eta_2}+A_9^*e^{\eta3+\eta_3^*}.
\end{aligned} \tag{48}$$

In these equations, $\eta_1 = k_1x - \omega_1 t + \eta_{10}$, $\eta_1^* = -k_1^* x - \omega_1^* t + \eta_{10}^*$, $\eta_2 = k_2x - \omega_2 t + \eta_{20}$, $\eta_2^* = -k_2^* x - \omega_2^* t + \eta_{20}^*$, $\eta_3 = k_3x - \omega_3 t + \eta_{30}$, $\eta_3^* = -k_3^* x - \omega_3^* t + \eta_{30}^*$, and k_1, k_2 and k_3 are arbitrary complex constants. Form Eqs. (42)-(45), we know

$$\begin{aligned}
&\omega_1 = -\frac{i}{k_1}, \omega_1^* = \frac{i}{k_1^*},\\
&\omega_2 = -\frac{i}{k_2}, \omega_2^* = \frac{i}{k_2^*},\\
&\omega_3 = -\frac{i}{k_3}, \omega_3^* = \frac{i}{k_3^*},
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
&A_1 = \frac{ik_1}{(k_1 - k_1^*)(-\omega_1 - \omega_1^*)}, A_1^* = \frac{-ik_1^*}{(k_1^* - k_1)(-\omega_1^* - \omega_1)},\\
&A_2 = \frac{ik_1}{(k_1 - k_2^*)(-\omega_1 - \omega_2^*)}, A_2^* = \frac{-ik_1^*}{(k_1^* - k_2)(-\omega_1^* - \omega_2)},\\
&A_3 = \frac{ik_2}{(k_2 - k_1^*)(-\omega_2 - \omega_1^*)}, A_3^* = \frac{-ik_2^*}{(k_2^* - k_1)(-\omega_2^* - \omega_1)},\\
&A_4 = \frac{ik_2}{(k_2 - k_2^*)(-\omega_2 - \omega_2^*)}, A_4^* = \frac{-ik_2^*}{(k_2^* - k_2)(-\omega_2^* - \omega_2)},\\
&A_5 = \frac{ik_1}{(k_1 - k_3^*)(-\omega_1 - \omega_3^*)}, A_5^* = \frac{-ik_1^*}{(k_1^* - k_3)(-\omega_1^* - \omega_3)},\\
&A_6 = \frac{ik_2}{(k_2 - k_3^*)(-\omega_2 - \omega_3^*)}, A_6^* = \frac{-ik_2^*}{(k_2^* - k_3)(-\omega_2^* - \omega_3)},\\
&A_7 = \frac{ik_3}{(k_3 - k_1^*)(-\omega_3 - \omega_1^*)}, A_7^* = \frac{-ik_3^*}{(k_3^* - k_1)(-\omega_3^* - \omega_1)},\\
&A_8 = \frac{ik_3}{(k_3 - k_2^*)(-\omega_3 - \omega_2^*)}, A_8^* = \frac{-ik_3^*}{(k_3^* - k_2)(-\omega_3^* - \omega_2)},\\
&A_9 = \frac{ik_3}{(k_3 - k_3^*)(-\omega_3 - \omega_3^*)}, A_9^* = \frac{-ik_3^*}{(k_3^* - k_3)(-\omega_3^* - \omega_3)}.
\end{aligned} \tag{50}$$

Thus, a set of equations for the unknown functions $G_1(x,t)$, $G_1^*(-x,t)$, $F_2(x,t)$ and $F_2^*(-x,t)$ are obtained. In order to get the function G_3 and its parity transformed complex conjugate G_3^*, substituting the expressions for G_1 and F_2 into Eq. (43), G_3 and G_3^* are given in the form

$$\begin{aligned}
G_3 =& B_1 e^{\eta_1+\eta_2+\eta_1^*} + B_2 e^{\eta_1+\eta_2+\eta_2^*} + B_3 e^{\eta_1+\eta_2+\eta_3^*} + B_4 e^{\eta_1+\eta_3+\eta_1^*} + B_5 e^{\eta_1+\eta_3+\eta_2^*}\\
&+ B_6 e^{\eta_1+\eta_3+\eta_3^*} + B_7 e^{\eta_2+\eta_3+\eta_1^*} + B_8 e^{\eta_2+\eta_3+\eta_2^*} + B_9 e^{\eta_2+\eta_3+\eta_3^*},
\end{aligned} \tag{51}$$

$$\begin{aligned}
G_3^* =& B_1^* e^{\eta_1^*+\eta_2^*+\eta_1} + B_2^* e^{\eta_1^*+\eta_2^*+\eta_2} + B_3^* e^{\eta_1^*+\eta_2^*+\eta_3} + B_4^* e^{\eta_1^*+\eta_3^*+\eta_1} + B_5^* e^{\eta_1^*+\eta_3^*+\eta_2}\\
&+ B_6^* e^{\eta_1^*+\eta_3^*+\eta_3} + B_7^* e^{\eta_2^*+\eta_3^*+\eta_1} + B_8^* e^{\eta_2^*+\eta_3^*+\eta_2} + B_9^* e^{\eta_2^*+\eta_3^*+\eta_3},
\end{aligned} \tag{52}$$

where

$$
\begin{aligned}
B_1 &= -\frac{(k_1-k_2)^2 k_1^{*3}}{(k_2-k_1^*)^2(k_1-k_1^*)^2}, B_1^* = -\frac{(k_1^*-k_2^*)^2 k_1^3}{(k_2^*-k_1)^2(k_1^*-k_1)^2},\\
B_2 &= -\frac{(k_1-k_2)^2 k_2^{*3}}{(k_2-k_2^*)^2(k_1-k_2^*)^2}, B_2^* = -\frac{(k_1^*-k_2^*)^2 k_2^3}{(k_2^*-k_2)^2(k_1^*-k_2)^2},\\
B_3 &= -\frac{(k_1-k_2)^2 k_3^{*3}}{(k_2-k_3^*)^2(k_1-k_3^*)^2}, B_1^* = -\frac{(k_1^*-k_2^*)^2 k_3^3}{(k_2^*-k_3)^2(k_1^*-k_3)^2},\\
B_4 &= -\frac{(k_1-k_3)^2 k_1^{*3}}{(k_3-k_1^*)^2(k_1-k_1^*)^2}, B_4^* = -\frac{(k_1^*-k_3^*)^2 k_1^3}{(k_3^*-k_1)^2(k_1^*-k_1)^2},\\
B_5 &= -\frac{(k_1-k_3)^2 k_2^{*3}}{(k_3-k_2^*)^2(k_1-k_2^*)^2}, B_5^* = -\frac{(k_1^*-k_3^*)^2 k_2^3}{(k_3^*-k_2)^2(k_1^*-k_2)^2},\\
B_6 &= -\frac{(k_1-k_3)^2 k_3^{*3}}{(k_3-k_3^*)^2(k_1-k_3^*)^2}, B_6^* = -\frac{(k_1^*-k_3^*)^2 k_3^3}{(k_3^*-k_3)^2(k_1^*-k_3)^2},\\
B_7 &= -\frac{(k_2-k_3)^2 k_1^{*3}}{(k_3-k_1^*)^2(k_2-k_1^*)^2}, B_7^* = -\frac{(k_2^*-k_3^*)^2 k_1^3}{(k_3^*-k_1)^2(k_2^*-k_1)^2},\\
B_8 &= -\frac{(k_2-k_3)^2 k_2^{*3}}{(k_3-k_2^*)^2(k_2-k_2^*)^2}, B_8^* = -\frac{(k_2^*-k_3^*)^2 k_2^3}{(k_3^*-k_2)^2(k_2^*-k_2)^2},\\
B_9 &= -\frac{(k_2-k_3)^2 k_3^{*3}}{(k_3-k_3^*)^2(k_2-k_3^*)^2}, B_9^* = -\frac{(k_2^*-k_3^*)^2 k_3^3}{(k_3^*-k_3)^2(k_2^*-k_3)^2}.
\end{aligned} \tag{53}
$$

Then substituting the expressions of G_1, G_1^*, G_3, G_3^*, F_2 and F_2^* into Eq. (46), the functions F_4 and F_4^* are derived as follows

$$
\begin{aligned}
F_4 =& C_1 e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*} + C_2 e^{\eta_1+\eta_2+\eta_1^*+\eta_3^*} + C_3 e^{\eta_1+\eta_2+\eta_2^*+\eta_3^*} + C_4 e^{\eta_1+\eta_3+\eta_1^*+\eta_2^*} +\\
& C_5 e^{\eta_1+\eta_3+\eta_1^*+\eta_3^*} + C_6 e^{\eta_1+\eta_3+\eta_2^*+\eta_3^*} + C_7 e^{\eta_2+\eta_3+\eta_1^*+\eta_2^*} + C_8 e^{\eta_2+\eta_3+\eta_1^*+\eta_3^*} +\\
& C_9 e^{\eta_2+\eta_3+\eta_2^*+\eta_3^*},
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
F_4^* =& C_1^* e^{\eta_1^*+\eta_2^*+\eta_1+\eta_2} + C_2^* e^{\eta_1^*+\eta_2^*+\eta_1+\eta_3} + C_3^* e^{\eta_1^*+\eta_2^*+\eta_2+\eta_3} + C_4^* e^{\eta_1^*+\eta_3^*+\eta_1+\eta_2} +\\
& C_5^* e^{\eta_1^*+\eta_3^*+\eta_1+\eta_3} + C_6^* e^{\eta_1^*+\eta_3^*+\eta_2+\eta_3} + C_7^* e^{\eta_2^*+\eta_3^*+\eta_1+\eta_2} + C_8^* e^{\eta_2^*+\eta_3^*+\eta_1+\eta_3} +\\
& C_9^* e^{\eta_2^*+\eta_3^*+\eta_2+\eta_3},
\end{aligned} \tag{55}
$$

where

$$
\begin{aligned}
C_1 &= \frac{(k_1^*-k_2^*)^2(k_1-k_2)^2 k_1^2 k_1^* k_2^2 k_2^*}{(k_2-k_2^*)^2(k_1-k_2^*)^2(k_2-k_1^*)^2(k_1-k_1^*)^2},\\
C_2 &= \frac{(k_1^*-k_3^*)^2(k_1-k_2)^2 k_1^2 k_1^* k_2^2 k_3^*}{(k_2-k_3^*)^2(k_1-k_3^*)^2(k_2-k_1^*)^2(k_1-k_1^*)^2},\\
C_3 &= \frac{(k_2^*-k_3^*)^2(k_1-k_2)^2 k_1^2 k_2^* k_2^2 k_3^*}{(k_2-k_3^*)^2(k_1-k_3^*)^2(k_2-k_2^*)^2(k_1-k_2^*)^2},\\
C_4 &= \frac{(k_1^*-k_2^*)^2(k_1-k_3)^2 k_1^2 k_1^* k_3^2 k_2^*}{(k_3-k_2^*)^2(k_1-k_2^*)^2(k_3-k_1^*)^2(k_1-k_1^*)^2},
\end{aligned}
$$

$$C_5=\frac{(k_1^*-k_3^*)^2(k_1-k_3)^2k_1^2k_1^*k_3^2k_3^*}{(k_3-k_3^*)^2(k_1-k_3^*)^2(k_3-k_1^*)^2(k_1-k_1^*)^2},$$
$$C_6=\frac{(k_2^*-k_3^*)^2(k_1-k_3)^2k_1^2k_2^*k_3^2k_3^*}{(k_3-k_3^*)^2(k_1-k_3^*)^2(k_3-k_2^*)^2(k_1-k_2^*)^2},$$
$$C_7=\frac{(k_1^*-k_2^*)^2(k_2-k_3)^2k_2^2k_1^*k_3^2k_2^*}{(k_3-k_2^*)^2(k_2-k_2^*)^2(k_3-k_1^*)^2(k_2-k_1^*)^2},$$
$$C_8=\frac{(k_1^*-k_3^*)^2(k_2-k_3)^2k_2^2k_1^*k_3^2k_3^*}{(k_3-k_3^*)^2(k_2-k_3^*)^2(k_3-k_1^*)^2(k_2-k_1^*)^2},$$
$$C_9=\frac{(k_2^*-k_3^*)^2(k_2-k_3)^2k_2^2k_2^*k_3^2k_3^*}{(k_3-k_3^*)^2(k_2-k_3^*)^2(k_3-k_2^*)^2(k_2-k_2^*)^2}, \tag{56}$$

and

$$C_1^*=\frac{(k_1-k_2)^2(k_1^*-k_2^*)^2k_1^{*2}k_1k_2^{*2}k_2}{(k_2-k_2^*)^2(k_1^*-k_2)^2(k_2^*-k_1)^2(k_1^*-k_1)^2},$$
$$C_2^*=\frac{(k_1-k_3)^2(k_1^*-k_2^*)^2k_1^{*2}k_1k_2^{*2}k_3}{(k_2^*-k_3)^2(k_1^*-k_3)^2(k_2^*-k_1)^2(k_1^*-k_1)^2},$$
$$C_3^*=\frac{(k_2-k_3)^2(k_1^*-k_2^*)^2k_1^{*2}k_2k_2^{*2}k_3}{(k_2^*-k_3)^2(k_1^*-k_3)^2(k_2^*-k_2)^2(k_1^*-k_2)^2},$$
$$C_4^*=\frac{(k_1-k_2)^2(k_1^*-k_3^*)^2k_1^{*2}k_1k_3^{*2}k_2}{(k_3^*-k_2)^2(k_1^*-k_2)^2(k_3^*-k_1)^2(k_1^*-k_1)^2},$$
$$C_5^*=\frac{(k_1-k_3)^2(k_1^*-k_3^*)^2k_1^{*2}k_1k_3^{*2}k_3}{(k_3^*-k_3)^2(k_1^*-k_3)^2(k_3^*-k_1)^2(k_1^*-k_1)^2},$$
$$C_6^*=\frac{(k_2-k_3)^2(k_1^*-k_3^*)^2k_1^{*2}k_2k_3^{*2}k_3}{(k_3^*-k_3)^2(k_1^*-k_3)^2(k_3^*-k_2)^2(k_1^*-k_2)^2},$$
$$C_7^*=\frac{(k_1-k_2)^2(k_2^*-k_3^*)^2k_2^{*2}k_1k_3^{*2}k_2}{(k_3^*-k_2)^2(k_2^*-k_2)^2(k_3^*-k_1)^2(k_2^*-k_1)^2},$$
$$C_8^*=\frac{(k_1-k_3)^2(k_2^*-k_3^*)^2k_2^{*2}k_1k_3^{*2}k_3}{(k_3^*-k_3)^2(k_2^*-k_3)^2(k_3^*-k_1)^2(k_2^*-k_1)^2},$$
$$C_9^*=\frac{(k_2-k_3)^2(k_2^*-k_3^*)^2k_2^{*2}k_2k_3^{*2}k_3}{(k_3^*-k_3)^2(k_2^*-k_3)^2(k_3^*-k_2)^2(k_2^*-k_2)^2}. \tag{57}$$

So as to derive the expression of G_5, we substitute the expressions for G_1, G_1^*, G_3, G_3^*, F_2, F_2^*, F_4 and F_4^* into Eq. (44), the functions G_5 and G_5^* are given as follows

$$G_5=D_1e^{\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*}+D_2e^{\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_3^*}+D_3e^{\eta_1+\eta_2+\eta_3+\eta_2^*+\eta_3^*}, \tag{58}$$
$$G_5^*=D_1^*e^{\eta_1^*+\eta_2^*+\eta_3^*+\eta_1+\eta_2}+D_2^*e^{\eta_1^*+\eta_2^*+\eta_3^*+\eta_1+\eta_3}+D_3^*e^{\eta_1^*+\eta_2^*+\eta_3^*+\eta_2+\eta_3}, \tag{59}$$

where

$$D_1=\frac{(k_1^*-k_2^*)^2(k_2-k_3)^2(k_1-k_3)^2(k_1-k_2)^2k_2^{*3}k_1^{*3}}{(k_2-k_1^*)^2(k_2-k_2^*)^2(k_3-k_2^*)^2(k_1-k_1^*)^2(k_3-k_1^*)^2(k_1-k_2^*)^2},$$
$$D_2=\frac{(k_1^*-k_3^*)^2(k_2-k_3)^2(k_1-k_3)^2(k_1-k_2)^2k_3^{*3}k_1^{*3}}{(k_3-k_1^*)^2(k_3-k_3^*)^2(k_2-k_3^*)^2(k_1-k_1^*)^2(k_2-k_1^*)^2(k_1-k_3^*)^2}, \tag{60}$$
$$D_3=\frac{(k_2^*-k_3^*)^2(k_2-k_3)^2(k_1-k_3)^2(k_1-k_2)^2k_3^{*3}k_2^{*3}}{(k_3-k_2^*)^2(k_3-k_3^*)^2(k_2-k_3^*)^2(k_1-k_2^*)^2(k_2-k_2^*)^2(k_1-k_3^*)^2},$$

and

$$
\begin{aligned}
D_1^* &= \frac{(k_1-k_2)^2(k_2^*-k_3^*)^2(k_1^*-k_3^*)^2(k_1^*-k_2^*)^2k_2^3k_1^3}{(k_2^*-k_1)^2(k_2^*-k_2)^2(k_3^*-k_2)^2(k_1^*-k_1)^2(k_3^*-k_1)^2(k_1^*-k_2)^2},\\
D_2^* &= \frac{(k_1-k_3)^2(k_2^*-k_3^*)^2(k_1^*-k_3^*)^2(k_1^*-k_2^*)^2k_3^3k_1^3}{(k_3^*-k_1)^2(k_3^*-k_3)^2(k_2^*-k_3)^2(k_1^*-k_1)^2(k_2^*-k_1)^2(k_1^*-k_3)^2},\\
D_3^* &= \frac{(k_2-k_3)^2(k_2^*-k_3^*)^2(k_1^*-k_3^*)^2(k_1^*-k_2^*)^2k_3^3k_2^3}{(k_3^*-k_2)^2(k_3^*-k_3)^2(k_2^*-k_3)^2(k_1^*-k_2)^2(k_2^*-k_2)^2(k_1^*-k_3)^2}.
\end{aligned} \tag{61}
$$

Then, substituting the expressions for G_1, G_1^*, G_3, G_3^*, G_5, G_5^*, F_2, F_2^*, F_4 and F_4^* into Eq. (47), the functions F_6 and F_6^* are given as follows

$$F_6 = E_1 e^{\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*+\eta_3^*}, \tag{62}$$

$$F_6^* = E_1^* e^{\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*+\eta_3^*}, \tag{63}$$

where

$$
\begin{aligned}
E_1 &= -\frac{M}{N},\\
E_1^* &= -\frac{M^*}{N^*},\\
M &= k_3^*k_2^*k_1^*(k_2^*-k_3^*)^2k_1^2(k_2-k_3)^2(k_1^*-k_3^*)^2(k_1-k_3)^2k_2^2(k_1-k_2)^2(k_1^*-k_2^*)^2k_3^2,\\
N &=(k_1^*-k_2)^2(k_1-k_3^*)^2(k_2^*-k_3)^2(k_1-k_1^*)^2(k_2-k_2^*)^2(k_3-k_3^*)^2(k_1-k_2^*)^2(k_1^*-k_3)^2\\
&\quad (k_2-k_3^*)^2.
\end{aligned} \tag{64}
$$

The general nonlocal three-soliton solution of the reverse space nonlocal FL equation (1) is given as follows

$$u(x,t) = \frac{G_1+G_3+G_5}{1+F_2+F_4+F_6}. \tag{65}$$

According to the bilinear form of parity transformed complex conjugate equation, the parity transformed complex conjugate field is derived in the form

$$u^*(-x,t) = \frac{G_1^*+G_3^*+G_5^*}{1+F_2^*+F_4^*+F_6^*}. \tag{66}$$

In order to derive three-soliton solutions of the reverse time and reverse space-time nonlocal FL equation, we substitute transformations $x \to -ix$, $t \to it$ into three-soliton solutions Eqs. (65)-(66) of the reverse space nonlocal FL equation. These solutions are presented as follows

$$a)\ u(x,t) = \frac{G_1^{(1)}+G_3^{(1)}+G_5^{(1)}}{1+F_2^{(1)}+F_4^{(1)}+F_6^{(1)}}, \tag{67}$$

$$u^*(x,-t) = \frac{G_1^{*(1)}+G_3^{*(1)}+G_5^{*(1)}}{1+F_2^{*(1)}+F_4^{*(1)}+F_6^{*(1)}}, \tag{68}$$

$$b)\ u(x,t) = \frac{G_1^{(2)}+G_3^{(2)}+G_5^{(2)}}{1+F_2^{(2)}+F_4^{(2)}+F_6^{(2)}}, \tag{69}$$

$$u^*(-x,-t) = \frac{G_1^{*(2)}+G_3^{*(2)}+G_5^{*(2)}}{1+F_2^{*(2)}+F_4^{*(2)}+F_6^{*(2)}}. \tag{70}$$

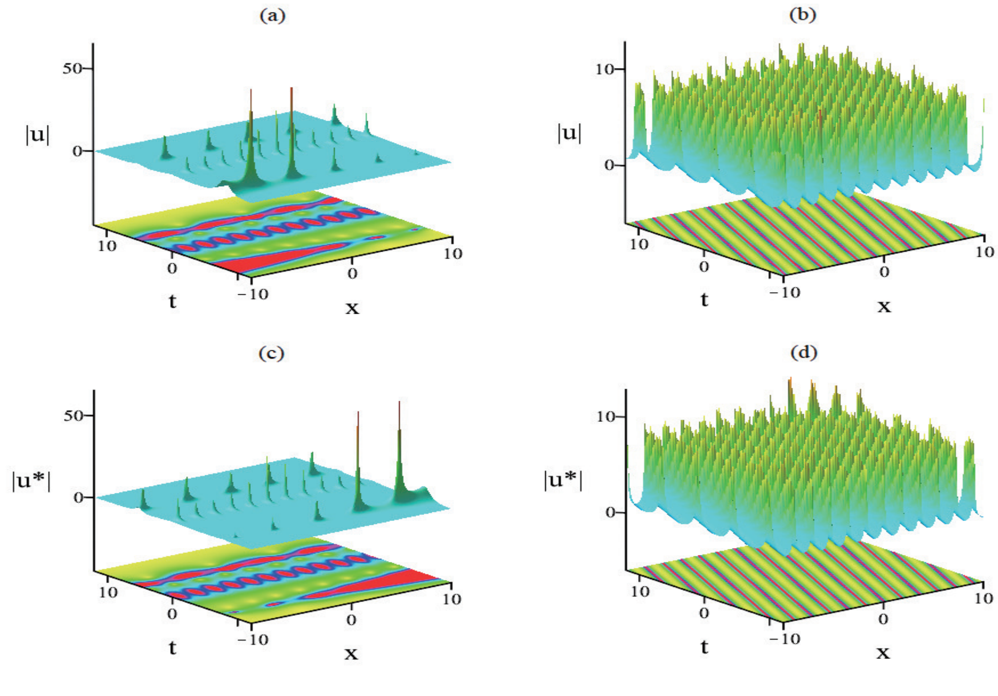

Figure 7: Three-soliton solutions of reverse space and reverse space-time nonlocal FL equations (with parameters: $k_1 = 1.75i$, $k_1^* = -1.75i$, $k_2 = -0.75i$, $k_2^* = 0.75i$, $k_3 = -0.19 - 0.75i$, $k_3^* = -0.19 + 0.75i$, $\eta_{10} = \eta_{10}^* = -1.5$, $\eta_{20} = \eta_{20}^* = 6$, $\eta_{30} = \eta_{30}^* = -5$). (a) and (c) describe the reverse space FL equation; (b) and (d) describe the reverse space-time FL equation.

where

$$
\begin{aligned}
G_1^{(1)} &= e^{\xi_1} + e^{\xi_2} + e^{\xi_3}, \\
G_3^{(1)} &= B_1 e^{\xi_1+\xi_2+\xi_1^*} + B_2 e^{\xi_1+\xi_2+\xi_2^*} + B_3 e^{\xi_1+\xi_2+\xi_3^*} + B_4 e^{\xi_1+\xi_3+\xi_1^*} + B_5 e^{\xi_1+\xi_3+\xi_2^*} + \\
&\quad B_6 e^{\xi_1+\xi_3+\xi_3^*} + B_7 e^{\xi_2+\xi_3+\xi_1^*} + B_8 e^{\xi_2+\xi_3+\xi_2^*} + B_9 e^{\xi_2+\xi_3+\xi_3^*}, \\
G_5^{(1)} &= D_1 e^{\xi_1+\xi_2+\xi_3+\xi_1^*+\xi_2^*} + D_2 e^{\xi_1+\xi_2+\xi_3+\xi_1^*+\xi_3^*} + D_3 e^{\xi_1+\xi_2+\xi_3+\xi_2^*+\xi_3^*}, \\
F_2^{(1)} &= A_1 e^{\xi_1+\xi_1^*} + A_2 e^{\xi_1+\xi_2^*} + A_3 e^{\xi_1+\xi_3^*} + A_4 e^{\xi_2+\xi_1^*} + A_5 e^{\xi_2+\xi_2^*} + A_6 e^{\xi_2+\xi_3^*} + A_7 e^{\xi_3+\xi_1^*} \\
&\quad + A_8 e^{\xi_3+\xi_2^*} + A_9 e^{\xi_3+\xi_3^*}, \\
F_4^{(1)} &= C_1 e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*} + C_2 e^{\xi_1+\xi_2+\xi_1^*+\xi_3^*} + C_3 e^{\xi_1+\xi_2+\xi_2^*+\xi_3^*} + C_4 e^{\xi_1+\xi_3+\xi_1^*+\xi_2^*} + \\
&\quad C_5 e^{\xi_1+\xi_3+\xi_1^*+\xi_3^*} + C_6 e^{\xi_1+\xi_3+\xi_2^*+\xi_3^*} + C_7 e^{\xi_2+\xi_3+\xi_1^*+\xi_2^*} + C_8 e^{\xi_2+\xi_3+\xi_1^*+\xi_3^*} + \\
&\quad C_9 e^{\xi_2+\xi_3+\xi_2^*+\xi_3^*}, \\
F_6^{(1)} &= E_1 e^{\xi_1+\xi_2+\xi_3+\xi_1^*+\xi_2^*+\xi_3^*}, \\
\xi_1 &= -ik_1x - i\omega_1 t + \eta_{10}, \xi_1^* = ik_1^*x - i\omega_1^* t + \eta_{10}^*, \\
\xi_2 &= -ik_2x - i\omega_2 t + \eta_{20}, \xi_2^* = ik_2^*x - i\omega_2^* t + \eta_{20}^*, \\
\xi_3 &= -ik_3x - i\omega_3 t + \eta_{30}, \xi_3^* = ik_3^*x - i\omega_3^* t + \eta_{30}^*,
\end{aligned}
$$

and

$$
\begin{aligned}
G_1^{(2)} &= e^{\zeta_1} + e^{\zeta_2} + e^{\zeta_3},\\
G_3^{(2)} &= B_1 e^{\zeta_1+\zeta_2+\zeta_1^*} + B_2 e^{\zeta_1+\zeta_2+\zeta_2^*} + B_3 e^{\zeta_1+\zeta_2+\zeta_3^*} + B_4 e^{\zeta_1+\zeta_3+\zeta_1^*} + B_5 e^{\zeta_1+\zeta_3+\zeta_2^*} +\\
&\quad B_6 e^{\zeta_1+\zeta_3+\zeta_3^*} + B_7 e^{\zeta_2+\zeta_3+\zeta_1^*} + B_8 e^{\zeta_2+\zeta_3+\zeta_2^*} + B_9 e^{\zeta_2+\zeta_3+\zeta_3^*},\\
G_5^{(2)} &= D_1 e^{\zeta_1+\zeta_2+\zeta_3+\zeta_1^*+\zeta_2^*} + D_2 e^{\zeta_1+\zeta_2+\zeta_3+\zeta_1^*+\zeta_3^*} + D_3 e^{\zeta_1+\zeta_2+\zeta_3+\zeta_2^*+\zeta_3^*},\\
F_2^{(2)} &= A_1 e^{\zeta_1+\zeta_1^*} + A_2 e^{\zeta_1+\zeta_2^*} + A_3 e^{\zeta_1+\zeta_3^*} + A_4 e^{\zeta_2+\zeta_1^*} + A_5 e^{\zeta_2+\zeta_2^*} + A_6 e^{\zeta_2+\zeta_3^*} + A_7 e^{\zeta_3+\zeta_1^*}\\
&\quad + A_8 e^{\zeta_3+\zeta_2^*} + A_9 e^{\zeta_3+\zeta_3^*},\\
F_4^{(2)} &= C_1 e^{\zeta_1+\zeta_2+\zeta_1^*+\zeta_2^*} + C_2 e^{\zeta_1+\zeta_2+\zeta_1^*+\zeta_3^*} + C_3 e^{\zeta_1+\zeta_2+\zeta_2^*+\zeta_3^*} + C_4 e^{\zeta_1+\zeta_3+\zeta_1^*+\zeta_2^*} +\\
&\quad C_5 e^{\zeta_1+\zeta_3+\zeta_1^*+\zeta_3^*} + C_6 e^{\zeta_1+\zeta_3+\zeta_2^*+\zeta_3^*} + C_7 e^{\zeta_2+\zeta_3+\zeta_1^*+\zeta_2^*} + C_8 e^{\zeta_2+\zeta_3+\zeta_1^*+\zeta_3^*} +\\
&\quad C_9 e^{\zeta_2+\zeta_3+\zeta_2^*+\zeta_3^*},\\
F_6^{(2)} &= E_1 e^{\zeta_1+\zeta_2+\zeta_3+\zeta_1^*+\zeta_2^*+\zeta_3^*},\\
\zeta_1 &= -k_1 x - i\omega_1 t + \eta_{10}, \zeta_1^* = k_1^* x - i\omega_1^* t + \eta_{10}^*,\\
\zeta_2 &= -k_2 x - i\omega_2 t + \eta_{20}, \zeta_2^* = k_2^* x - i\omega_2^* t + \eta_{20}^*,\\
\zeta_3 &= -k_3 x - i\omega_3 t + \eta_{30}, \zeta_3^* = k_3^* x - i\omega_3^* t + \eta_{30}^*.
\end{aligned}
$$

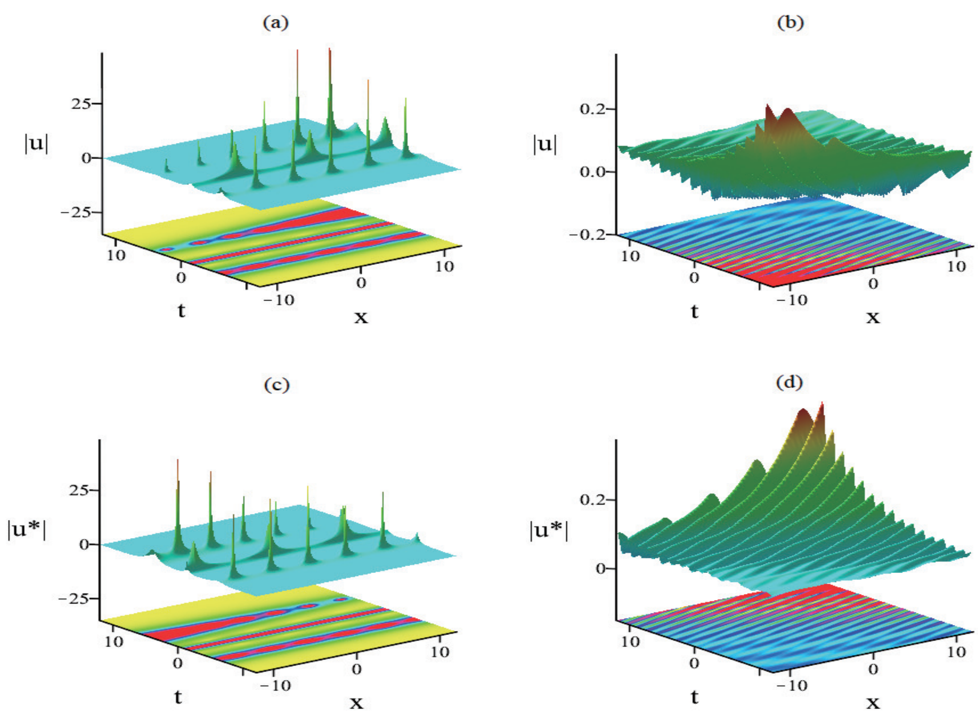

Figure 8: Three-soliton solutions of reverse space and reverse space-time nonlocal FL equations (with parameters: $k_1 = 0.7i$, $k_1^* = -0.7i$, $k_2 = 0.1 - 0.8i$, $k_2^* = 0.1 + 0.8i$, $k_3 = -0.35i$, $k_3^* = 0.35i$, $\eta_{10} = \eta_{10}^* = 2.5$, $\eta_{20} = \eta_{20}^* = 1$, $\eta_{30} = \eta_{30}^* = -3$). (a) and (c) describe the reverse space FL equation; (b) and (d) describe the reverse space-time FL equation.

Then some figures are presented to describe the three-soliton solutions (65)-(70) of three types of nonlocal FL equations (see Fig.7-Fig.10). In these figures, (a) and (b) are the profiles of $|u|$, (c) and (d) are the profiles of $|u^*|$. Figure 7 and figure 8 show the comparison between the reverse space FL equation and the reverse space-time FL equation. And figure 9 and figure 10 show the difference between the reverse time FL equation and the reverse space-time FL equation. These figures have the same parameters k_1, k_2, k_3, η_{10}, η_{20} and η_{30} for different equations. Through these pictures, the difference between three-soliton solutions of two different nonlocal FL equations can be observed intuitively. It is obvious that $|u|$ and $|u^*|$ of the reverse space/time FL equation have the same shapes as spatial/time evolution, but their enhancing shapes are antipodal, and profiles of the reverse space FL equation and the reverse time FL equation present three breather-like solitons. The solutions of the reverse space-time FL equation are periodic, and in Fig.8 and Fig.10 the results exhibit the periodic oscillations with exponential growth trend, and $|u(x,t)|$ and $|u^*(-x,-t)|$ have the opposite enhancing directions as time evolution. Through these figures, the shapes of three-soliton solutions of the reverse space/time FL equation are parallel with the x or t axis, however three-soliton solution of the reverse space-time FL equation can be viewed the parallel superposition of time and space local breather-like solitons.

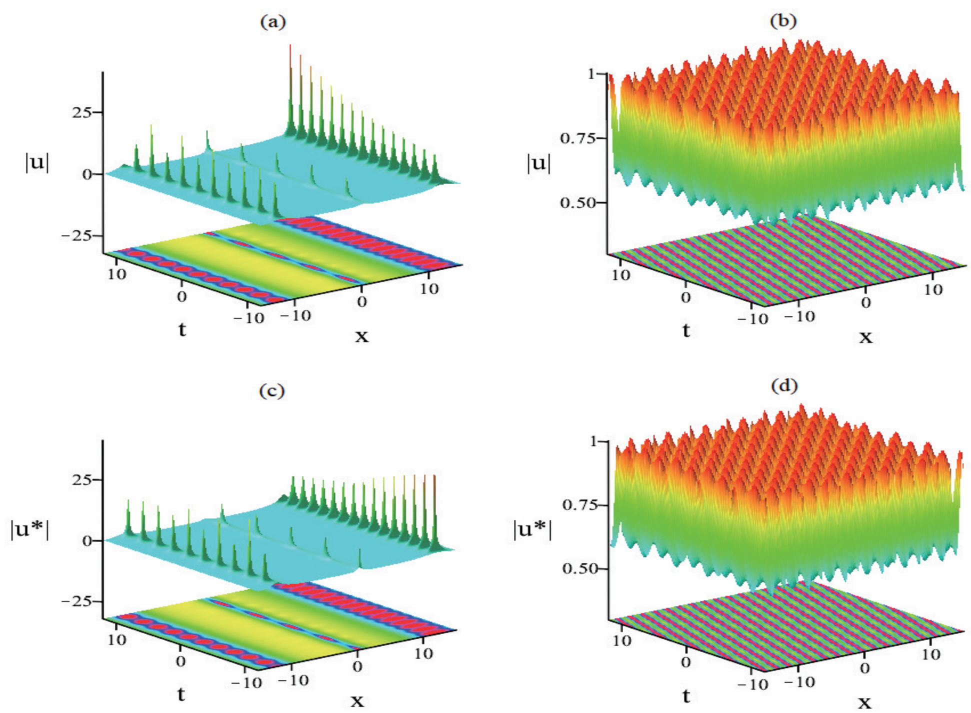

Figure 9: Three-soliton solutions of reverse time and reverse space-time nonlocal FL equations (with parameters: $k_1 = 1.7i$, $k_1^* = -1.7i$, $k_2 = -0.5i$, $k_2^* = 0.5i$, $k_3 = -0.75i$, $k_3^* = 0.75i$, $\eta_{10} = \eta_{10}^* = -1.5$, $\eta_{20} = \eta_{20}^* = 6$, $\eta_{30} = \eta_{30}^* = -5$). (a) and (c) describe the reverse time FL equation; (b) and (d) describe the reverse space-time FL equation.

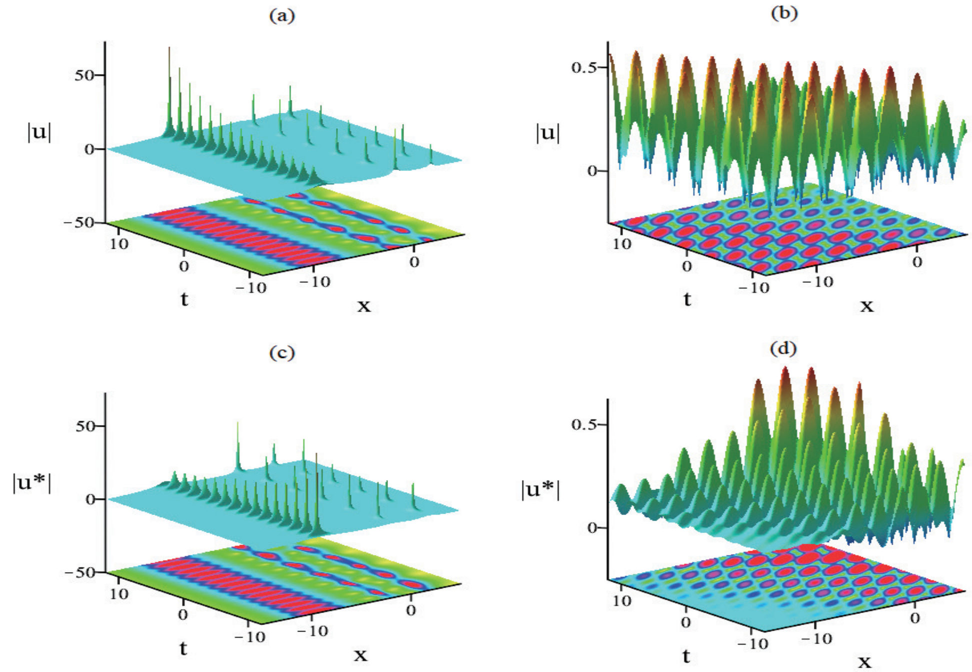

Figure 10: Three solitons solutions of reverse time and reverse space-time nonlocal FL equations (with parameters: $k_1 = 0.5i$, $k_1^* = -0.5i$, $k_2 = 0.25 - 1.5i$, $k_2^* = 0.25 + 1.5i$, $k_3 = -1.22i$, $k_3^* = 1.22i$, $\eta_{10} = \eta_{10}^* = 2.5$, $\eta_{20} = \eta_{20}^* = 1$, $\eta_{30} = \eta_{30}^* = 1$). (a) and (c) describe the reverse time FL equation; (b) and (d) describe the reverse space-time FL equation.

3 Asymptotic analysis

3.1 Asymptotic analysis on two-soliton solution of the reverse space FL equation

Through asymptotic analysis in [34], it shows that when solitons undergo multiple collisions, there exists possibility of soliton's shape restoration. Asymptotic analysis is used to investigate the elastic and inelastic interactions between the bound solitons and the regular one soliton [35].

Considering the above two-soliton solution Eq. (36), without loss of generality, we assume that $\eta_{10} = \eta_{20} = 0$ and $k_1/k_2 > 0$. For fixed η_1, note that $\eta_2 + \eta_2^* = 2\mathrm{Re}(\frac{k_2}{k_1}\eta_1) + 2\mathrm{Re}(\frac{k_2}{k_1}\omega_1 - \omega_2)t$, and suppose $\mathrm{Re}(\frac{k_2}{k_1}\omega_1 - \omega_2) > 0$.

i) Taking limit $t \to -\infty$: $\eta_1 + \eta_1^* \sim 0$, $\eta_2 + \eta_2^* \sim -\infty$, the asymptotic expressions for the two solitons before interaction can be given by

$$u^{1-} \sim \frac{1}{2}e^{\frac{\eta_1 - \eta_1^* - \alpha_1}{2}}\mathrm{sech}(\frac{\eta_1 + \eta_1^* + \alpha_1}{2}), \tag{71}$$
$$e^{\alpha_1} = A_1 = \frac{ik_1}{(k_1 - k_1^*)(-\omega_1 - \omega_1^*)}.$$

ii) Taking limit $t \to +\infty$: $\eta_1 + \eta_1^* \sim 0$, $\eta_2 + \eta_2^* \sim +\infty$, the asymptotic expressions for the two solitons after interaction can be given by

$$u^{1+} \sim \frac{B_2}{2A_4} e^{\frac{\eta_1 - \eta_1^* - \alpha_2}{2}} \mathrm{sech}(\frac{\eta_1 + \eta_1^* + \alpha_2}{2}), \tag{72}$$
$$e^{\alpha_2} = \frac{C_1}{A_4} = -\frac{(k_1 - k_2)^2 (k_1^* - k_2^*)^2 k_1^2 k_1^*}{(k_1 - k_1^*)^2 (k_1 - k_2^*)^2 (k_2 - k_1^*)^2}.$$

For fixed η_2, note that $\eta_1 + \eta_1^* = 2\mathrm{Re}(\frac{k_1}{k_2}\eta_2) + 2\mathrm{Re}(\frac{k_1}{k_2}\omega_2 - \omega_1)t$, and it is obvious that $\mathrm{Re}(\frac{k_1}{k_2}\omega_2 - \omega_1) < 0$.

i) Taking limit $t \to -\infty$: $\eta_1 + \eta_1^* \sim +\infty$, $\eta_2 + \eta_2^* \sim 0$, the asymptotic expressions for the two solitons before interaction can be given by

$$u^{2-} \sim \frac{B_1}{2A_1} e^{\frac{\eta_2 - \eta_2^* - \alpha_3}{2}} \mathrm{sech}(\frac{\eta_2 + \eta_2^* + \alpha_3}{2}), \tag{73}$$
$$e^{\alpha_3} = \frac{C_1}{A_1} = -\frac{(k_1 - k_2)^2 (k_1^* - k_2^*)^2 k_2^2 k_2^*}{(k_1 - k_2^*)^2 (k_2 - k_1^*)^2 (k_2 - k_2^*)^2}.$$

ii) Taking limit $t \to +\infty$: $\eta_1 + \eta_1^* \sim -\infty$, $\eta_2 + \eta_2^* \sim 0$, the asymptotic expressions for the two solitons after interaction can be given by

$$u^{2+} \sim \frac{1}{2} e^{\frac{\eta_2 - \eta_2^* - \alpha_4}{2}} \mathrm{sech}(\frac{\eta_2 + \eta_2^* + \alpha_4}{2}), \tag{74}$$
$$e^{\alpha_4} = A_4 = \frac{ik_2}{(k_2 - k_2^*)(-\omega_2 - \omega_2^*)}.$$

Comparing the asymptotic expressions of two-soliton solution between before interaction and after interaction, we find that k_1, k_1^*, k_2 and k_2^* accord with the conditions

$$\frac{k_1^2 |k_2 - k_1^*||k_1^* - k_2^*|}{k_1^{*2} |k_1 - k_2||k_1 - k_2^*|} = 1 \quad \text{and} \quad \frac{k_2^2 |k_1 - k_2^*||k_1^* - k_2^*|}{k_2^{*2} |k_1 - k_2||k_2 - k_1^*|} = 1, \tag{75}$$

the relations of amplitudes can be obtained

$$\mathrm{Am}^{1-} = \mathrm{Am}^{1+} \quad \text{and} \quad \mathrm{Am}^{2-} = \mathrm{Am}^{2+}, \tag{76}$$

where Am^{1-} and Am^{2-} denote the amplitudes for the two solitons before the interaction, while Am^{1+} and Am^{2+} denote the amplitudes for the two solitons after the interaction. When k_1, k_1^*, k_2 and k_2^* do not accord with conditions (75), it can be yields

$$\mathrm{Am}^{1-} \neq \mathrm{Am}^{1+} \quad \text{and} \quad \mathrm{Am}^{2-} \neq \mathrm{Am}^{2+}. \tag{77}$$

Through expressions (76) and (77), it is obvious that the elastic interaction for two-soliton solution of the reverse space nonlocal FL equation appears under conditions (75), inelastic interaction for two-soliton solution of the reverse space nonlocal FL equation arises beyond conditions (75).

3.2 Asymptotic analysis on three-soliton solution of the reverse space FL equation

Considering the above three-soliton solution Eq. (65), without loss of generality, we assume that $\eta_{10}=\eta_{20}=\eta_{30}=0$, $k_1/k_2>0$, $k_2/k_3>0$ and $k_1/k_3>0$. For fixed η_1, note that $\eta_2+\eta_2^*=2\mathrm{Re}(\frac{k_2}{k_1}\eta_1)+2\mathrm{Re}(\frac{k_2}{k_1}\omega_1-\omega_2)t$ and $\eta_3+\eta_3^*=2\mathrm{Re}(\frac{k_3}{k_1}\eta_1)+2\mathrm{Re}(\frac{k_3}{k_1}\omega_1-\omega_3)t$, and suppose $\mathrm{Re}(\frac{k_2}{k_1}\omega_1-\omega_2)>0$ and $\mathrm{Re}(\frac{k_3}{k_1}\omega_1-\omega_3)>0$.

i) Taking limit $t\to-\infty$: $\eta_1+\eta_1^*\sim 0$, $\eta_2+\eta_2^*\sim-\infty$, $\eta_3+\eta_3^*\sim-\infty$, the asymptotic expressions for the three solitons before interaction can be given by

$$u^{1-}\sim\frac{1}{2}e^{\frac{\eta_1-\eta_1^*-\alpha_1}{2}}\mathrm{sech}(\frac{\eta_1+\eta_1^*+\alpha_1}{2}),\tag{78}$$
$$e^{\alpha_1}=A_1=\frac{ik_1}{(k_1-k_1^*)(-\omega_1-\omega_1^*)}.$$

ii) Taking limit $t\to+\infty$: $\eta_1+\eta_1^*\sim 0$, $\eta_2+\eta_2^*\sim+\infty$, $\eta_3+\eta_3^*\sim+\infty$, the asymptotic expressions for the three solitons after interaction can be given by

$$u^{1+}\sim\frac{D_3}{2C_9}e^{\frac{\eta_1-\eta_1^*-\alpha_5}{2}}\mathrm{sech}(\frac{\eta_1+\eta_1^*+\alpha_5}{2}),\tag{79}$$
$$e^{\alpha_5}=\frac{E_1}{C_9}=-\frac{(k_1-k_2)^2(k_1-k_3)^2(k_1^*-k_2^*)^2(k_1^*-k_3^*)^2k_1^2k_1^*}{(k_1-k_1^*)^2(k_1-k_2^*)^2(k_1-k_3^*)^2(k_2-k_1^*)^2(k_3-k_1^*)^2}.$$

For fixed η_2, note that $\eta_1+\eta_1^*=2\mathrm{Re}(\frac{k_1}{k_2}\eta_2)+2\mathrm{Re}(\frac{k_1}{k_2}\omega_2-\omega_1)t$ and $\eta_3+\eta_3^*=2\mathrm{Re}(\frac{k_3}{k_2}\eta_2)+2\mathrm{Re}(\frac{k_3}{k_2}\omega_2-\omega_3)t$. Supposing $\mathrm{Re}(\frac{k_3}{k_2}\omega_2-\omega_3)>0$, it is obvious that $\mathrm{Re}(\frac{k_1}{k_2}\omega_2-\omega_1)<0$.

i) Taking limit $t\to-\infty$: $\eta_1+\eta_1^*\sim+\infty$, $\eta_2+\eta_2^*\sim 0$, $\eta_3+\eta_3^*\sim-\infty$, the asymptotic expressions for the three solitons before interaction can be given by

$$u^{2-}\sim\frac{B_1}{2A_1}e^{\frac{\eta_2-\eta_2^*-\alpha_6}{2}}\mathrm{sech}(\frac{\eta_2+\eta_2^*+\alpha_6}{2}),\tag{80}$$
$$e^{\alpha_6}=\frac{C_1}{A_1}=-\frac{(k_1-k_2)^2(k_1^*-k_2^*)^2k_2^2k_2^*}{(k_1-k_2^*)^2(k_2-k_1^*)^2(k_2-k_2^*)^2}.$$

ii) Taking limit $t\to+\infty$: $\eta_1+\eta_1^*\sim-\infty$, $\eta_2+\eta_2^*\sim 0$, $\eta_3+\eta_3^*\sim+\infty$, the asymptotic expressions for the three solitons after interaction can be given by

$$u^{2+}\sim\frac{B_9}{2A_9}e^{\frac{\eta_2-\eta_2^*-\alpha_7}{2}}\mathrm{sech}(\frac{\eta_2+\eta_2^*+\alpha_7}{2}),\tag{81}$$
$$e^{\alpha_7}=\frac{C_9}{A_9}=-\frac{(k_2-k_3)^2(k_2^*-k_3^*)^2k_2^2k_2^*}{(k_2-k_2^*)^2(k_2-k_3^*)^2(k_3-k_2^*)^2}.$$

For fixed η_3, note that $\eta_1+\eta_1^*=2\mathrm{Re}(\frac{k_1}{k_3}\eta_3)+2\mathrm{Re}(\frac{k_1}{k_3}\omega_3-\omega_1)t$ and $\eta_2+\eta_2^*=2\mathrm{Re}(\frac{k_2}{k_3}\eta_3)+2\mathrm{Re}(\frac{k_2}{k_3}\omega_3-\omega_2)t$. It is obvious that $\mathrm{Re}(\frac{k_1}{k_3}\omega_3-\omega_1)<0$ and $\mathrm{Re}(\frac{k_2}{k_3}\omega_3-\omega_2)<0$.

i) Taking limit $t \to -\infty$: $\eta_1 + \eta_1^* \sim +\infty$, $\eta_2 + \eta_2^* \sim +\infty$, $\eta_3 + \eta_3^* \sim 0$, the asymptotic expressions for the three solitons before interaction can be given by

$$u^{3-} \sim \frac{D_1}{2C_1} e^{\frac{\eta_3 - \eta_3^* - \alpha_8}{2}} \operatorname{sech}(\frac{\eta_3 + \eta_3^* + \alpha_8}{2}), \tag{82}$$
$$e^{\alpha_8} = \frac{E_1}{C_1} = -\frac{(k_1 - k_3)^2 (k_1^* - k_3^*)^2 (k_2 - k_3)^2 (k_2^* - k_3^*)^2 k_3^2 k_3^*}{(k_1 - k_3^*)^2 (k_2 - k_3^*)^2 (k_3 - k_1^*)^2 (k_3 - k_2^*)^2 (k_3 - k_3^*)^2}.$$

ii) Taking limit $t \to +\infty$: $\eta_1 + \eta_1^* \sim -\infty$, $\eta_2 + \eta_2^* \sim -\infty$, $\eta_3 + \eta_3^* \sim 0$, the asymptotic expressions for the three solitons after interaction can be given by

$$u^{3+} \sim \frac{1}{2} e^{\frac{\eta_3 - \eta_3^* - \alpha_9}{2}} \operatorname{sech}(\frac{\eta_3 + \eta_3^* + \alpha_9}{2}), \tag{83}$$
$$e^{\alpha_9} = A_9 = \frac{ik_3}{(k_3 - k_3^*)(-\omega_3 - \omega_3^*)}.$$

Comparing the asymptotic expressions of three-soliton solution between before interaction and after interaction, we find that k_1, k_1^*, k_2, k_2^*, k_3 and k_3^* accord with the conditions

$$\frac{k_2^2 k_3^2 |k_1 - k_2^*||k_1 - k_3^*||k_1^* - k_2^*||k_1^* - k_3^*|}{k_2^{*2} k_3^{*2} |k_1 - k_2||k_1 - k_3||k_2 - k_1^*||k_3 - k_1^*|} = 1, \tag{84}$$
$$\frac{k_1^2 k_3^{*2} |k_2 - k_3||k_3 - k_2^*||k_2 - k_1^*||k_1^* - k_2^*|}{k_1^{*2} k_3^2 |k_1 - k_2||k_1 - k_2^*||k_2 - k_3^*||k_2^* - k_3^*|} = 1, \tag{85}$$
$$\frac{k_1^2 k_2^2 |k_1^* - k_3^*||k_2^* - k_3^*||k_3 - k_1^*||k_3 - k_2^*|}{k_1^{*2} k_2^{*2} |k_1 - k_3||k_1 - k_3^*||k_2 - k_3||k_2 - k_3^*|} = 1, \tag{86}$$

the relations of amplitudes can be obtained

$$\mathrm{Am}^{1-} = \mathrm{Am}^{1+}, \quad \mathrm{Am}^{2-} = \mathrm{Am}^{2+} \quad \text{and} \quad \mathrm{Am}^{3-} = \mathrm{Am}^{3+}, \tag{87}$$

where Am^{1-}, Am^{2-} and Am^{3-} denote the amplitudes for the three solitons before the interaction, while Am^{1+}, Am^{2+} and Am^{3+} denote the amplitudes for the three solitons after the interaction. When k_1, k_1^*, k_2, k_2^*, k_3 and k_3^* do not accord with conditions (84)-(86), we have

$$\mathrm{Am}^{1-} \neq \mathrm{Am}^{1+}, \quad \mathrm{Am}^{2-} \neq \mathrm{Am}^{2+} \quad \text{and} \quad \mathrm{Am}^{3-} \neq \mathrm{Am}^{3+}. \tag{88}$$

Through expressions (87) and (88), it is obvious that the elastic interaction for three-soliton of the reverse space nonlocal FL equation appears under conditions (84)-(86), inelastic interaction for three-soliton of the reverse space nonlocal FL equation arises beyond conditions (84)-(86).

4 Lax pair and conservation laws for three types of nonlocal FL equations

4.1 Lax pair and integrability

In this subsection, the integrability of nonlocal FL equations will be shown by finding their Lax pairs which constructed from matrix generalization. The Lax pair for the reverse space nonlocal

FL equation (1) can be expressed as follows

$$\Psi_{S,x} = U_1 \Psi_S, \qquad \Psi_{S,t} = V_1 \Psi_S. \tag{89}$$

with

$$U_1 = \begin{pmatrix} \frac{1}{2}\lambda^2 & -\lambda u_x(x,t) \\ \lambda u_x^*(-x,t) & -\frac{1}{2}\lambda^2 \end{pmatrix},$$
$$V_1 = \begin{pmatrix} \frac{i}{2\lambda^2} - iu(x,t)u^*(-x,t) & \frac{i}{\lambda}u(x,t) \\ \frac{i}{\lambda}u^*(-x,t) & -\frac{i}{2\lambda^2} + iu(x,t)u^*(-x,t) \end{pmatrix},$$

where $\Psi_S = (\psi_{S,1}, \psi_{S,2})^T$ is a column vector function, and Ψ_T and Ψ_{ST} below are also column vector functions. The compatibility condition of the Lax pair, which is zero curvature equation $U_{1t} - V_{1x} + [U_1, V_1] = 0$, leads to Eq.(1). These variable transformations (2) and (3) allow us to derive the Lax pair of the reverse time and reverse space-time nonlocal FL equations from that of the reverse space one. The Lax pair for the reverse time nonlocal FL equation (4) is derived as follows

$$\Psi_{T,x} = U_2 \Psi_T, \qquad \Psi_{T,t} = V_2 \Psi_T. \tag{90}$$

with

$$U_2 = \begin{pmatrix} -\frac{i}{2}\lambda^2 & -\lambda u_x(x,t) \\ \lambda u_x^*(x,-t) & \frac{i}{2}\lambda^2 \end{pmatrix},$$
$$V_2 = \begin{pmatrix} -\frac{1}{2\lambda^2} + u(x,t)u^*(x,-t) & -\frac{1}{\lambda}u(x,t) \\ -\frac{1}{\lambda}u^*(x,-t) & \frac{1}{2\lambda^2} - u(x,t)u^*(x,-t) \end{pmatrix}.$$

The Lax pair for the reverse space-time nonlocal FL equation (5) is shown as follows

$$\Psi_{ST,x} = U_3 \Psi_{ST}, \qquad \Psi_{ST,t} = V_3 \Psi_{ST}. \tag{91}$$

with

$$U_3 = \begin{pmatrix} -\frac{1}{2}\lambda^2 & -\lambda u_x(x,t) \\ \lambda u_x^*(-x,-t) & \frac{1}{2}\lambda^2 \end{pmatrix},$$
$$V_3 = \begin{pmatrix} -\frac{1}{2\lambda^2} + u(x,t)u^*(-x,-t) & -\frac{1}{\lambda}u(x,t) \\ -\frac{1}{\lambda}u^*(-x,-t) & \frac{1}{2\lambda^2} - u(x,t)u^*(-x,-t) \end{pmatrix}.$$

The transformation relationship between these equations provides an effective method for us to derive the Lax pairs of different equations. In fact, given the solutions of the reverse space nonlocal FL equation, the solutions of reverse time and reverse space-time counterparts can be derived from the principle. However, if not, then the solutions of reverse time and reverses pace-time nonlocal FL equation may be derive desired solutions by other methods.

4.2 Conservation laws

Based on the Lax pair, the infinitely many conservation laws are constructed in both positive and negative orders. We consider the associated spectral problem of the reverse space nonlocal FL equation

$$\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right)_x=\left(\begin{array}{cc}\frac{1}{2}\lambda^2 & -\lambda u_x\\ \lambda u_x^* & -\frac{1}{2}\lambda^2\end{array}\right)\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right), \tag{92}$$

and associate time evolution equation

$$\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right)_t=\left(\begin{array}{cc}\frac{i}{2\lambda^2}-iuu^* & \frac{i}{\lambda}u\\ \frac{i}{\lambda}u^* & -\frac{i}{2\lambda^2}+iuu^*\end{array}\right)\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right). \tag{93}$$

They satisfy the following expression

$$u_x(\frac{\psi_2}{\psi_1})_x=\lambda u_x u_x^*-\lambda^2 u_x\frac{\psi_2}{\psi_1}+\lambda u_x^2(\frac{\psi_2}{\psi_1})^2, \tag{94}$$

$$u(\frac{\psi_2}{\psi_1})_t=\frac{i}{\lambda}uu^*+(-\frac{i}{\lambda^2}+2iuu^*)u\frac{\psi_2}{\psi_1}-\frac{i}{\lambda}u^2(\frac{\psi_2}{\psi_1})^2. \tag{95}$$

The expression of $\frac{\psi_2}{\psi_1}$ is given as follows

$$\frac{\psi_2}{\psi_1}=\sum_{i=1}^{\infty}\mathbf{P}_i\lambda^{-2i+1}. \tag{96}$$

Substituting (96) into Eq. (94), and comparing the coefficients of λ, we obtain

$$P_1=u_x^*, \tag{97}$$

$$P_{i+1}=-P_{i,x}+\sum_{j=1}^{i}u_xP_jP_{i+1-j}.(i=1,2,...) \tag{98}$$

It can be easily shown that ψ_1 satisfies

$$(\mathrm{ln}\psi_1)_{xt}=(\mathrm{ln}\psi_1)_{tx}. \tag{99}$$

Hence, the conservation laws are derived as follows

$$(-\lambda u_x\frac{\psi_2}{\psi_1})_t=(-iuu^*+i\frac{1}{\lambda}u\frac{\psi_2}{\psi_1})_x, \tag{100}$$

which can be written as

$$(u_xP_i)_t=-(iuP_{i-1})_x,(i=1,2,...), \tag{101}$$
$$P_0=-u^*.$$

Among thses conservation laws, the first two are listed below

$$(u_x u_x^*)_t = (iuu^*)_x, \tag{102}$$

$$[u_x(-u_{xx}^* + u_x u_x^{*2})]_t = (-iuu_x^*)_x. \tag{103}$$

On the other hand, substituting the expansion

$$\frac{\psi_2}{\psi_1} = \sum_{i=1}^{\infty} \mathbf{Q}_i \lambda^{2i-1} \tag{104}$$

into Eq. (95) and comparing the coefficients of λ, one obtains

$$Q_1 = u^*, \tag{105}$$

$$Q_{i+1} = iQ_{i,t} + 2uu^* Q_i - \sum_{j=1}^{i} uP_j P_{i+1-j} \quad (i = 1, 2, ...). \tag{106}$$

Then other conservation laws are given as follows

$$(u_x Q_i)_t = -i(uP_{i+1})_x \quad (i = 1, 2, ...). \tag{107}$$

Among thses conservation laws, the first two are listed below

$$(u_x u^*)_t = -i[u(iu_t^* + uu^{*2})]_x, \tag{108}$$

$$[u_x(iu_t^* + uu^{*2})]_t = -i[u(-u_{tt}^* + i(uu^{*2})_t)]_x. \tag{109}$$

The transformations Eqs.(2)-(3) allows us to derive the conversation laws of the reverse time and reverse space-time nonlocal FL equation from those of the reverse space ones. The first two conversation laws for the reverse time nonlocal FL equation (4) are derived as follows

$$(-iu_x u_x^*)_t = (uu^*)_x, \tag{110}$$

$$[u_x(u_{xx} - iu_x u_x^{*2})]_t = (iuu_x^*)_x, \tag{111}$$

and

$$(u_x u^*)_t = [u(u_t^* + uu^{*2})]_x, \tag{112}$$

$$[u_x(u_t^* + uu^{*2})]_t = [u(u_{tt}^* + (uu^{*2})_t)]_x. \tag{113}$$

The first two conversation laws for the reverse space-time nonlocal FL equation (5) are derived as follows

$$(u_x u_x^*)_t = (uu^*)_x, \tag{114}$$

$$[u_x(u_{xx}^* + u_x u_x^{*2})]_t = (uu_x^*)_x, \tag{115}$$

and

$$(u_x u^*)_t = [u(u_t^* + uu^{*2})]_x, \tag{116}$$

$$[u_x(u_t^* + uu^{*2})]_t = [u(u_{tt}^* + (uu^{*2})_t)]_x. \tag{117}$$

So, through the transformation relationship between these equations, it is effective to provide the conversation laws of different equations. However the prerequisite for doing these things is knowing the Lax pairs of these equations.

5 Conclusions

In this paper, three types of nonlocal Fokas-Lenells equations are considered by means of the Hirota bilinear method. The one-, two- and three-soliton solutions of the reverse time and reverse space-time nonlocal FL equation are converted from those of the reverse space ones. Furthermore, the graphical representations are presented showing the shape of solution more visually, and the physical interpretation of the obtained figures is discussed for different choices of the parameters that occur in the solutions. Then, asymptotic analysis of two- and three-soliton solutions of reverse space nonlocal FL equation are given to understand the long time asymptotic behavior. The Lax integrability of three types of nonlocal FL equations is investigated using variable transformations, and infinitely many conservation laws are constructed based on the Lax pairs of different equations. These results might be useful to comprehend some physical phenomena and inspire some novel physical applications on other nonlinear system.

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Compliance with ethical standards
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